MATH 114 MIDTERM 2 SOLUTIONS

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True/False

(1) $f_y(a,b) = \lim_{y\to b} \frac{f(a,y)-f(a,b)}{y-b}$.

Answer: True. This is essentially the definition of f_y, the partial derivative of f with respect to y. You may recognize this in an alternate form:

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h},$$

but, if we let $h = y - b$, then this equation reduces to the one given in the problem.

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(2) If $f(x,y)$ is differentiable at some point (a,b), then

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

Answer: True. If f is differentiable at (a,b), then f is certainly continuous at (a,b). By definition, if f is continuous at (a,b), then

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

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(3) If a function $f(x,y)$ has continuous first partial derivatives at some point, then the function is differentiable at that point.

Answer: The correct answer is actually "false," although we ended up accepting either answer as correct. To see that this statement is false, consider the following function:

$$f(x,y) = \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$

Then f is the constant function 1 along both axes, and so the partial derivatives $f_x(0,0) = 0$ and $f_y(0,0) = 0$. However, it is clear that f is not even continuous at the origin, let alone differentiable there, so f is an example of a function with continuous partial derivatives at the origin that is not differentiable at the origin. In order for a function f to be differentiable at a point, it must have continuous partial derivatives in a *neighborhood* of the point.

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(4) If $f(x, y) \to L$ as $(x, y) \to (a, b)$ along every straight line passing through (a, b), then $\lim_{(x,y)\to(a,b)} f(x, y) = L$.

Answer: False. In order for the limit to exist, the limit along *every* path through that point must agree, not just the straight-line paths. For an example of a function for which the straight-line paths all have the same limits but the limit does not exist, see problem 1 from the solution key to the second practice midterm (http://www.math.upenn.edu/˜fithian/114S05/smt2key.pdf).

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(5) If a vector function $\mathbf{r}(t)$ is of constant length (i.e. $|\mathbf{r}(t)| = \text{constant}$), then the function and its first derivative have the property $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

Answer: True. If $|\mathbf{r}(t)| = \text{constant}$, then

$$|\mathbf{r}(t)|^2 = \text{ constant}$$

as well. Hence,

$$\begin{aligned} 0 = \frac{d}{dt}(|\mathbf{r}(t)|^2) &= \frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{r}(t)) \\ &= \frac{d\mathbf{r}}{dt} \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt} \\ &= 2\mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt}. \end{aligned}$$

Therefore, $\mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt} = 0$, which is to say that

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0.$$

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(6) Match the functions to the surfaces and level curves.

Answer: Function 1, $z = x^2 + y^2$, has graph A and level curves given in E, function 2, $z = \frac{-5x}{\sqrt{x^2+y^2+1}}$, has graph C and level curves given in D, and function 3, $z = \frac{1}{4x^2+y^2}$, has graph B and level curves given in F.

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Multiple Choice

(1) Let $f(x, y) = y \sin(xy)$. Find $f_y(0, \pi/3)$.

Answer: B. Computing directly using the product rule, we see that

$$f_y = y \cos(xy) \cdot x + \sin(xy) = xy \cos(xy) + \sin(xy).$$

Plugging in $(x, y) = (0, \pi/3)$,

$$f_y(0, \pi/3) = (0)(\pi/3) \cos(0) + \sin(0) = 0.$$

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(2) What is the domain of the function $z = \sqrt{1-(x^2+y^2)}$.

Answer: C. In order for the square root to be well defined, we must have that $1-(x^2+y^2) \geq 0$, which is to say that

$$x^2 + y^2 \leq 1.$$

This simply describes all points on and interior to the circle $x^2+y^2 = 1$, which is option C.

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(3) Evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}.$$

Answer: F. To see that this limit does not exist, consider the limit along the paths (i) where $x = 0$ and (ii) where $y = x$:

$$(i) \qquad \lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{xy}{x^2+xy+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0;$$

$$(ii) \qquad \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{xy}{x^2+xy+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2+x^2} = \lim_{x\to 0} \frac{x^2}{3x^2} = \frac{1}{3}.$$

Since these two limits don't agree, we see that the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$ does not exist.

