

MATH 104 HW 8

CLAY SHONKWILER

§7.1

6. Evaluate the integral

$$\int_{\pi/4}^{\pi/3} \frac{\sec^2 z}{\tan z} dz.$$

Answer: Let $u = \tan z$. Then $du = \sec^2 z dz$, so the above integral becomes

$$\int_1^{\sqrt{3}} \frac{du}{u} = \ln |u| \Big|_1^{\sqrt{3}} = \ln \sqrt{3} - \ln 1 = \frac{1}{2} \ln 3.$$

35. Evaluate

$$\int_1^{e^{\pi/3}} \frac{dx}{x \cos(\ln x)}.$$

Answer: Let $u = \ln x$. Then $du = \frac{1}{x} dx$, so the above integral becomes

$$\int_0^{\pi/3} \frac{du}{\cos u} = \int_0^{\pi/3} \sec u du = \ln |\sec u + \tan u| \Big|_0^{\pi/3} = \ln \left(\frac{1}{2} + \sqrt{3} \right) - \ln(1) = \ln \left(\frac{1}{2} + \sqrt{3} \right).$$

38. Evaluate

$$\int_2^4 \frac{2dx}{x^2 - 6x + 10}.$$

Answer: Note that

$$(x^2 - 6x) + 10 = (x^2 - 6x + 9) + 10 - 9 = (x - 3)^2 + 1,$$

so we can re-write the integral as

$$\int_2^4 \frac{2dx}{(x - 3)^2 + 1}.$$

Now, letting $u = x - 3$, this in turn is equal to

$$2 \int_{-1}^1 \frac{du}{u^2 + 1^2} = 2 [\tan^{-1} u]_{-1}^1 = 2 \left(\frac{\pi}{4} - \frac{-\pi}{4} \right) = \pi.$$

44. Evaluate

$$\int (\csc x - \tan x)^2 dx.$$

Answer: First, note that

$$\int (\csc x - \tan x)^2 dx = \int (\csc^2 x - 2 \csc x \tan x + \tan^2 x) dx.$$

Now, $\int \csc x dx = -\cot x + C_1$. Further,

$$\csc x \tan x = \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x} = \frac{1}{\cos x} = \sec x$$

and $\int \sec x = \ln |\sec x + \tan x| + C_2$, so $\int 2 \csc x \tan x = 2 \ln |\sec x + \tan x| + C_2$. Finally,

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C_3.$$

Thus, if $C = C_1 + C_2 + C_3$,

$$\int (\csc x - \tan x)^2 dx = \cot x + \ln |\sec x + \tan x| + \tan x - x + C.$$

48. Evaluate

$$\int \frac{x^2}{x^2 + 1} dx.$$

Answer: Using long division,

$$\int \frac{x^2}{x^2 + 1} dx = \int \left[1 - \frac{1}{x^2 + 1} \right] dx = x - \tan^{-1} x + C.$$

55. Evaluate

$$\int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x} dx.$$

Answer: Splitting the numerator, we see that

$$\begin{aligned} \int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x} dx &= \int_0^{\pi/4} \frac{1}{\cos^2 x} dx + \int_0^{\pi/4} \frac{\sin x}{\cos^2 x} dx \\ &= \int_0^{\pi/4} \sec^2 x dx + \int \sec x \tan x dx \\ &= \tan x + \sec x \Big|_0^{\pi/4} \\ &= (1 + \sqrt{2}) - (0 + 1) \\ &= \sqrt{2} \end{aligned}$$

58. Evaluate

$$\int \frac{1}{1 + \cos x} dx.$$

Answer: Note that we can multiply by a form of 1 in the following way:

$$\begin{aligned}
 \int \frac{1}{1 + \cos x} dx &= \int \frac{1}{1 + \cos x} \cdot \frac{1 - \cos x}{1 - \cos x} dx \\
 &= \int \frac{1 - \cos x}{1 - \cos^2 x} dx \\
 &= \int \frac{1 - \cos x}{\sin^2 x} dx \\
 &= \int \frac{1}{\sin^2 x} dx - \int \frac{\cos x}{\sin^2 x} dx \\
 &= \int \csc^2 x dx - \int \csc x \cot x dx \\
 &= -\cot x + \csc x + C.
 \end{aligned}$$

83.

