MATH 104 HW 9

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§7.3

12. Evaluate
$$\int \frac{2x+1}{x^2-7x+12}dx$$

Answer: Note that $x^2 - 7x + 12 = (x-4)(x-3)$, so we set $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-3} + \frac{B}{x-4}$ and solve for A and B:
$$2x + 1 = A(x-4) + B(x-3).$$
Letting $x = 4$, we see that
$$2(4) + 1 = B(1) = B,$$
so $B = 9$. Letting $x = 3$,
$$2(3) + 1 = A(-1) = -A,$$
so $A = -7$. Hence,
$$\begin{aligned}\int \frac{2x+1}{x^2-7x+12}dx &= \int \frac{-7}{x-3}dx + \int \frac{9}{x-4}dx \\ &= -7\ln|x-3| + 9\ln|x-4| + C.\end{aligned}$$

14. Evaluate
$$\int_{1/2}^{1} \frac{y+4}{y^2+y}dy.$$

Answer: Note that $y^2 + y = y(y+1)$, so we set $\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1}$ and solve for A and B:
$$y + 4 = A(y+1) + By.$$
Letting $y = 0$, we see that
$$4 = A(1) = A.$$
Letting $y = -1$, we see that
$$-1 + 4 = B(-1) = -B,$$

so $B = -3$. Hence,

$$\begin{aligned}\int_{1/2}^{1} \frac{y+4}{y^2+y} dy &= \int_{1/2}^{1} \frac{4}{y} - \int_{1/2}^{1} \frac{3}{y+1} \\ &= 4\ln|y| - 3\ln|y+1|]_{1/2}^{1} \\ &= (4\ln 1 - 3\ln 2) - (4\ln 1/2 - 3\ln 3/2) \\ &= 3\ln 3/2 - 3\ln 2 - 4\ln 1/2.\end{aligned}$$

18. Evaluate

$$\int_{-1}^{0} \frac{x^3 dx}{x^2-2x+1}.$$

Answer: Using the division algorithm (which I don't reproduce here),

$$\frac{x^3}{x^2-2x+1} = x+2+\frac{3x-2}{x^2-2x+1}.$$

Note that $x^2-2x+1 = (x-1)^2$, so we set $\frac{3x-2}{x^2-2x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$ and solve for A and B:

$$3x-2 = A(x-1)+B.$$

Letting $x = 1$,

$$3(1)-2 = B,$$

so $B = 1$. Letting $x = 2$,

$$3(2)-2 = A(1)+B = A+2,$$

so $A = 2$. Hence,

$$\begin{aligned}\int_{-1}^{0} \frac{3x-2}{x^2-2x+1} dx &= \int_{-1}^{0} \left[\frac{2}{x-1} + \frac{1}{(x-1)^2}\right] dx \\ &= 2\ \ln|x-1|]_{-1}^{0} + \left.\frac{-1}{x-1}\right]_{-1}^{0} \\ &= (2\ln 1 - 2\ln 2) + \left(1 - \frac{1}{2}\right) \\ &= \frac{1}{2} - 2\ln 2.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{-1}^{0} \frac{x^3}{x^2-2x+1} dx &= \int_{-1}^{0} \left[x+2+\frac{2}{x-1} + \frac{1}{(x-1)^2}\right] dx \\ &= \left[\frac{x^2}{2}\right]_{-1}^{0} + [2x]_{-1}^{0} + \left(\frac{1}{2} - 2\ln 2\right) \\ &= \left(0 - \frac{1}{2}\right) + (0+2) + \left(\frac{1}{2} - 2\ln 2\right) \\ &= 2 - 2\ln 2.\end{aligned}$$

24. Evaluate

$$\int \frac{8x^2+8x+2}{(4x^2+1)^2}dx.$$

Answer: Note that $(4x^2+1)^2$ is already factored, so we let $\frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2}$ and solve for A, B, C, D:

$$8x^2+8x+2 = (Ax+B)(4x^2+1)+(Cx+D) = 4Ax^3+4Bx^2+(A+C)x+B+D.$$

Hence, $A = 0$, $4B = 8$, so $B = 2$, $8 = A+C = C$ and $2 = B+D = 2+D$, so $D = 0$. Hence,

$$\int \frac{8x^2+8x+2}{(4x^2+1)^2}dx = \int \left[\frac{8}{4x^2+1} + \frac{8x}{(4x^2+1)^2}\right] dx.$$

