MATH 104 HW 4

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§5.6

10. Find the lateral surface area of the cone generated by revolving the line segment $y = x/2$, $0 \leq x \leq 4$, about the y-axis. Check your answer with the geometry formula.

Answer: Since we're revolving about the y-axis, we need to re-write the line as $x = 2y$. Then

$$\frac{dx}{dy} = 2$$

and so $\left(\frac{dx}{dy}\right)^2 = 4$. As x ranges from 0 to 4, $y = x/2$ ranges from 0 to 2; therefore,

$$\begin{aligned} SA &= \int_0^2 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= \int_0^2 2\pi(2y)\sqrt{1+4}dy \\ &= 2\pi\sqrt{5}\int_0^2 2y dy \\ &= 2\pi\sqrt{5}y^2\Big]_0^2 \\ &= 2\pi\sqrt{5}(4-0) \\ &= 8\pi\sqrt{5}. \end{aligned}$$

16. Find the area of the surface generated by revolving the curve $y = \sqrt{x+1}$, $1 \leq x \leq 5$ about the x-axis.

Answer: Since we're revolving about the x-axis, we can integrate in terms of x. Now,

$$\frac{dy}{dx} = \frac{1}{2}(x+1)^{-1/2} = \frac{1}{2\sqrt{x+1}},$$

so

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4(x+1)}.$$

Therefore,

$$\begin{aligned} SA &= \int_1^5 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\ &= \int_1^5 2\pi\sqrt{x+1}\sqrt{1+\frac{1}{4(x+1)}}dx \\ &= \int_1^5 \pi\sqrt{4(x+1)\left(1+\frac{1}{4(x+1)}\right)}dx \\ &= \int_1^5 \pi\sqrt{4(x+1)+1}dx \\ &= \int_1^5 \pi\sqrt{4x+5}dx. \end{aligned}$$

Now, if we let $u = 4x+5$, then $du = 4dx$. Thus,

$$\begin{aligned} SA &= \frac{\pi}{4}\int_1^5 \sqrt{4x+5}(4)dx \\ &= \frac{\pi}{4}\int_9^{25}\sqrt{u}du \\ &= \frac{\pi}{4}\frac{2}{3}u^{3/2}\Big]_9^{25} \\ &= \frac{\pi}{6}\left[25\sqrt{25}-9\sqrt{9}\right] \\ &= \frac{98\pi}{6} \\ &= \frac{49\pi}{3} \end{aligned}$$

20. Find the area of the surface generated by revolving $x = \sqrt{2y-1}$, $5/8 \le y \le 1$ about the y-axis.

Answer: Since $x = \sqrt{2y-1}$,

$$\frac{dx}{dy} = \frac{1}{2}(2y-1)^{-1/2}\cdot(2) = \frac{1}{\sqrt{2y-1}},$$

so

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{2y-1}.$$

Therefore,

$$\begin{aligned}
SA &= \int_{5/8}^{1} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \\
&= \int_{5/8}^{1} 2\pi\sqrt{2y-1}\sqrt{1+\frac{1}{2y-1}}dy \\
&= 2\pi \int_{5/8}^{1} \sqrt{(2y-1)+1}dy \\
&= 2\pi \int_{5/8}^{1} \sqrt{2y}dy \\
&= 2\sqrt{2}\pi\frac{2}{3}y^{3/2}\Big]_{5/8}^{1} \\
&= \frac{4\sqrt{2}\pi}{3}\left[1-\frac{5\sqrt{5}}{8\sqrt{8}}\right] \\
&= \frac{4\sqrt{2}\pi}{3} - \frac{5\sqrt{5}\pi}{3}.
\end{aligned}$$

26. Find the area of the surface generated by revolving about the x-axis the portion of the astroid $x^{2/3}+y^{2/3}=1$ shown here.

