MATH 104 HW 4

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§6.1

13. Let $f(x) = x^2 + 1$, $x \geq 0$. Find a formula for f^{-1}.

Answer: To find f^{-1}, we let $y = f(x) = x^2 + 1$ and switch x and y, yielding $x = y^2 + 1$. Now, solve for y. Hence, $y^2 = x - 1$, so $y = \pm\sqrt{x-1}$. Since we're only dealing with $x \geq 0$, we're only interested in the positive square root, so $f^{-1}(x) = \sqrt{x-1}$.

16. Let $f(x) = x^2 - 2x + 1$, $x \geq 1$. Find a formula for f^{-1}.

Answer: Again, we let $y = f(x) = x^2 - 2x + 1$. Switch x and y to get $x = y^2 - 2y + 1$ and solve for y. Factoring the righthand side, we see that $x = (y-1)^2$. Hence,

$$y - 1 = \pm\sqrt{x},$$

so $y = \pm\sqrt{x} + 1$. Since we're only concerned with $x \geq 1$, we're dealing with the positive square root, so $f^{-1}(x) = \sqrt{x} + 1$.

26. Let $f(x) = (1/5)x + 7$, $a = -1$.

- **(a):** Find $f^{-1}(x)$.
- **(b):** Graph f and f^{-1} together.
- **(c):** Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

Answer:

- **(a):** Let $y = f(x) = (1/5)x + 7$. Then, switching x and y yields $x = (1/5)y + 7$, which reduces to $(1/5)y = x - 7$ or $y = 5(x-7) = 5x - 35$. Hence, $f^{-1}(x) = 5x - 35$.
- **(b):** This is a straight line with slope 5 and y-intercept at -35.
- **(c):** Using the above formulas for f and f^{-1}, we see that $df/dx = 1/5$ and $df^{-1}/dx = 5$. Hence, $df^{-1}/dx = 1/(df/dx)$.

33. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(2, 4)$ and has a slope of $1/3$ there. Find the value of df^{-1}/dx at $x = 4$.

Answer: Since $y = f(x)$ passes through $(2, 4)$, we know that $f(2) = 4$. Hence, $f^{-1}(4) = 2$. Now,

$$\left.\frac{df^{-1}}{dx}\right|_{x=4} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(4)=2}} = \frac{1}{\frac{1}{3}} = 3.$$

Thus, the value of df^{-1}/dx at $x = 4$ is 3.

§6.2

5. Let $y = \ln 3x$ and find the derivative of y with respect to x.
Answer: By the chain rule,

$$\frac{dy}{dx} = \frac{1}{3x} \cdot \frac{d}{dx}[3x] = \frac{1}{3x} \cdot 3 = \frac{1}{x}.$$

16. Let $y = t\sqrt{\ln t}$ and find the derivative of y with respect to t.
Answer: By the product rule,

$$\frac{dy}{dt} = \frac{d}{dt}[t]\sqrt{\ln t} + t \cdot \frac{d}{dt}\left[\sqrt{\ln t}\right] = \sqrt{\ln t} + t \cdot \frac{d}{dt}\left[\sqrt{\ln t}\right].$$

Now, using the power rule and the chain rule,

$$\frac{d}{dt}\left[\sqrt{\ln t}\right] = \frac{1}{2}(\ln t)^{-1/2} \cdot \frac{d}{dt}[\ln t] = \frac{1}{2\sqrt{\ln t}} \cdot \frac{1}{t} = \frac{1}{2t\sqrt{\ln t}}.$$

Therefore, we see that

$$\frac{dy}{dt} = \sqrt{\ln t} + t\frac{d}{dt}\left[\sqrt{\ln t}\right] = \sqrt{\ln t} + t\left(\frac{1}{2t\sqrt{\ln t}}\right) = \sqrt{\ln t} + \frac{1}{2\sqrt{\ln t}}.$$

