MATH 104 HW 10

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§8.4

2. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} e^{-n}.$$

Answer: We can re-write the series as

$$\sum_{n=1}^{\infty} \frac{1}{e} \cdot \left(\frac{1}{e}\right)^{n-1},$$

which is a geometric series with $a = r = \frac{1}{e}$, so the series converges to

$$\frac{\frac{1}{e}}{1 - \frac{1}{e}} = \frac{\frac{1}{e}}{\frac{e-1}{e}} = \frac{1}{e-1}.$$

3. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{n}{n+1}.$$

Answer: Note that, by L'Hôpital's Rule,

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1} = 1.$$

Hence, by the nth term test, the series diverges.

6. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}}.$$

Answer: We can re-write this series as

$$\sum_{n=1}^{\infty} -2 \cdot \frac{1}{n^{3/2}} = -2 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

Since $3/2 > 1$, this is a convergent p-series.

20. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}.$$

Answer: We can re-write this series as

$$\sum_{n=1}^{\infty} \frac{1}{\ln 3} \cdot \left(\frac{1}{\ln 3}\right)^{n-1}.$$

This is a geometric series with $a = r = \ln 3 \approx 1.09 > 1$, so it diverges.

27. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}.$$

Answer: We apply the integral test:

$$\int_1^{\infty} \frac{8\tan^{-1} x}{1+x^2} dx = \lim_{b\to\infty} \int_1^b 8 \int_1^b \frac{\tan^{-1} x}{1+x^2} dx$$

Now, letting $u = \tan^{-1} x$, $du = \frac{1}{1+x^2}$, so the above integral becomes

$$\begin{aligned}
\lim_{b\to\infty} 8 \int_{\pi/4}^{\tan^{-1} b} u du &= \lim_{b\to\infty} 8 \left[\frac{u^2}{2}\right]_{\pi/4}^{\tan^{-1} b} \\
&= \lim_{b\to\infty} 8 \left[\frac{\left(\tan^{-1} b\right)^2}{2} - \frac{(\pi/4)^2}{2}\right] \\
&= 8 \left[\frac{(\pi/2)^2}{2} - \frac{(\pi/4)^2}{2}\right] \\
&= 8 \left[\frac{\pi^2}{8} - \frac{\pi^2}{32}\right] \\
&= 8 \left[\frac{3\pi^2}{32}\right] \\
&= \frac{3\pi^2}{4}.
\end{aligned}$$

Therefore, since $\int_1^{\infty} \frac{8\tan^{-1} x}{1+x^2} dx$ converges, the series $\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$ also converges, by the integral test.

39.

(a): Show that

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p} \quad (p \text{ a positive constant})$$

converges if and only if $p > 1$.

Answer: Suppose $p = 1$. Let $u = \ln x$. Then $du = \frac{1}{x} dx$ and so

$$\begin{aligned}\int_2^\infty \frac{dx}{x \ln x} &= \int_{\ln 2}^\infty \frac{du}{u} \\ &= \lim_{b\to\infty} \int_{\ln 2}^b \frac{du}{u} \\ &= \lim_{b\to\infty} \ln|u|\Big]_{\ln 2}^b \\ &= \lim_{b\to\infty} [\ln b - \ln(\ln 2)] \\ &= \infty.\end{aligned}$$

On the other hand, if $p < 1$, then again we let $u = \ln x$, and so $du = \frac{1}{x} dx$ and

$$\begin{aligned}\int_2^\infty \frac{dx}{x(\ln x)^p} &= \int_{\ln 2}^\infty \frac{du}{u^p} \\ &= \lim_{b\to\infty} \int_{\ln 2}^b \frac{du}{u^p} \\ &= \lim_{b\to\infty} \frac{u^{-p+1}}{-p+1}\Big]_{\ln 2}^b \\ &= \lim_{b\to\infty} \left[\frac{b^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}\right] \\ &= \infty\end{aligned}$$

since $1 - p > 0$. Finally, if $p > 1$, then the above computation of the integral still applies; however,

$$\lim_{b\to\infty} \left[\frac{b^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}\right] = -\frac{(\ln 2)^{1-p}}{1-p}$$

since $1 - p < 0$. Therefore, we conclude that $\int_2^\infty \frac{dx}{x(\ln x)^p}$ converges if and only if $p > 1$.

(b): What implications does the fact in (a) have for the convergence of the series

$$\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}?$$

Answer: The result proved in (a) demonstrates, by the integral test, that this series converges if and only if $p > 1$.

§8.5

2. Does the following series converge or diverge?

$$\sum_{n=1}^\infty \frac{3}{n + \sqrt{n}}.$$

Answer: Since for all $n \geq 1$, $\sqrt{n} \leq n$, we know that $n+\sqrt{n} \leq n+n = 2n$. Therefore,

$$\frac{3}{n+\sqrt{n}} \geq \frac{3}{2n}.$$

Now,

$$\sum_{n=1}^{\infty} \frac{3}{2n} = \sum_{n=1}^{\infty} \frac{3}{2} \cdot \frac{1}{n}$$

is a p-series with $p = 1$, and so diverges. Hence, since $\frac{3}{n+\sqrt{n}} \geq \frac{3}{2n}$ for all $n \geq 1$, the series $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ diverges as well.

6. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}.$$

Answer: Since constant terms will tend not to mean very much as n gets big, we compare the terms of this series with $\frac{n}{n^2\sqrt{n}} = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$. Now,

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is a p-series with $p = 3/2 > 1$, so it converges. Hence, we expect the original series should converge. However, since $\frac{n+1}{n^2\sqrt{n}} \geq \frac{1}{n^{3/2}}$, a direct comparison won't demonstrate this, so we do a limit comparison instead:

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2\sqrt{n}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{3/2}(n+1)}{n^2\sqrt{n}} = \lim_{n\to\infty} \frac{n+1}{n} = 1 > 0.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ acts like the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is to say it also converges.

14. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}.$$

Answer: Since $\ln n$ grows very, very slowly, we limit compare the terms of this series to the terms of the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$:

$$\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^{3/2}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} (\ln n)^2 = \infty.$$

Since $\sum \frac{1}{n^{3/2}}$ converges, this doesn't tell us anything. However, it still seems like $(\ln n)^2$ shouldn't prevent the series from converging, so let's compare the terms of this series to the terms of another convergent series, $\sum_1 n^{5/4}$:

$$\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^{3/2}}}{\frac{1}{n^{5/4}}} = \lim_{n\to\infty} \frac{n^{5/4}(\ln n)^2}{n^{3/2}}$$
$$= \lim_{n\to\infty} \frac{(\ln n)^2}{n^{1/4}}$$

By L'Hôpital's Rule, this limit is equal to

$$\lim_{n\to\infty} \frac{2\ln n \cdot \frac{1}{n}}{\frac{1}{4}\cdot\frac{1}{n^{3/4}}} = \lim_{n\to\infty} 8\frac{\ln n}{n^{1/4}}.$$

Again using L'Hôpital's Rule, this limit is equal to

$$\lim_{n\to\infty} 8\frac{\frac{1}{n}}{\frac{1}{4}\cdot\frac{1}{n^{3/4}}} = \lim_{n\to\infty} 32\frac{1}{n^{1/4}} = 0.$$

Since $\sum \frac{1}{n^{5/4}}$ converges (it is a p-series with $p = 5/4 > 1$), this tells us that the series

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$$

converges as well.

24. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{3^{n-1}+1}{3^n}.$$

Answer: Re-write the series as

$$\sum_{n=1}^{\infty} \left[\frac{3^{n-1}}{3^n} + \frac{1}{3^n}\right] = \sum_{n=1}^{\infty} \left[\frac{1}{3} + \frac{1}{3^n}\right].$$

Now,

$$\lim_{n\to\infty} \left[\frac{1}{3} + \frac{1}{3^n}\right] = \frac{1}{3} + 0 = \frac{1}{3} \neq 0,$$

so we see that the terms of the series do not converge to 0. Therefore, by the nth term test, the series does not converge.

38. If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty}(a_n/n)$?

Answer: Yes. Note that, for all $n \geq 1$,

$$\frac{a_n}{n} \leq a_n.$$

Therefore, since $\sum_{n=1}^{\infty} a_n$ converges, the direct comparison test tells us that

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

converges as well.

§8.6

2. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} n^2 e^{-n}.$$

Answer: Using the ratio test,

$$\lim_{n\to\infty} \frac{(n+1)^2 e^{-(n+1)}}{n^2 e^{-n}} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2 e}.$$

By two applications of L'Hôpital's Rule, this limit is equal to

$$\lim_{n\to\infty} \frac{2(n+1)}{2ne} = \lim_{n\to\infty} \frac{2}{2e} = \frac{1}{e} < 1.$$

Therefore, by the ratio test, the series converges.

4. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{n!}{10^n}.$$

Answer: Again, we use the ratio test:

$$\lim_{n=1}^{\infty} \frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}} = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \lim_{n\to\infty} \frac{n+1}{10} = \infty > 1,$$

so we see, by the ratio test, that the series diverges.

8. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}.$$

Answer: It's tempting here to use the nth root test; however, since the terms of the series are not all nonnegative, the nth root test does not apply. However,

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n} = \sum_{n=1}^{\infty} \frac{-2}{3}\left(\frac{-2}{3}\right)^{n-1}$$

is a geometric series with $a = r = \frac{-2}{3}$ which has absolute value less than 1, so the series converges to

$$\frac{\frac{-2}{3}}{1 - \left(\frac{-2}{3}\right)} = \frac{\frac{-2}{3}}{\frac{5}{3}} = \frac{-2}{5}.$$

16. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{n \ln n}{2^n}.$$

Answer: Using the nth root test,

$$\lim_{n\to\infty} \sqrt[n]{\frac{n \ln n}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}\,\sqrt[n]{\ln n}}{2} = \frac{1}{2} \lim_{n\to\infty} (\ln n)^{1/n}.$$

Now,

$$\lim_{n\to\infty} \ln{(\ln n)}^{1/n} = \lim_{n\to\infty} \frac{1}{n}\ln(\ln n) = \lim_{n\to\infty} \frac{\ln(\ln n)}{n} = \lim_{n\to\infty} \frac{\frac{1}{\ln n}\cdot\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n\ln n} = 0,$$

so we see that

$$\lim_{n\to\infty} \sqrt[n]{\frac{n\ln n}{2^n}} = \frac{1}{2}\lim_{n\to\infty} (\ln n)^{1/2} = \frac{1}{2}\cdot e^0 = \frac{1}{2} < 1.$$

Therefore, by the nth root test, the series converges.

23. Does the following series converge or diverge?

$$\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}.$$

Answer: Using the nth root test,

$$\lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\ln n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, the series converges by the nth root test.

29. Let $a_1 = \frac{1}{3}$ and $a_{n+1} = \frac{3n-1}{2n+5}a_n$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge?

Answer: Using the ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{3n-1}{2n+5}a_n}{a_n} = \lim_{n\to\infty} \frac{3n-1}{2n+5}.$$

By L'Hôpital's Rule, this limit is equal to

$$\lim_{n\to\infty} \frac{3}{2} = \frac{3}{2} > 1,$$

so, by the ratio test, the series diverges.

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