(a): Evaluate $\int \cos^3 \theta d\theta$.

Answer: Note that

$$\int \cos^3 \theta d\theta = \int (1 - \sin^2 \theta) \cos \theta d\theta = \int \cos \theta d\theta - \int \sin^2 \theta \cos \theta d\theta.$$

Now, $\int \cos \theta d\theta = \sin \theta + C_1$. Further, if $u = \sin \theta$, then $du = \cos \theta d\theta$, so we can re-write the second integral as

$$\int u^2 du = \frac{u^3}{3} + C_2 = \frac{\sin^3 \theta}{3} + C_2.$$

Hence, if $C = C_1 + C_2$,

$$\int \cos^3 \theta d\theta = \sin \theta + \frac{\sin^3 \theta}{3} + C.$$

(b): Evaluate $\int \cos^5 \theta d\theta$.

Answer: Note that $\cos^5 \theta = \cos^2 \theta \cos^3 \theta = (1 - \sin^2 \theta) \cos^3 \theta$, so

$$\int \cos^5 \theta d\theta = \int \cos^3 \theta d\theta - \int \sin^2 \theta \cos^3 \theta d\theta.$$

Now, we already computed $\int \cos^3 \theta d\theta$ in (a). On the other hand,

$$\int \sin^2 \theta \cos^3 \theta d\theta = \int \sin^2 \theta (1 - \sin^2 \theta) \cos \theta d\theta = \int \sin^2 \theta \cos \theta d\theta - \int \sin^4 \theta \cos \theta d\theta.$$

In each case, we let $u = \sin \theta$, so this integral becomes $\frac{\sin^3 \theta}{3} - \frac{\sin^5 \theta}{5} + C_1$. Hence, combining the above, we see that

$$\int \cos^5 \theta d\theta = \sin \theta + \frac{\sin^3 \theta}{3} - \left(\frac{\sin^3 \theta}{3} - \frac{\sin^5 \theta}{5} \right) + C = \sin \theta + \frac{\sin^5 \theta}{5} + C.$$

(c): Without actually evaluating the integral, explain how you would evaluate $\int \cos^9 \theta d\theta$.

Answer: We would proceed as in the above cases, making substitutions for $\cos^2 \theta$ and using what we know about the integrals of \cos to lower powers.

§7.2

7. Evaluate

$$\int \tan^{-1} y dy$$

Answer: Let $u = \tan^{-1} y$ and $dv = dy$. Then $du = \frac{1}{1+y^2} dy$ and $v = y$, so

$$\int \tan^{-1} y dy = y \tan^{-1} y - \int \frac{y dy}{1+y^2}.$$

Now letting $u = 1 + y^2$, $du = 2y dy$, so

$$\int \frac{y dy}{1+y^2} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C_1 = \frac{1}{2} \ln(1+y^2) + C_1.$$

Therefore,

$$\int \tan^{-1} y dy = y \tan^{-1} y - \frac{1}{2} \ln(1+y^2) + C.$$

10. Evaluate

$$\int x^3 e^x dx.$$

Answer: Let $u = x^3$ and $dv = e^x dx$. Then $du = 3x^2 dx$ and $v = e^x$, so

$$\int x^3 e^x dx = x^3 e^x - 3 \int x^2 e^x dx.$$

Now, letting $u = x^2$ and $dv = e^x dx$, $du = 2x dx$ and $v = e^x$, so

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx.$$

Next, letting $u = x$ and $dv = e^x$, we see that $du = dx$ and $v = e^x$, so

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x.$$

Therefore, putting it all together,

$$\int x^3 e^x dx = x^3 e^x - 3(x^2 e^x - 2(x e^x - e^x)) = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C.$$

16. Evaluate

$$\int t^2 e^{4t} dt.$$

Answer: Let $u = t^2$ and $dv = e^{4t} dt$. Then $du = 2t dt$ and $v = \frac{e^{4t}}{4}$. Hence,

$$\int t^2 e^{4t} dt = t^2 \frac{e^{4t}}{4} - \frac{1}{2} \int t e^{4t} dt.$$

Letting $u = t$ and $dv = e^{4t}dt$, $du = dt$ and $v = \frac{e^{4t}}{4}$. Thus,

$$\int te^{4t} dt = t \frac{e^{4t}}{4} - \frac{1}{4} \int e^{4t} dt = t \frac{e^{4t}}{4} - \frac{e^{4t}}{16}.$$

