MATH 104 QUIZ IV SOLUTIONS

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(1) Suppose you win the PowerBall jackpot and are given the following options: take the $200,000,000 jackpot as a 20 year annuity (that is, you will receive $10,000,000 per year for the next 20 years) or take a $100,000,000 lump sum (for the purpose of this problem, we're ignoring the hefty taxes you'd have to pay on either of these amounts). Suppose, further, that if you take the lump sum, you can invest it at 7% per year, compounded continuously. How long would it take you to double the $100,000,000 lump sum (use the estimate $\ln 2 \approx 0.7$). Which option is the better investment?

Answer: We want to find t such that

$$200,000,000 = 100,000,000e^{0.07t}.$$

Dividing by 100,000,000, we have

$$2 = e^{0.07t}.$$

Taking the natural log of both sides,

$$\ln 2 = 0.07t,$$

so

$$t = \frac{\ln 2}{0.07} \approx \frac{0.7}{0.07} = 10,$$

so you can double your money in ten years. Hence, taking the lump sum and investing it at 7% will give you the same $200,000,000 as the annuity in half the time.

(2) Evaluate

$$\lim_{x \to 0} \frac{\sin x - x}{x^2}$$

Answer: Since $\sin 0 = 0$, we see that both top and bottom go to 0. Hence, by L'Hôpital's Rule,

$$\lim_{x \to 0} \frac{\sin x - x}{x^2} = \lim_{x \to \infty} \frac{\cos x - 1}{2x}.$$

Now, $\cos 0 = 1$, so, again, the top and the bottom both converge to 0. Therefore, we apply L'Hôpital's Rule again to see that

$$\lim_{x \to 0} \frac{\cos x - 1}{2x} = \lim_{x \to 0} \frac{-\sin x}{2} = 0.$$

(3) True or false?
$$e^x = o(x^x)$$

Answer: Let's the compare the two as x gets large:
$$\lim_{x \to \infty} \frac{e^x}{x^x} = \lim_{x \to \infty} \left(\frac{e}{x}\right)^x.$$

Now, as $x \to \infty$, $\frac{e}{x}$ gets very small, so
$$\lim_{x \to \infty} \left(\frac{e}{x}\right)^x = 0.$$

Hence, it is indeed true that $e^x = o(x^x)$.

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