MATH 104 HW 7

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§6.8

2. Use reference triangles to find the following angles.
Answer:

(a): $\tan^{-1}(-1)$

If $\tan\theta = -1$, then that means the two non-hypotenuse sides of the triangle are the same length, with the "opposite" side lying below the x-axis (since the range for $\tan^{-1}$ is between $-\pi/2$ and $\pi/2$). In particular, $\tan^{-1}(-1) = -\pi/4$.

(b): $tan^{-1}(\sqrt{3})$

Since $\sin\pi/3 = \sqrt{3}/2$ and $\cos\pi/3 = 1/2$, we see that $\tan\pi/3 = \frac{\sin\pi/3}{\cos\pi/3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$, so $\tan^{-1}(\sqrt{3}) = \pi/3$.

(c): $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$.

Since $\sin(-\pi/6) = 1/2$ and $\cos(-\pi/6) = \sqrt{3}/2$, we see that $\tan(\pi/6) = -1/\sqrt{3}$, so $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\pi/6$.

14. Given that $\alpha = \tan^{-1}(4/3)$, find $\sin\alpha$, $\cos\alpha$, $\sec\alpha$, $\csc\alpha$ and $\cot\alpha$.
Answer: The easiest to determine is $\cot\alpha = \frac{1}{\tan\alpha} = 3/4$. Now, if $\tan\alpha = 4/3$, then we can draw a triangle with side opposite α of length 4 and side adjacent to α of length 3. Then the hypotenuse of this triangle is $\sqrt{4^2+3^2} = \sqrt{25} = 5$. Hence, $\sin\alpha = 4/5$ and $\cos\alpha = 3/5$. In turn, this means $\csc\alpha = 5/4$ and $\sec\alpha = 5/3$.

16. Given that $\alpha = \sec^{-1}(-\sqrt{13}/2)$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\csc\alpha$ and $\cot\alpha$.
Answer: $\cos\alpha = \frac{1}{\sec\alpha} = -2/\sqrt{13}$. Hence, we're dealing with a triangle with adjacent side of length 2 and hypotenuse of length $\sqrt{13}$, so the opposite side is of length $\sqrt{\left(\sqrt{13}\right)^2 - 2^2} = \sqrt{9} = 3$. Furthermore, since $\cos\alpha$ is negative, the relevant triangle has it's "adjacent" side lying along the negative x-axis (i.e. to the left of the origin). Therefore, $\sin\alpha = 3/\sqrt{13}$ and $\tan\alpha = 3/-2 = -3/2$. Hence, $\csc\alpha = \sqrt{13}/3$ and $\cot\alpha = -2/3$.

18. Find $\sec\left(\cos^{-1}\frac{1}{2}\right)$
Answer: Let $\alpha = \cos^{-1}\frac{1}{2}$. Then $\cos\left(\cos^{-1}\ \alpha\right) = \frac{1}{2}$. Hence, $\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{\frac{1}{2}} = 2$.

30. Evaluate $\sec(\tan^{-1}2x)$.

Answer: Let $\alpha = \tan^{-1} 2x$. Then we can think of α being an angle in a right triangle with opposite side of length $2x$ and adjacent side of length 1. This leads to a hypotenuse of length $\sqrt{(2x)^2 + 1^2} = \sqrt{4x^2+1}$. Therefore, $\cos\alpha = \frac{1}{\sqrt{4x^2+1}}$, so

$$\sec(\tan^{-1} 2x) = \sec\alpha = \frac{1}{\cos\alpha} = \sqrt{4x^2+1}.$$

32. Evaluate $\tan\left(\sec^{-1}\frac{y}{5}\right)$.

Answer: Let $\alpha = \sec^{-1}\frac{y}{5}$. Then $\cos\alpha = \frac{5}{y}$, so we can think of α as being an angle in a triangle with adjacent side of length 5 and hypotenuse of length y. Thus, the opposite side is of length $\sqrt{y^2 - 5^2} = \sqrt{y^2 - 25}$. Therefore,

$$\tan\left(\sec^{-1}\frac{y}{5}\right) = \tan\alpha = \frac{\sqrt{y^2-25}}{5}.$$

§6.9

4. Let $y = \sin^{-1}(1-t)$. Find the derivative of y with respect to t

Answer: Using the chain rule,

$$\frac{dy}{dt} = \frac{1}{\sqrt{1-(1-t)^2}} \cdot \frac{d}{dt}(1-t) = \frac{1-t}{\sqrt{1-(1-2t+t^2)}} = \frac{1-t}{\sqrt{2t-t^2}}.$$