Answer: As per the hint in the book, since the astroid is symmetric, we will revolve the first-quadrant portion about the x-axis and double the result. Now, since we're revolving about the x-axis, we want to integrate in terms of x, so we re-write the expression for the astroid as

$$y = (1-x^{2/3})^{3/2}.$$

Then

$$\frac{dy}{dx} = \frac{3}{2}(1-x^{2/3})^{1/2}\cdot(-\frac{2}{3}x^{-1/3}) = -\frac{\sqrt{1-x^{2/3}}}{\sqrt[3]{x}}.$$

Thus,

$$\left(\frac{dy}{dx}\right)^2 = \frac{1-x^{2/3}}{x^{2/3}}$$

and so

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{1-x^{2/3}}{x^{2/3}} = \frac{1}{x^{2/3}}.$$

Therefore,

$$\begin{aligned} SA &= \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx \\ &= \int_0^1 2\pi(1 - x^{2/3})^{3/2}\sqrt{\frac{1}{x^{2/3}}}dx \\ &= \int_0^1 2\pi(1 - x^{2/3})^{3/2}\frac{1}{\sqrt[3]{x}}dx. \end{aligned}$$

If we let $u = 1 - x^{2/3}$, then $du = -\frac{2}{3}x^{-1/3}dx$, so we see that

$$\begin{aligned} SA &= -\frac{3}{2}\int_0^1 2\pi(1 - x^{2/3})^{3/2}\frac{-2}{3}x^{-1/3}dx \\ &= -3\pi\int_1^0 u^{3/2}du \\ &= 3\pi\int_0^1 u^{3/2}du \\ &= 3\pi\frac{2}{5}u^{5/2}\Big]_0^1 \\ &= \frac{6\pi}{5}. \end{aligned}$$

§5.7

2. The ends of a log are placed on two scales. One scale reads 100 kg and the other 200 kg. Where is the log's center of mass?

Answer: The log's center of mass is located 2/3 of the way down the log, towards the heavier end.

5. A rod has given density function $\delta(x) = 4$ and lies along the x-axis between $x = 0$ and $x = 2$. Find the rod's moment about the origin, mass, and center of mass.

Answer: The rod's moment is given by

$$\begin{aligned} M_0 &= \int_0^2 x\delta(x)dx \\ &= \int_0^2 x(4)dx \\ &= 2x^2\Big]_0^2 \\ &= 8kg \cdot m. \end{aligned}$$

The mass of the rod is given by

$$\begin{aligned} M &= \int_0^2 \delta(x)dx \\ &= \int_0^4 4dx \\ &= 4x]_0^2 \\ &= 8kg. \end{aligned}$$

Therefore, the center of mass of the rod is

$$\bar{x} = \frac{M_0}{M} = \frac{8kg \cdot m}{8kg} = 1m$$

from the origin.

12. A rod has given density function $\delta(x) = \begin{cases} x+1 & 0 \le x < 1 \\ 2 & 1 \le x \le 2 \end{cases}$. Find the rod's moment about the origin, mass and center of mass.

Answer: The rod's moment is given by the sum of two integrals:

$$\begin{aligned} M_0 &= \int_0^1 x\delta(x)dx + \int_1^2 x\delta(x)dx \\ &= \int_0^1 x(x+1)dx + \int_1^2 x(2)dx \\ &= \int_0^1 (x^2+x)dx + \int_1^2 2xdx \\ &= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 + \left[x^2\right]_1^2 \\ &= \left[\left(\frac{1}{3} + \frac{1}{2}\right) - 0\right] + [4-1] \\ &= \frac{5}{6} + 3 \\ &= \frac{23}{6} kg \cdot m. \end{aligned}$$

The mass of the rod is again given by the sum of two integrals:

$$\begin{aligned} M &= \int_0^1 \delta(x)dx + \int_1^2 \delta(x)dx \\ &= \int_0^1 (x+1)dx + \int_1^2 2dx \\ &= \left[\frac{x^2}{2} + x\right]_0^1 + [2x]_1^2 \\ &= \left[\left(\frac{1}{2}+1\right) - 0\right] + [4-2] \\ &= \frac{3}{2} + 2 \\ &= \frac{7}{2}kg. \end{aligned}$$

Therefore, the rod's center of mass is

$$\bar{x} = \frac{M_0}{M} = \frac{\frac{23}{6}kg \cdot m}{\frac{7}{2}kg} = \frac{23}{21}m$$

from the origin.

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