MATH 104 HW 12

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§8.7

4. Does the alternating series converge or diverge?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{10^n}{n^{10}}.$$

Answer: $\frac{10^n}{n^{10}} \to \infty$ as $n \to \infty$, so the series diverges.

6. Does the alternating series converge or diverge?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$$

Answer: Since $\frac{\ln n}{n} \to 0$ as $n \to \infty$, this alternating series converges.

8. Does the alternating series converge or diverge?

$$\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)$$

Answer: Note that successive terms decrease ($1+\frac{1}{n+1} < 1+\frac{1}{n}$) and, since $\ln 1 = 0$, the terms go to zero, so this series converges.

14. Does the series converge absolutely, converge conditionally, or diverge?

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}.$$

Answer: Since the terms are going to zero and decreasing, the series converges. However,

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{1+\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{1+\sqrt{n}}$$

diverges (do a limit comparison with $\sum \frac{1}{\sqrt{n}}$), so the series only converges conditionally.

18. Does the series converge absolutely, converge conditionally, or diverge?

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin n}{n^2}.$$

Answer: Note that

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{\sin n}{n^2}\right| = \sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right| \leq \sum_{n=1}^{\infty}\frac{1}{n^2},$$

which is a convergent p-series, so the series converges absolutely.

22. Does the series converge absolutely, converge conditionally, or diverge?

$$\sum_{n=1}^{\infty}\frac{(-2)^{n+1}}{n+5^n}.$$

Answer: Note that

$$\sum_{n=1}^{\infty}\left|\frac{(-2)^{n+1}}{n+5^n}\right| = \sum_{n=1}^{\infty}\frac{2^{n+1}}{n+5^n} \leq \sum_{n=1}^{\infty}\frac{2^{n+1}}{5^n} = \sum_{n=1}^{\infty}\frac{4}{5}\left(\frac{2}{5}\right)^{n-1},$$

which is a convergent geometric series, so the series converges absolutely.

30. Does the series converge absolutely, converge conditionally, or diverge?

$$\sum_{n=1}^{\infty}(-5)^{-n}.$$

Answer: Note that

$$\sum_{n=1}^{\infty}\left|(-5)^{-n}\right| = \sum_{n=1}^{\infty}\left|(-1)^{-n}\frac{1}{5^n}\right| = \sum_{n=1}^{\infty}\frac{1}{5^n},$$

which is a convergent geometric series, so the series converges absolutely.

46. Estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the entire series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{10^n}.$$

Answer: By the Alternating Series Estimation Theorem, if S denotes the sum of the series, s_n denotes the nth partial sum and $u_n = \frac{1}{10^n}$, then

$$|L - s_4| < u_5 = \frac{1}{10^5} = 0.0001.$$

So the error involved in using the first four terms to estimate the sum is less than 0.0001.

51.

(a): The series

$$\frac{1}{3} - \frac{1}{2} + \frac{1}{9} - \frac{1}{4} + \frac{1}{27} - \frac{1}{8} + \cdots + \frac{1}{3^n} - \frac{1}{2^n} + \cdots$$

does not meet one of the conditions of Theorem 8. Which is it?

Answer: It does not meet condition 2, that $u_{n+1} \leq u_n$ for all n sufficiently large.

(b): Find the sum of the series in (a)
Answer:

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{3^n}-\frac{1}{2^n}\right) &= \sum_{n=1}^{\infty}\frac{1}{3^n}-\sum_{n=1}^{\infty}\frac{1}{2^n}\\ &= \sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{1}{3}\right)^{n-1}-\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^{n}\\ &= \frac{\frac{1}{3}}{1-\frac{1}{3}}-\frac{\frac{1}{2}}{1-\frac{1}{2}}\\ &= \frac{\frac{1}{3}}{\frac{2}{3}}-\frac{\frac{1}{2}}{\frac{1}{2}}\\ &= \frac{1}{2}-1\\ &= -\frac{1}{2}.\end{aligned}$$

§8.8

2. Find the power series' radius of convergence and interval of convergence. For what values of x does the series converge absolutely and for which does it converge conditionally?

$$\sum_{n=0}^{\infty}(x+5)^n.$$

Answer: Using the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(x+5)^{n+1}}{(x+5)^n}\right| = \lim_{n\to\infty}|x+5|.$$

Now, $|x+5|<1$ when $-6<x<-4$, so the radius of convergence is 1. Checking the endpoints, when $x=-6$, the series becomes

$$\sum_{n=0}^{\infty}(-1)^n$$

which diverges. When $x=-4$, the series becomes

$$\sum_{n=0}^{\infty}1^n$$

which diverges. Therefore, the interval of convergence is $-6<x<-4$. For all these values of x, the series converges absolutely.