Thus,

$$\int t^2 e^{4t} dt = t^2 \frac{e^{4t}}{4} - \frac{1}{2} \left[t \frac{e^{4t}}{4} - \frac{e^{4t}}{16} \right] = \frac{t^2 e^{4t}}{4} - \frac{te^{4t}}{8} + \frac{e^{4t}}{32} + C.$$

22. Evaluate

$$\int e^{-y} \cos y dy$$

Answer: Let $u = \cos y$ and $dv = e^{-y} dy$. Then $du = -\sin y dy$ and $v = -e^{-y}$, so

$$\int e^{-y} \cos y dy = -e^{-y} \cos y - \int e^{-y} \sin y dy.$$

Now, let $u = \sin y$ and $dv = e^{-y} dy$. Then $du = \cos y$ and $v = -e^{-y}$, so

$$\int e^{-y} \sin y dy = -e^{-y} \sin y + \int e^{-y} \cos y dy.$$

Therefore,

$$\int e^{-y} \cos y dy = -e^{-y} \cos y - \left(-e^{-y} \sin y + \int e^{-y} \cos y dy \right) = -e^{-y} \cos y + e^{-y} \sin y - \int e^{-y} \cos y dy.$$

Adding $\int e^{-y} \cos y dy$ to both sides, we see that

$$2 \int e^{-y} \cos y dy = e^{-y} \sin y - e^{-y} \cos y + C_1,$$

so, if $C = C_1/2$,

$$\int e^{-y} \cos y dy = \frac{e^{-y} \sin y - e^{-y} \cos y}{2} + C = \frac{\sin y - \cos y}{2e^y} + C.$$

26. Evaluate

$$\int_0^1 x \sqrt{1-x} dx.$$

Answer: Let $u = x$ and $dv = \sqrt{1-x} dx$. Then $du = dx$ and $v = -\frac{2}{3}(1-x)^{3/2}$. Hence,

$$\begin{aligned} \int_0^1 x \sqrt{1-x} dx &= -\frac{2}{3} x(1-x)^{3/2} \Big|_0^1 + \frac{2}{3} \int_0^1 (1-x)^{3/2} dx \\ &= -\frac{2}{3} x(1-x)^{3/2} - \frac{2}{3} \cdot \frac{2}{5} (1-x)^{5/2} \Big|_0^1 \\ &= (0-0) - \left(0 - \frac{4}{15} \right) \\ &= \frac{4}{15}. \end{aligned}$$

27. Evaluate

$$\int_0^{\pi/3} x \tan^2 x dx.$$

Answer: Note that

$$\int_0^{\pi/3} x \tan^2 x dx = \int_0^{\pi/3} x(\sec^2 x - 1) dx = \int_0^{\pi/3} x \sec^2 x dx - \int_0^{\pi/3} x dx.$$

Now, $\int_0^{\pi/3} x dx = \frac{x^2}{2} \Big|_0^{\pi/3} = \frac{\pi^2}{18}$. On the other hand, if $u = x$ and $dv = \sec^2 x dx$, then $du = dx$ and $v = \tan x$, so

$$\begin{aligned} \int_0^{\pi/3} x \tan^2 x dx &= x \tan x \Big|_0^{\pi/3} - \int_0^{\pi/3} \tan x dx \\ &= \frac{\pi}{3} \sqrt{3} - [\ln |\sec x|]_0^{\pi/3} \\ &= \frac{\pi}{3} \sqrt{3} - (\ln 2 - \ln 1) \\ &= \frac{\pi}{3} \sqrt{3} - \ln 2. \end{aligned}$$

Therefore,

$$\int_0^{\pi/3} x \tan^2 x dx = \frac{\pi}{3} \sqrt{3} - \ln 2 - \frac{\pi^2}{18}.$$

DRL 3E3A, UNIVERSITY OF PENNSYLVANIA
E-mail address: shonkwil@math.upenn.edu