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(4) Let $f(x,y) = (x^3+y^4)^5$; find the value of $f_{yx} - f_{xy}$ at the point $(1,2)$.

Answer: E. Remember that mixed partial commute; that is, for any differentiable function f, $f_{yx} = f_{xy}$, so $f_{yx} - f_{xy} = 0$. If you didn't remember this fact, you can compute directly. First, we need to know the first partials:

$$f_x = 5(x^3+y^4)^4 \cdot (3x^2) = 15x^2(x^3+y^4)^4,$$

$$f_y = 5(x^3+y^4)^4 \cdot (4y^3) = 20y^3(x^3+y^4)^4.$$

Then, we compute the mixed partials:

$$f_{yx} = \frac{\partial f_x}{\partial y} = 15x^2 \cdot 4(x^3+y^4)^3 \cdot (4y^3) = 240x^2y^3(x^3+y^4)^3,$$

$$f_{xy} = \frac{\partial f_y}{\partial x} = 20y^3 \cdot 4(x^3+y^4)^3 \cdot (3x^2) = 240x^2y^3(x^3+y^4)^3.$$

We see that these agree at all points, so $f_{yx} - f_{xy} = 0$ at all points, including the point $(1,2)$.

♣

(5) $e^x = 3\sin y$. Use implicit differentiation to find $\frac{dy}{dx}$ when $(x, y) = (0, 0)$.

Answer: B. Differentiating implicitly, we see that

$$e^x = 3\cos y \frac{dy}{dx},$$

so

$$\left.\frac{dy}{dx}\right|_{(0,0)} = \frac{e^0}{3\cos 0} = \frac{1}{3}.$$

♣

(6) Find the differential of $z = 2x^2 y$.

Answer: C. Recall the definition of the differential:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Then we see that the differential will be

$$4xydx + 2x^2 dy.$$

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(7) Let $z = xy^2 + x^3 y$ and let $x = x(t)$ and $y = y(t)$ such that $x(1) = 1$, $y(1) = 2$, $x'(1) = 3$ and $y'(1) = 4$. Find $\frac{dz}{dt}$ when $t = 1$.

Answer: E. By the chain rule, we know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y} dydt. \tag{1}$$

Now, $\frac{\partial z}{\partial x} = y^2 + 3x^2 y$; evaluating at $t = 1$, we're dealing with the point $(1, 2)$. Hence,

$$\left.\frac{\partial z}{\partial x}\right|_{(1,2)} = 2^2 + 3(1)^2(2) = 10.$$

On the other hand, $\frac{\partial z}{\partial y} = 2xy + x^3$. Evaluating at $t = 1$, we see that

$$\left.\frac{\partial z}{\partial x}\right|_{(1,2)} = 2(1)(2) + 1^3 = 5.$$

Therefore, plugging these numbers into (1) yields:

$$\left.\frac{dz}{dt}\right|_{t=1} = 10x'(1) + 5y'(1) = 10(3) + 5(4) = 50.$$

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(8) Find $\mathbf{v}(1/2)$ of a particle if $\mathbf{a}(t) = \mathbf{k}$ and $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$.

Answer: F. We know that $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt}$, so

$$\mathbf{v}(t) \int \mathbf{a}(t) dt = t\mathbf{k} + \mathbf{c},$$

where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. In other words,

$$\mathbf{v}(t) = c_1\mathbf{i} + c_2\mathbf{j} + (t + c_3)\mathbf{k}.$$

Now, we know that

$$\mathbf{i} - \mathbf{j} = \mathbf{v}(0) = c_1\mathbf{i} + c_2\mathbf{j} + (0 + c_3)\mathbf{k} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k},$$

so $c_1 = 1$, $c_2 = -1$ and $c_3 = 0$. Hence,

$$\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + t\mathbf{k}.$$

Plugging in $t = 1/2$ yields our final answer,

$$\mathbf{v}(1/2) = \mathbf{i} - \mathbf{j} + \frac{1}{2}\mathbf{k}.$$

♣

(9) Find the distance from the origin to the plane $3x + 2y + z = 6$.
Answer: E. To find the distance from the origin to the plane, we need first to find a point P in the plane. Since $3(1) + 2(1) + 1 = 6$, the point $P = (1, 1, 1)$ is in the plane. Now, we form the vector $\mathbf{P0}$ from P to the origin:

$$\mathbf{P0} = (0 - 1)\mathbf{i} + (0 - 1)\mathbf{j} + (0 - 1)\mathbf{k} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$$