Now,

$$\int \frac{8}{4x^2+1}dx = 8\int \frac{dx}{4x^2+1} = 8 \cdot \tan^{-1}(2x) + C_1.$$

On the other hand, if $u = 4x^2+1$, then $du = 8xdx$, so

$$\int \frac{8x}{(4x^2+1)^2}dx = \int \frac{du}{u^2} = \frac{-1}{u} + C_2 = \frac{-1}{4x^2+1} + C_2.$$

Therefore, if $C = C_1 + C_2$,

$$\int \frac{8x^2+8x+2}{(4x^2+1)^2}dx = 8\tan^{-1}(2x) - \frac{1}{4x^2+1} + C.$$

32. Evaluate

$$\int \frac{16x^3}{4x^2-4x+1}dx.$$

Answer: Using the division algorithm,

$$\frac{16x^3}{4x^2-4x+1} = 4x+4+\frac{12x-4}{4x^2-4x+1}.$$

Note that $4x^2-4x+1 = (2x-1)^2$, so we set $\frac{12x-4}{4x^2-4x+1} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2}$ and solve for A and B:

$$12x-4 = A(2x-1)+B = 2Ax+B-A.$$

Hence, $12 = 2A$, so $A = 6$, meaning $-4 = B-A = B-6$, so $B = 2$. Thus,

$$\begin{aligned}\int \frac{12x-4}{4x^2-4x+1}dx &= \int \left[\frac{6}{2x-1} + \frac{2}{(2x-1)^2}\right] dx \\ &= 3\ln|2x-1| - \frac{1}{2x-1} + C.\end{aligned}$$

Hence,

$$\begin{aligned}\int \frac{16x^3}{4x^2-4x+1}dx &= \int \left[4x+4+\frac{12x-4}{4x^2-4x+1}\right] dx \\ &= 2x^2+4x+3\ln|2x-1| - \frac{1}{2x-1} + C.\end{aligned}$$

§7.4

8. Evaluate

$$\int \sqrt{1-9t^2}dt.$$

Answer: Let $3t = \sin\theta$. Then $3dt = \cos\theta d\theta$, so $dt = \frac{1}{3}\cos\theta d\theta$. Note that $\theta = \sin^{-1} 3t$, so $\cos\theta = \sqrt{1-9t^2}$. Hence, we can re-write the integral as

$$\begin{aligned} \frac{1}{3}\int \sqrt{1-\sin^2\theta}\cos\theta d\theta &= \frac{1}{3}\int \sqrt{\cos^2\theta}\cos\theta d\theta \\ &= \frac{1}{3}\int \cos^2\theta d\theta \\ &= \frac{1}{3}\int \frac{1+\cos 2\theta}{2}d\theta \\ &= \frac{1}{6}\left[\theta + \frac{\sin 2\theta}{2}\right] + C \\ &= \frac{\theta}{6} + \frac{2\sin\theta\cos\theta}{12} + C \\ &= \frac{\sin^{-1} 3t}{6} + \frac{3t\sqrt{1-9t^2}}{6} + C. \end{aligned}$$

12. Evaluate

$$\int \frac{\sqrt{y^2-25}}{y^3}dy, \quad y > 5.$$

Answer: Let $y = 5\sec\theta$. Then $dy = 5\sec\theta\tan\theta d\theta$, $\theta = \sec^{-1}\frac{y}{5}$ and $\sin\theta = \frac{\sqrt{y^2-25}}{y}$. Hence, we can re-write the integral as

$$\begin{aligned}
\int \frac{\sqrt{25\sec^2\theta - 25}}{125\sec^3\theta}\cdot 5\sec\theta\tan\theta d\theta &= \int \frac{\sqrt{25\tan^2\theta}}{25\sec^2\theta}\tan\theta d\theta \\
&= \int \frac{5\tan\theta}{25\sec^2\theta}\tan\theta d\theta \\
&= \frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}d\theta \\
&= \frac{1}{5}\int \frac{\sec^2\theta - 1}{\sec^2\theta}d\theta \\
&= \frac{1}{5}\int \left[1 - \frac{1}{\sec^2\theta}\right] d\theta \\
&= \frac{1}{5}\int \left[1 - \cos^2\theta\right] d\theta \\
&= \frac{1}{5}\int \left[1 - \frac{1+\cos 2\theta}{2}\right] d\theta \\
&= \frac{1}{5}\int \left[\frac{1}{2} - \frac{\cos 2\theta}{2}\right] d\theta \\
&= \frac{1}{5}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right] + C \\
&= \frac{1}{10}\left[\theta - \sin\theta\cos\theta\right] + C \\
&= \frac{1}{10}\left[\sec^{-1}\frac{y}{5} - \frac{\sqrt{y^2-25}}{y}\cdot\frac{5}{y}\right] + C \\
&= \frac{1}{10}\left[\sec^{-1}\frac{y}{5} - \frac{5\sqrt{y^2-25}}{y^2}\right] + C.
\end{aligned}$$