12. Let $y = \cot^{-1}\sqrt{t-1}$. Find the derivative of y with respect to t.

Answer: By the chain rule,

$$\begin{aligned} \frac{dy}{dt} &= \frac{1}{\left(\sqrt{t-1}\right)^1 + 1} \cdot \frac{d}{dt}\left(\sqrt{t-1}\right) \\ &= \frac{1}{t-1+1} \cdot \frac{1}{2\sqrt{t-1}} \\ &= \frac{1}{2\sqrt{t}\sqrt{t-1}} \\ &= \frac{1}{2\sqrt{t^2-t}}. \end{aligned}$$

36. Evaluate the integral

$$\int \frac{6dr}{\sqrt{4-(r+1)^2}}$$

Answer: Let $u = r+1$. Then $du = dr$, so we can re-write the integral as

$$6\int \frac{du}{2^2 - u^2} = 6\left[\sin^{-1}\left(\frac{u}{2}\right) + C_1\right] = 6\sin^{-1}\left(\frac{r+1}{2}\right) + C_2,$$

where C_2 is some constant.

54. Evaluate the integral

$$\int_2^4 \frac{2dx}{x^2 - 6x + 10}.$$

Answer: Complete the square:

$$x^2 - 6x + 10 = (x^2 - 6x) + 10 = (x^2 - 6x + 9) + 10 - 9 = (x-3)^2 + 1.$$

Now, if $u = x - 3$, then $du = dx$, so we can re-write the integral as

$$2\int_2^4 \frac{dx}{(x-3)^2+1} = 2\int_{-1}^1 \frac{du}{u^2+1} = 2\left[\tan^{-1} u\right]_{-1}^1 = 2\left(\pi/4 - (-\pi/4)\right) = \pi.$$

60. Evaluate the integral

$$\int \frac{\sqrt{\tan^{-1} x}dx}{1+x^2}.$$

Answer: Let $u = \tan^{-1} x$. Then $du = \frac{dx}{1+x^2}$, so we can re-write the integral as

$$\int \sqrt{u}du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(\tan^{-1} x\right)^{3/2} + C.$$

§6.11

2. Show that each of the following is a solution to the differential equation $y' = y^2$

(a): $y = -\frac{1}{x}$.

Answer: If $y = -\frac{1}{x}$, then

$$\frac{dy}{dx} = -\frac{-1}{x^2} = \frac{1}{x^2} = \left(\frac{-1}{x}\right)^2 = y^2,$$

so y is a solution to $y' = y^2$.

(b): $y = -\frac{1}{x+3}$.

Answer: If $y = -\frac{1}{x+3}$, then

$$\frac{dy}{dx} = -\frac{-1}{(x+3)^2} = \frac{1}{(x+3)^2} = \left(\frac{-1}{x+3}\right)^2 = y^2,$$

so y is a solution to $y' = y^2$.

(c): $y = -\frac{1}{x+C}$.

Answer: If $y = -\frac{1}{x+C}$, then

$$\frac{dy}{dx} = -\frac{-1}{(x+C)^2} = \frac{1}{(x+C)^2} = \left(\frac{-1}{x+C}\right)^2 = y^2,$$

so y is a solution to $y' = y^2$.

13. Solve the differential equation

$$\frac{dy}{dx} = e^{x-y}.$$

Answer: Using the properties of exponentials, we re-write this as

$$\frac{dy}{dx} = e^x e^{-y}.$$

Then, multiplying both sides by e^y and dx, we see that

$$e^y dy = e^x dx.$$

Integrating both sides yields that

$$e^y = e^x + C$$

for some constant C. Taking the natural logarithm of both sides,

$$y = \ln(e^x + C)$$

16. Solve the differential equation

$$e^x \frac{dy}{dx} + 2e^x y = 1.$$

Answer: First we put the differential equation is standard form by dividing everything by e^x:

$$\frac{dy}{dx} + 2y = e^{-x}.$$

Thus, $P(x) = 2$ and $Q(x) = e^{-x}$. Hence,

$$\int P(x)dx = \int 2dx = 2x,$$

so

$$v(x) = e^{\int P(x)dx} = e^2 x.$$

Therefore,

$$y = \frac{1}{v(x)} \int v(x)Q(x)dx = \frac{1}{e^{2x}} \int e^{2x}e^{-x}dx = e^{-2x} \int e^x dx = e^{-2x}\left[e^x + C\right] = \frac{1}{e^x} + \frac{C}{e^{2x}}.$$