Now, the distance from the origin to the plane is given by $\left|\mathbf{P0} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|$, where $\mathbf{n}$ is a normal vector to the plane. For example, we know that

$$\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

is normal to the plane, so we may as well use that. Note that

$$|\mathbf{n}| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}.$$

Therefore, the distance from the origin to the plane is:

$$\left|\mathbf{P0} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \frac{1}{\sqrt{14}}|(-\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (3\mathbf{i} + 2\mathbf{j} + \mathbf{k})| = \frac{1}{\sqrt{14}}|-3 - 2 - 1| = \frac{6}{\sqrt{14}}.$$

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(10) If $u = x^{yz}$, find $\frac{\partial u}{\partial z}$.
Answer: C. Remember that, if a is a constant,

$$\frac{d}{dt}\left(a^{g(t)}\right) = a^{g(t)}(\ln a)\frac{dg}{dt}.$$

When we're finding the partial with respect to z, we simply treat x and y as constants, so this tells us that

$$\frac{\partial u}{\partial z} = x^{yz}(\ln x)\frac{\partial}{\partial z}(yz) = yx^{yz}\ln x.$$

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Free Response

(1) Show that

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 - y^2}{x^2 + 2y^2}$$

does not exist.

Proof. Let's take the limit along the path $y = kx$. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{2x^2 - y^2}{x^2 + 2y^2} = \lim_{x\to 0} \frac{2x^2 - (kx)^2}{x^2 + 2(kx^2)} = \lim_{x\to 0} \frac{x^2(2-k^2)}{x^2(1+2k^2)} = \lim_{x\to 0} \frac{2-k^2}{1+2k^2} = \frac{2-k^2}{1+2k^2}.$$

Now, if we plug in different values of k, we'll get different limits. For example, when $k = 1$, we get

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{2x^2 - y^2}{x^2 + 2y^2} = \frac{2-1}{1+2} = \frac{1}{3},$$

whereas when $k = 0$, we get

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{2x^2 - y^2}{x^2 + 2y^2} = \frac{2-0}{1+0} = 2.$$

Since we get different limits along different paths, the limit does not exist. □

(2) The radius of a right circular cylinder is increasing at the rate of 2 cm/sec and the height is decreasing at the rate of 4cm/sec. At what rate is the volume changing at the instant when the radius is 4 cm and the height is 10 cm?

Answer: Since we're asked to find the rate at which the volume is changing, we should be trying to compute $\frac{dV}{dt}$. Now, remember that

$$V = \pi r^2 h.$$

By the chain rule, we know that

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}.$$

Since $\frac{dr}{dt}$ and $\frac{dh}{dt}$ are given, we need only compute the partial derivatives of V. Now,

$$\frac{\partial V}{\partial r} = 2\pi r h$$

and

$$\frac{\partial V}{\partial h} = \pi r^2.$$

Hence, when $r = 4$ and $h = 10$,

$$\frac{dV}{dt} = (2\pi r h)\frac{dr}{dt} + (\pi r^2)\frac{dh}{dt} = 2\pi(4)(10)(2) + \pi(4)^2(-4) = 160\pi - 64\pi = 96\pi,$$

so the volume is changing at a rate of $96\pi\text{cm}^3/\text{sec}$ at this instant in time.

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(3) Find a linearization of $f(x, y) = x^3y^4$ at the point $(1, 1)$. Use your linearization to find an approximate value for $f(1, 1, 0.9)$.