26. Evaluate

$$\int \frac{6dt}{(9t^2+1)^2}.$$

Answer: Let $3t = \tan\theta$. Then $3dt = \sec^2\theta d\theta$, so $dt = \frac{1}{3}\sec^2\theta d\theta$. Also, $\theta = \tan^{-1}3t$, $\sin\theta = \frac{3t}{\sqrt{1+9t^2}}$ and $\cos\theta = \frac{1}{\sqrt{1+9t^2}}$. Therefore, we can re-write the integral as

$$\begin{aligned}
\int \frac{2\sec^2\theta d\theta}{(\tan^2\theta+1)^2} &= \int \frac{2\sec^2\theta d\theta}{\sec^4\theta} \\
&= \int \frac{2d\theta}{\sec^2\theta} \\
&= 2\int \cos^2\theta d\theta \\
&= 2\int \frac{1+\cos 2\theta}{2} d\theta \\
&= \int (1+\cos 2\theta) d\theta \\
&= \theta + \frac{\sin 2\theta}{2} + C \\
&= \theta + \sin\theta\cos\theta + C \\
&= \tan^{-1} 3t + \frac{3t}{\sqrt{1+9t^2}} \cdot \frac{1}{\sqrt{1+9t^2}} + C \\
&= \tan^{-1} 3t + \frac{3t}{1+9t^2} + C.
\end{aligned}$$

36. Evaluate

$$\int \frac{dx}{\sqrt{1-x^2}}.$$

Answer: Let $x = \sin\theta$. Then $dx = \cos\theta d\theta$, . Thus, we can re-write the integral as

$$\begin{aligned}
\int \frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} &= \int \frac{\cos\theta d\theta}{\sqrt{\cos^2\theta}} \\
&= \int d\theta \\
&= \theta + C \\
&= \sin^{-1} x + C.
\end{aligned}$$

40. Solve the initial value problem

$$(x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1.$$

Answer: Dividing by $(x^2+1)^2$, we see that

$$\frac{dy}{dx} = (x^2+1)^{-3/2}.$$

Hence,

$$y = \int (x^2+1)^{-3/2} dx.$$

Let $x = \tan\theta$. Then $dx = \sec^2\theta d\theta$, so we can re-write the integral as

$$\begin{aligned} y &= \int (\tan^2\theta + 1)^{-3/2} \sec^2\theta d\theta \\ &= \int \frac{\sec^2\theta d\theta}{(\sec^2\theta)^{3/2}} \\ &= \int \frac{\sec^2\theta d\theta}{\sec^3\theta} \\ &= \int \frac{d\theta}{\sec\theta} \\ &= \int \cos\theta d\theta \\ &= \sin\theta + C. \end{aligned}$$

Now, since $\theta = \tan^{-1} x$, $\sin\theta = \frac{x}{\sqrt{x^1+1}}$, so

$$y = \frac{x}{\sqrt{x^2+1}} + C.$$

Using the initial value,

$$1 = y(0) = \frac{0}{1} + C = 0 + C = C,$$

so we conclude that

$$y = \frac{x}{\sqrt{x^2+1}} + 1.$$

42. Find the volume of the solid generated by revolving about the x-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y = 2/(1+x^2)$, and the line $x = 1$.