38. Let $y = \sqrt{(x^2+1)(x-1)^2}$. Use logarithmic differentiation to find the derivative of y with respect to x.

Answer: To use logarithmic differentiation, we take the natural log of both sides:

$$\begin{aligned}
\ln y = \ln\sqrt{(x^2+1)(x-1)^2} &= \ln\left((x^2+1)(x-1)^2\right)^{1/2} \\
&= \frac{1}{2}\ln\left((x^2+1)(x-1)^2\right) \\
&= \frac{1}{2}\left[\ln(x^2+1) + \ln(x-1)^2\right] \\
&= \frac{1}{2}\left[\ln(x^2+1) + 2\ln(x-1)\right].
\end{aligned}$$

Now, differentiating both sides with respect to x, we see that

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x^2+1} \cdot 2x + 2\frac{1}{x-1}\right] &= \frac{x}{x^2+1} + \frac{1}{x-1} \\
&= \frac{x(x-1)}{(x^2+1)(x-1)} + \frac{x^2+1}{(x^2+1)(x-1)} \\
&= \frac{2x^2-x+1}{(x^2+1)(x-1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dy}{dx} &= y\left(\frac{2x^2-x+1}{(x^2+1)(x-1)}\right) \\ &= \sqrt{(x^2+1)(x-1)^2}\left(\frac{2x^2-x+1}{(x^2+1)(x-1)}\right) \\ &= \frac{(x-1)(2x^2-x+1)}{\sqrt{(x^2+1)(x-1)^2}}.\end{aligned}$$

54. Evaluate the integral

$$\int \frac{8rdr}{4r^2-5}.$$

Answer: Let $u = 4r^2 - 5$. Then $du = 8rdr$, so we can re-write the integral as

$$\int \frac{du}{u} = \ln|u| = \ln|4r^2-5|.$$

64. Evaluate the integral

$$\int_{\pi/4}^{\pi/2} \cot t dt.$$

Answer: Recall that $\cot t = \frac{\cos t}{\sin t}$. Let $u = \sin t$. Then $du = \cos t dt$, so we can re-write the integral as

$$\int_{\pi/4}^{\pi/2} \frac{\cos t}{\sin t} dt = \int_{\sqrt{2}/2}^{1} \frac{du}{u} = \ln|u|]_{\sqrt{2}/2}^{1} = \ln 1 - \ln\sqrt{2}/2 = -\ln\sqrt{2}/2.$$

§6.3

18. Let $y = e^{2x/3}$. Find the derivative of y with respect to x.

Answer: Using the chain rule,

$$\frac{dy}{dx} = e^{2x/3}\frac{d}{dx}(2x/3) = \frac{2}{3}e^{2x/3}.$$

38. Suppose $\ln xy = e^{x+y}$. Find dy/dx.

Answer: To find this derivative, we must use implicit differentiation. If we take the derivative of the left side, then we get, by the chain rule and product rule,

$$\frac{1}{xy}\left(y + x\frac{dy}{dx}\right) = \frac{1}{x} + \frac{1}{y}\frac{dy}{dx}.$$

Differentiating the right side yields

$$e^{x+y}\left(1 + \frac{dy}{dx}\right) = e^{x+y} + e^{x+y}\frac{dy}{dx}.$$

Therefore, if we differentiate both sides simultaneously, we see that

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = e^{x+y} + e^{x+y}\frac{dy}{dx}.$$

Now, solving for dy/dx, we have

$$\left(\frac{1}{y} + e^{x+y}\right)\frac{dy}{dx} = e^{x+y} - \frac{1}{x}$$

and so

$$\frac{dy}{dx} = \frac{e^{x+y} - \frac{1}{x}}{\frac{1}{y} + e^{x+y}}.$$

44. Evaluate the integral

$$\int_{-\ln 2}^{0} e^{-x} dx$$

Answer: Let $u = -x$. Then $du = -dx$, so we can re-write the integral as

$$-\int_{-\ln 2}^{0} e^{-x}(-dx) = -\int_{\ln 2}^{0} e^{u} du = -\left[e^u\right]_{\ln 2}^{0} = -e^0 + e^{\ln 2} = -1 + 2 = 1.$$