Answer: Remember that the linearization at a point (x_0, y_0) is, by definition, given by

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Now, $f_x = 3x^2y^4$ and $f_y = 4x^3y^3$, so

$$f_x(1, 1) = 3(1)^2(1)^4 = 3$$

and

$$f_y(1, 1) = 4(1)^3(1)^3 = 4.$$

Also, $f(1, 1) = (1)^3(1)^4 = 1$, so

$$L(x, y) = f(1, 1) + 3(x-1) + 4(y-1) = 1 + 3(x-1) + 4(y-1) = 3x + 4y - 6.$$

To find an approximate value for $f(1.1, 0.9)$, we simply plug $(x, y) = (1.1, 0.9)$ into the above equation:

$$L(1.1, 0.9) = 3(1.1) + 4(0.9) - 6 = 3.3 + 3.6 - 6 = 6.9 - 6 = 0.9.$$

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(4) If $\mathbf{r}(t) = (t^2 + 3)\mathbf{i} + (2t^2 - 3t + 5)\mathbf{j}$ describes the motion of a particle, find:

(a) the velocity when $t = 0$

(b) the speed when $t = 0$

(c) the normal and tangential components of the acceleration when $t = 0$

(d) the curvature when $t = 0$

Answer: We work one step at a time:

(a) If $\mathbf{r}(t)$ is as given, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + (4t - 3)\mathbf{j}.$$

Hence,

$$\mathbf{v}(3) = 2(3)\mathbf{i} + (4(3) - 3)\mathbf{j} = 6\mathbf{i} + 9\mathbf{j}.$$

(b) In general, the speed will be given by

$$|\mathbf{v}(t)| = \sqrt{(2t)^2 + (4t-3)^2} = \sqrt{4t^2 + (16t^2 - 24t + 9)} = \sqrt{20t^2 - 24t + 9}.$$

Hence,

$$|\mathbf{v}(3)| = \sqrt{20(9) - 24(3) + 9} = \sqrt{180 - 72 + 9} = \sqrt{117} = 3\sqrt{13}.$$

(c) Recall that $a_{\mathbf{T}} = \frac{d|\mathbf{v}|}{dt}$. We computed $|\mathbf{v}(\mathbf{t})|$ in part (b) above, so, using that computation, we see that

$$a_{\mathbf{T}} = \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}(20t^2 - 24t + 9)^{-1/2} \cdot (40t - 24) = \frac{20t - 12}{\sqrt{20t^2 - 24t + 9}}.$$

Plugging in $t = 3$, we see that, at this point,

$$a_{\mathbf{T}} = \frac{20(3) - 12}{\sqrt{20(9) - 24(3) + 9}} = \frac{48}{\sqrt{117}} = \frac{48}{3\sqrt{13}} = \frac{16}{\sqrt{13}}.$$

Now, there are a number of different ways to compute $a_{\mathbf{N}}$; since we haven't computed κ yet, we'll use the fact that

$$a_{\mathbf{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathbf{T}}^2}.$$

Now,

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = 2\mathbf{i} + 4\mathbf{j},$$

so

$$|\mathbf{a}(t)| = \sqrt{2^2 + 4^2} = \sqrt{20}.$$

Therefore,

$$\begin{aligned} a_{\mathbf{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathbf{T}}^2} &= \sqrt{20 - \left(\frac{16}{\sqrt{13}}\right)^2} \\ &= \sqrt{20 - \frac{256}{13}} \\ &= \sqrt{\frac{260}{13} - \frac{256}{13}} \\ &= \sqrt{\frac{4}{13}} \\ &= \frac{2}{\sqrt{13}}. \end{aligned}$$

(d) There are a couple of different ways we could compute κ. Perhaps the easiest is simply to remember that $a_{\mathbf{N}} = \kappa\,|\mathbf{v}|^2$. Hence, when $t = 3$,

$$\kappa = \frac{a_{\mathbf{N}}}{|\mathbf{v}|^2} = \frac{\frac{2}{\sqrt{13}}}{(3\sqrt{13})^2} = \frac{2}{9(13)\sqrt{13}} = \frac{2}{117\sqrt{13}} = \frac{6}{117\sqrt{117}}.$$

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