22. Solve the differential equation

$$\sqrt{x}\frac{dy}{dx} = e^{y+\sqrt{x}}, \quad x > 0.$$

Answer: Using the properties of exponentials, we can re-write this as

$$\sqrt{x}\frac{dy}{dx} = e^y e^{\sqrt{x}}.$$

Now, we separate variables:

$$e^{-y}dy = \frac{e^{\sqrt{x}}}{\sqrt{x}}dx.$$

Integrating the left side yields $\int e^{-y}dy = -e^{-y} + C_1$ for some constant C_1. To integrate the righthand side, let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}dx$, so we re-write the righthand side as

$$2\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}dx = 2\int e^u du = 2e^u + C_2 = 2e^{\sqrt{x}} + C_2.$$

Therefore, if we let $C = C_2 - C_1$ and combine the results just found, we see that integrating both sides of the above separated differential equation yields

$$-e^{-y} = 2e^{\sqrt{x}} + C$$

or

$$e^{-y} = -2e^{\sqrt{x}} - C.$$

If we want to solve for y explicitly, we take the natural logarithm of both sides, yielding

$$-y = \ln\left(-2e^{\sqrt{x}} - C\right)$$

or

$$y = -\ln\left(-2e^{\sqrt{x}} - C\right) = \ln\frac{1}{-2e^{\sqrt{x}} - C}.$$

24. Solve the differential equation

$$xy' - y = 2x\ln x.$$

Answer: This looks like a linear first order differential equation, so we put it into standard form by dividing by x:

$$y' + \frac{-1}{x}y = 2\ln x,$$

where $P(x) = \frac{-1}{x}$ and $Q(x) = 2\ln x$. Hence,

$$\int P(x)dx = \int \frac{-1}{x}dx = -\ln|x| = \ln\frac{1}{|x|},$$

so

$$v(x) = e^{\int P(x)dx} = e^{\ln\frac{1}{|x|}} = \frac{1}{|x|}.$$

If $x \geq 0$, then $|x| = x$, so we just use $v(x) = \frac{1}{x}$. Hence,

$$y = \frac{1}{v(x)}\int v(x)Q(x)dx = \frac{1}{\frac{1}{x}}\int \frac{1}{x}2\ln x dx = 2x\int \frac{\ln x}{x}dx.$$

On the other hand, if $x < 0$, then $|x| = -x$, so we can use $v(x) = \frac{-1}{x}$. In that case,

$$y = \frac{1}{v(x)}\int v(x)Q(x)dx = \frac{1}{\frac{-1}{x}}\int \frac{-1}{x}2\ln x dx = -x\int \frac{-1}{x}2\ln x dx = 2x\int \frac{1}{x}\ln x dx.$$

Either way, we get the same result, so we know that $y = 2x\int \frac{1}{x}\ln x dx$. To compute this integral, let $u = \ln x$. Then $du = \frac{1}{x}dx$, so we can re-write this integral as

$$\int u du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C,$$

so

$$y = 2x\left[\frac{(\ln x)^2}{2} + C\right] = x(\ln x)^2 + 2Cx.$$

36. Solve the initial value problem

$$t\frac{dy}{dt} + 2y = t^3, \quad t > 0 \quad y(2) = 1.$$

Answer: First, let's put this differential equation into standard form by dividing by t:

$$\frac{dy}{dt} + \frac{2}{t}y = t^2.$$

Then $P(t) = \frac{2}{t}$ and $Q(t) = t^2$. Hence,

$$\int P(t)dt = \int \frac{2}{t}dt = 2\int \frac{1}{t}dt = 2\ln|t| = \ln|t|^2.$$

Since $t > 0$, we can ignore the absolute value, meaning that

$$v(t) = e^{\int P(t)dt} = e^{\ln t^2} = t^2.$$

Therefore,

$$y = \frac{1}{v(t)}\int v(t)Q(t)dt = \frac{1}{t^2}\int t^2 \cdot t^2 dt = \frac{1}{t^2}\int t^4 dt = \frac{1}{t^2}\left[\frac{t^5}{5} + C\right] = \frac{t^3}{5} + \frac{C}{t^2}.$$

Now, using our initial condition, we know that

$$1 = y(2) = \frac{2^3}{5} + \frac{C}{2^2} = \frac{8}{5} + \frac{C}{4}.$$

Therefore,

$$\frac{C}{4} = 1 - \frac{8}{5} = \frac{-3}{5},$$

so $C = \frac{-12}{5}$. Therefore, we conclude that

$$y = \frac{t^3}{5} - \frac{12}{5t^2}.$$

40. Solve the initial value problem

$$\frac{dy}{dx} + xy = x, \quad y(0) = -6.$$

Answer: This differential equation is already in standard form, with $P(x) = x$ and $Q(x) = x$ as well. Then

$$\int P(x)dx = \int xdx = \frac{x^2}{2},$$

so

$$v(x) = e^{\int P(x)dx} = e^{\frac{x^2}{2}}.$$

Therefore,

$$y = \frac{1}{v(x)}\int v(x)Q(x)dx = \frac{1}{e^{x^2/2}}\int e^{x^2/2}xdx.$$

Thus, if $u = x^2/2$, $du = xdx$ and so we can re-write this integral as

$$\int e^u du = e^u + C = e^{x^2/2} + C.$$

Therefore,

$$y = \frac{1}{e^{x^2/2}}\int e^{x^2/2}xdx = \frac{1}{e^{x^2/2}}\left[e^{x^2/2} + C\right] = 1 + Ce^{-x^2/2}.$$