Answer: Using the disc method, the volume is given by

$$V = \int_0^1 \pi r^2 dx.$$

Now, $r = y = \frac{2}{1+x^2}$, so

$$V = \int_0^1 \pi \frac{4}{(1+x^2)^2} dx.$$

Let $x = \tan\theta$. Then $dx = \sec^2\theta d\theta$, so

$$\begin{aligned} V &= 4\pi \int_0^{\pi/4} \frac{1}{(1+\tan^2\theta)^2} \sec^2\theta d\theta \\ &= 4\pi \int_0^{\pi/4} \frac{1}{\sec^4\theta} \sec^2\theta d\theta \\ &= 4\pi \int_0^{\pi/4} \frac{1}{\sec^2\theta} d\theta \\ &= 4\pi \int_0^{\pi/4} \cos^2\theta d\theta \\ &= 4\pi \int_0^{\pi/4} \frac{1+\cos 2\theta}{2} d\theta \\ &= 2\pi \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} \\ &= 2\pi \left[\left(\frac{\pi}{4} + \frac{1}{2}\right) - 0\right] \\ &= \frac{\pi^2}{2} + \pi. \end{aligned}$$

§7.6

10. Evaluate

$$\int_{-\infty}^{2} \frac{2dx}{x^2+4}.$$

Answer: By definition,

$$\begin{aligned} \int_{-\infty}^{2} \frac{2dx}{x^2+4} &= \lim_{a\to-\infty} \int_a^2 \frac{2dx}{x^2+4} \\ &= \lim_{a\to-\infty} 2\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_a^2 \\ &= \lim_{a\to-\infty} \left[\tan^{-1} 1 - \tan^{-1}\left(\frac{a}{2}\right)\right] \\ &= \frac{\pi}{4} - \left(\frac{-\pi}{2}\right) \\ &= \frac{3\pi}{4}. \end{aligned}$$

16. Evaluate

$$\int_0^2 \frac{s+1}{\sqrt{4-s^2}} ds.$$

Answer: By definition

$$\begin{aligned}\int_0^2 \frac{s+1}{\sqrt{4-s^2}}ds &= \lim_{b\to 2^-}\int_0^b \frac{s+1}{\sqrt{4-s^2}}ds\\ &= \lim_{b\to 2^-}\int \left[\frac{s}{\sqrt{4-s^2}}+\frac{1}{\sqrt{4-s^2}}\right]ds\\ &= \lim_{b\to 2^-}\left[-\sqrt{4-s^2}+\sin^{-1}\left(\frac{s}{2}\right)\right]_0^b\\ &= \lim_{b\to 2^-}\left[\left(-\sqrt{4-b^2}+\sin^{-1}\left(\frac{b}{2}\right)\right)-\left(-\sqrt{4-0}+\sin^{-1}0\right)\right]\\ &= \frac{\pi}{2}+2.\end{aligned}$$

24. Evaluate

$$\int_{-\infty}^{\infty} 2xe^{-x^2}dx.$$

Answer: By definition,

$$\int_{-\infty}^{\infty} 2xe^{-x^2}dx = \int_{-\infty}^{0} 2xe^{-x^2}+\int_0^{\infty} 2xe^{-x^2}dx = \lim_{b\to-\infty}\int_b^0 2xe^{-x^2}dx+\lim_{c\to\infty}\int_0^c 2xe^{-x^2}dx.$$

Now, let $u=-x^2$. Then $du=-2xdx$, so

$$\int 2xe^{-x^2}dx = -\int e^u du = -e^u + C = -e^{-x^2}+C.$$

Thus,

$$\begin{aligned}\int_{-\infty}^{\infty} 2xe^{-x^2}dx &= \lim_{b\to-\infty}\int_b^0 2xe^{-x^2}dx+\lim_{c\to\infty}\int_0^c 2xe^{-x^2}dx\\ &= \lim_{b\to-\infty}\left[-e^{-x^2}\right]_b^0+\lim_{c\to\infty}\left[-e^{-x^2}\right]_0^c\\ &= \lim_{b\to-\infty}\left[-1+e^{-b^2}\right]+\lim_{c\to\infty}\left[-e^{-c^2}+1\right]\\ &= -1+0+0+1\\ &= 0.\end{aligned}$$

32. Evaluate

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}}.$$

Answer: Note that $\frac{1}{\sqrt{|x-1|}}$ is discontinuous at $x = 1$, so, by definition,

$$\begin{aligned}\int_0^2 \frac{dx}{\sqrt{|x-1|}} &= \int_0^1 \frac{dx}{\sqrt{|x-1|}} + \int_1^2 \frac{dx}{\sqrt{|x-1|}} \\ &= \int_0^1 \frac{dx}{\sqrt{-(x-1)}} + \int_1^2 \frac{dx}{\sqrt{x-1}} \\ &= \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{-(x-1)}} + \lim_{c\to 1^+} \int_c^2 \frac{dx}{\sqrt{x-1}}.\end{aligned}$$