6. Find the power series' radius of convergence and interval of convergence. For what values of x does the series converge absolutely and for which does it converge conditionally?

$$\sum_{n=0}^{\infty}(2x)^n.$$

Answer: Using the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(2x)^n}\right| = \lim_{n\to\infty}|2x|.$$

Now, $|2x| < 1$ when $-\frac{1}{2} < x < \frac{1}{2}$, so the radius of convergence is $\frac{1}{2}$. Checking the endpoints, when $x = -\frac{1}{2}$, the series becomes

$$\sum_{n=0}^{\infty}(-1)^n$$

which diverges. When $x = \frac{1}{2}$, the series becomes

$$\sum_{n=0}^{\infty}1^n$$

which diverges. Therefore, the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$. For all these values of x, the series converges absolutely.

12. Find the power series' radius of convergence and interval of convergence. For what values of x does the series converge absolutely and for which does it converge conditionally?

$$\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}.$$

Answer: Using the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{3^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n x^n}\right| = \lim_{n\to\infty}\left|\frac{3x}{n+1}\right| = 0.$$

Therefore, the series converges absolutely for all x.

13. Find the power series' radius of convergence and interval of convergence. For what values of x does the series converge absolutely and for which does it converge conditionally?

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!}.$$

Answer: Using the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{x^{2n+3}}{(n+1)!}\cdot\frac{n!}{x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = 0,$$

so the series converges absolutely for all values of x.

22. Find the power series' radius of convergence and interval of convergence. For what values of x does the series converge absolutely and for which does it converge conditionally?

$$\sum_{n=1}^{\infty}(\ln n)x^n.$$

Answer: Using the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(\ln(n+1))x^{n+1}}{(\ln n)x^n}\right| = \lim_{n\to\infty}\left|\frac{x\ln(n+1)}{\ln n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{xn}{n+1}\right| = |x|.$$

Now, $|x| < 1$ when $-1 < x < 1$, so the radius of convergence is 1. Checking the endpoints, when $x = -1$, the series becomes

$$\sum_{n=1}^{\infty}(\ln n)(-1)^n$$

which diverges since $\ln n \to \infty$ as $n \to \infty$. When $x = 1$, the series becomes

$$\sum_{n=1}^{\infty}\ln n$$

which diverges. Therefore, the interval of convergence is $-1 < x < 1$. For all these values of x, the series converges absolutely.

34. Find the series' interval of convergence and, within this interval, find the sum of the series as a function of x:

$$\sum_{n=0}^{\infty}\frac{(x+1)^n}{9^n}.$$

Answer: Using the nth root test,

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{(x+1)^n}{9^n}\right|} = \lim_{n\to\infty}\left|\frac{x+1}{9}\right| = \left|\frac{x+1}{9}\right|.$$

Now, $\left|\frac{x+1}{9}\right| < 1$ when $|x+1| < 9$, which is to say that $-10 < x < 8$. Now, checking the endpoints, when $x = -10$, the series becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{9^n}$$

which is a convergent alternating series. When $x = 8$, the series becomes

$$\sum_{n=0}^{\infty}\frac{1}{9^n}$$

which is a convergent geometric series. Therefore the interval of convergence of the power series is $-10 \le x \le 8$. Within this interval, the series is a geometric series with $r = \frac{x+1}{9}$, which has absolute value less than 1. Therefore,

$$\sum_{n=0}^{\infty} \frac{(x+1)^n}{9^n} = \sum_{n=0}^{\infty} \left(\frac{x+1}{9}\right)^n = \sum_{n=1}^{\infty} \left(\frac{x+1}{9}\right)^{n-1} = \frac{1}{1-\frac{x+1}{9}} = \frac{1}{\frac{9-(x+1)}{9}} = \frac{9}{8-x}.$$

42. The series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

converges to e^x for all x.

(a): Find a series for $(d/dx)e^x$.

Answer: We did this in class. See your notes from Tuesday.

(b): Find a series for $\int e^x dx$. Do you get the series for e^x? Explain your answer.

Answer: We integrate term-by-term:

$$\int e^x = \int \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)n!} + C = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} + C.$$

This is essentially the same as the series for e^x, except with the first term left off. However, the first term is a constant, and will get picked up by the C.

(c): Replace x by $-x$ in the series for e^x to find a series that converges to e^{-x} for all x. Then multiply the series for e^x and e^{-x} to find the first six terms of a series for $e^{-x} \cdot e^x$.

Answer: The series for e^{-x} is

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots.$$

Since we didn't talk about series multiplication in class at all, I don't expect you to be able to do the last part of this problem.

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