Now, using our initial value, we know that

$$-6 = y(0) = 1 + Ce^{-(0)^2/2} = 1 + Ce^0 = 1 + C,$$

so $C = -7$. Therefore, we conclude that

$$y = 1 - 7e^{-x^2/2}.$$

42. Use Theorem 4 to solve the following initial value problem for v as a function of t.

$$\frac{dv}{dt} + \frac{k}{m}v = 0, \quad v(0) = v_0$$

(where k and m are positive constants).

Answer: Since this equation is already in standard form, with $P(t) = \frac{k}{m}$ and $Q(t) = 0$, we can proceed:

$$\int P(t)dt = \int \frac{k}{m}dt = \frac{k}{m}t.$$

Now, we can't very well use $v(t)$ to be our integrating factor, since we're solving for $v(t)$, so we call our integrating factor $u(t)$:

$$u(t) = e^{\int P(t)dt} = e^{\frac{k}{m}t}.$$

Therefore,

$$v(t) = \frac{1}{u(t)}\int u(t)Q(t)dt = \frac{1}{e^{\frac{k}{m}t}}\int e^{\frac{k}{m}t}\cdot 0dt = e^{-\frac{k}{m}t}\int 0dt = e^{-\frac{k}{m}t}\cdot C.$$

Now, using our initial condition, we know that

$$v_0 = v(0) = Ce^{-\frac{k}{m}(0)} = Ce^0 = C,$$

so we conclude that

$$y = v_0e^{-\frac{k}{m}t}.$$

46. You have \$1000 with which to open an account and plan to add \$1000 per year. All funds in the account will earn 10% interest per year, compounded continuously. If the added deposits are also credited to your account continuously, the number of dollars x in your account at time t (years) will satisfy the initial value problem

$$\frac{dx}{dt} = 1000 + 0.10x, \quad x(0) = 1000.$$

(a): Solve the initial value problem for x as a function of t.

Answer: Re-arranging a bit, we see that

$$\frac{dx}{dt} + (-0.10)x = 1000,$$

which is an equation in standard form with $P(t) = -0.10$ and $Q(t) = 1000$. Thus,

$$\int P(t)dt = \int -0.10dt = -0.10t,$$

so

$$v(x) = e^{\int P(t)dt} = e^{-0.10t}.$$

Therefore,

$$x = \frac{1}{v(t)} \int v(t)Q(t)dt = \frac{1}{e^{-0.10t}} \int e^{-0.10t} 1000dt = 1000e^{0.10t} \int e^{-0.10t} dt.$$

Let $u = -0.10t$. Then $du = -0.10dt$, so we can re-write this integral as

$$-10 \int e^{-0.10t}(-0.10)dt = -10 \int e^u du = -10e^u + C = -10e^{-0.10t} + C.$$

Therefore,

$$x = 1000e^{0.10t} \left[-10e^{-0.10t} + C\right] = -10,000 + 1000Ce^{0.10t}.$$

Using our initial condition, we know that

$$1000 = x(0) = -10,000+1000Ce^{0.10(0)} = -10,000+1000Ce^0 = -10,000+1000C,$$

so $1000C = 11,000$, which means $C = \frac{11,000}{1000} = 11$. Therefore, $1000C = 11,000$, so

$$x = 11,000e^{0.10t} - 10,000.$$

(b): About how many years will it take for the amount in your account to reach \$100,000?

Answer: This will occur when

$$100,000 = x(t) = 11,000e^{0.10t} - 10,000,$$

or

$$110,000 = 11,000e^{0.10t}.$$

Dividing both sides by 11,000, this means

$$10 = e^{0.10t}.$$

Taking the natural log of both sides,

$$\ln 10 = 0.10t,$$

so

$$t = \frac{\ln 10}{0.10} = 10 \ln 91 \approx 23.03.$$

Hence, it will take slightly over 23 years for you to accumulate \$100,000 at this rate of savings (note that if you were putting \$1000 per year under your mattress instead of in an interest-earning account, it would take you 99 years from when you started to save this same amount).

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