55. Evaluate the integral

$$\int_{0}^{\pi/4} (1 + e^{\tan\theta})\sec^2\theta d\theta.$$

Answer: Let $u = \tan\theta$. Then $du = \sec^2\theta d\theta$. Hence, we can re-write the integral as

$$\int_0^1 (1 + e^u) du = \left[u + e^u\right]_0^1 = (1 + e) - (0 + 1) = e.$$

64. Suppose $\frac{dy}{dt} = e^{-t}\sec^2(\pi e^{-t})$ and that $y(\ln 4) = 2/\pi$. Solve this initial value problem for y.

Answer: If we let $u = \pi e^{-t}$, then $du/dt = -\pi e^{-t}$, so we can re-write the expression for dy/dt as

$$\frac{-1}{\pi}(-\pi e^{-t})\sec^2(\pi e^{-t}) = \frac{-1}{\pi}\sec^2 u \frac{du}{dt}.$$

Therefore, integrating with respect to u, we see that

$$y = \frac{-1}{\pi}\tan u + C = \frac{-1}{\pi}\tan(\pi e^{-t}) + C.$$

Using the initial value, we know that

$$\frac{2}{\pi} = y(\ln 4) = \frac{-1}{\pi}\tan(\pi e^{-\ln 4}) + C = \frac{-1}{\pi}\tan(\pi/4) + C = \frac{-1}{\pi} + C.$$

Hence, $C = \frac{2}{\pi} + \frac{1}{\pi} = \frac{3}{\pi}$. Therefore,

$$y = \frac{-1}{\pi}\tan(\pi e^{-t}) + \frac{3}{\pi}.$$

§6.4

13. Let $y = 5^{\sqrt{s}}$. Find the derivative of y with respect to s.
Answer: Using the chain rule,

$$\frac{dy}{ds} = 5^{\sqrt{s}} \ln 5 \cdot \frac{d}{ds}\left[\sqrt{s}\right] = 5^{\sqrt{s}} \ln 5 \frac{1}{2\sqrt{s}}.$$

18. Let $y = (\ln\theta)^\pi$. Find the derivative of y with respect to θ.
Answer: Using the generalized power rule and the chain rule,

$$\frac{dy}{d\theta} = \pi(\ln\theta)^{\pi-1} \cdot \frac{d}{d\theta}[\ln\theta] = \pi(\ln\theta)^{\pi-1}\frac{1}{\theta} = \frac{\pi(\ln\theta)^{\pi-1}}{\theta}.$$

25. Let $y = \log_4 x + \log_4 x^2$. Find the derivative of y with respect to x.
Answer: Using the chain rule,

$$\frac{dy}{dx} = \frac{1}{\ln 4}\frac{1}{x}\cdot\frac{d}{dx}[x] + \frac{1}{\ln 4}\frac{1}{x^2}\cdot\frac{d}{dx}[x^2] = \frac{1}{x\ln 4} + \frac{2x}{x^2\ln 4} = \frac{1}{x\ln 4} + \frac{2}{x\ln 4} = \frac{3}{x\ln 4}.$$

40. Let $y = x^{x+1}$. Use logarithmic differentiation to find the derivative of y with respect to x.
Answer: First, take the logarithm of both sides:

$$\ln y = \ln\left(x^{x+1}\right) = (x+1)\ln x = x\ln x + \ln x.$$

Differentiating, we see that

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + 1\cdot\ln x + \frac{1}{x} = 1 + \ln x + \frac{1}{x} = \ln x + \frac{x+1}{x}.$$

Therefore,

$$\frac{dy}{dx} = y\left(1 + \ln x + \frac{1}{x}\right) = x^{x+1}\left(\ln x + \frac{x+1}{x}\right).$$

58. Evaluate the integral

$$\int x^{\sqrt{2}-1}dx.$$

Answer: This is just the power rule in reverse:

$$\int x^{\sqrt{2}-1}dx = \frac{x^{\sqrt{2}}}{\sqrt{2}} + C.$$

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