In the first integral, let $u = -(x-1)$, so $du = -dx$, and in the second let $u = x - 1$ so $du = dx$. Then

$$\int \frac{dx}{\sqrt{-(x-1)}} = -\int \frac{du}{\sqrt{u}} = -2\sqrt{u} + C = -2\sqrt{-(x-1)} + C$$

and

$$\int \frac{dx}{\sqrt{x-1}} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{x-1} + C.$$

Hence,

$$\begin{aligned}\int_0^2 \frac{dx}{\sqrt{|x-1|}} &= \lim_{b\to 1^-} \left[-2\sqrt{-(x-1)}\right]_0^b + \lim_{c\to 1^+} \left[2\sqrt{x-1}\right]_c^2 \\ &= \lim_{b\to 1^-} \left[-2\sqrt{-(b-1)} + 2\right] + \lim_{c\to 1^+} \left[2 - 2\sqrt{c-1}\right] \\ &= 0 + 2 + 2 - 0 \\ &= 4.\end{aligned}$$

36. Determine whether the integral converges or diverges:

$$\int_0^{\pi/2} \cot\theta d\theta.$$

Answer: Note that $\cot\theta$ is discontinuous at $x = 0$. By definition, then,

$$\begin{aligned}\int_0^{\pi/2} \cot\theta d\theta &= \lim_{b\to 0^+} \int_b^{\pi/2} \cot\theta d\theta \\ &= \lim_{b\to 0^+} [\ln|\sin\theta|]_b^{\pi/2} \\ &= \lim_{b\to 0^+} [\ln 1 - \ln(\sin b)] \\ &= -\infty,\end{aligned}$$

so the integral diverges.

51. Determine whether the integral converges or diverges:

$$\int_1^{\infty} \frac{dx}{\sqrt{x^6+1}}.$$

Answer: (Note that I've changed the lower limit of integration from 0 to 1; I think 0 was a typo) As $x \to \infty$, $\sqrt{x^6+1}$ should act like x^3. In fact, since for all $x \geq 1$, $\sqrt{x^6+1} \geq x^3$, $\frac{1}{\sqrt{x^6+1}} \leq \frac{1}{x^3}$. Now, since

$$\int_1^\infty \frac{1}{x^3} dx$$

converges, the direct comparison test implies that $\int_1^\infty \frac{dx}{\sqrt{x^6+1}}$ also converges.

52. Determine whether the integral converges or diverges:

$$\int_2^\infty \frac{dx}{\sqrt{x^2-1}}.$$

Answer: As $x \to \infty$, $\sqrt{x^2-1}$ should act like x. Let's do a limit comparison:

$$\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^2-1}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{x}{x\sqrt{1-\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1.$$

Now, since $\int_2^\infty \frac{dx}{x}$ diverges, the limit comparison test tells us that $\int_2^x \frac{dx}{\sqrt{x^2-1}}$ diverges as well.

66. As Example 8 shows, the integral $\int_1^\infty (dx/x)$ diverges. This means that the integral

$$\int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx,$$

which measures the *surface area* of the solid of revolution traced out by revolving the curve $y = 1/x$, $1 \leq x$, about the x-axis, diverges also. by comparing the two integrals, we see that, for every finite value $b > 1$,

$$\int_1^b 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx > 2\pi \int_1^b \frac{1}{x}dx.$$

However, the integral

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

for the *volume* of the solid converges.

(a): Calculate it.

Answer: By definition,

$$\begin{aligned}\int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx &= \lim_{b\to\infty} \int_1^b \pi \left(\frac{1}{x}\right)^2 dx \\ &= \lim_{b\to\infty} \pi \int_1^b \frac{dx}{x^2} \\ &= \lim_{b\to\infty} \pi \left[\frac{-1}{x}\right]_1^b \\ &= \lim_{b\to\infty} \pi \left[\frac{-1}{b} + 1\right] \\ &= \pi.\end{aligned}$$

(b): This solid of revolution is sometimes described as a can that does not hold enough paint to cover its own interior. Think about that for a moment. It is common sense that a finite amount of paint cannot cover an infinite surface. But if we fill the horn with pain (a finite amount), then we *will* have covered an infinite surface. Explain the apparent contradiction.

Answer: The correspondence between amount of paint and surface area is slightly spurious. The unspoken assumption in the above apparent contradiction is that a layer of paint has no thickness. However, if this is true, then there's no problem with a finite volume of paint covering an infinite surface; since the layer of paint has no thickness, it also has no volume.

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