MATH 104 HW 10

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§8.1

5. Let $a_n = \frac{2^n}{2^{n+1}}$. Find the values of a_1, a_2, a_3 and a_4.
Answer:

$$a_1 = \frac{2^1}{2^2} = \frac{2}{4} = \frac{1}{2}$$
$$a_2 = \frac{2^2}{2^3} = \frac{4}{8} = \frac{1}{2}$$
$$a_1 = \frac{2^3}{2^4} = \frac{8}{16} = \frac{1}{2}$$
$$a_1 = \frac{2^4}{2^5} = \frac{16}{32} = \frac{1}{2}$$

12. Let $a_1 = 2$, $a_2 = -1$ and $a_{n+2} = a_{n+1}/a_n$. Find the first ten terms of the sequence:
Answer:

$$a_1 = 2$$
$$a_2 = -1$$
$$a_3 = \frac{a_2}{a_1} = \frac{-1}{2}$$
$$a_4 = \frac{a_3}{a_2} = \frac{\frac{-1}{2}}{-1} = \frac{1}{2}$$
$$a_5 = \frac{a_4}{a_3} = \frac{\frac{1}{2}}{\frac{-1}{2}} = -1$$
$$a_6 = \frac{a_5}{a_4} = \frac{-1}{\frac{1}{2}} = -2$$
$$a_7 = \frac{a_6}{a_5} = \frac{-2}{-1} = 2$$
$$a_8 = \frac{a_7}{a_6} = \frac{2}{-2} = -1$$
$$a_9 = \frac{a_8}{a_7} = \frac{-1}{2}$$
$$a_{10} = \frac{a_9}{a_8} = \frac{\frac{-1}{2}}{-1} = \frac{1}{2}$$

15. Find a formula for the nth term in the sequence $1, -4, 9, -16, 25, \ldots$.
Answer: The nth term of the sequence is

$$a_n = (-1)^{n-1} \cdot n^2.$$

21. Find a formula for the nth term in the sequence $1, 0, 1, 0, 1, \ldots$.
Answer: The nth term in the sequence is

$$a_n = \frac{1}{2} + \frac{(-1)^{n+1}}{2}.$$

38. Let

$$a_n = \frac{2^n - 1}{3^n}.$$

Does the sequence $\{a_n\}$ converge or diverge?
Answer: Note that

$$a_n = \frac{2^n - 1}{3^n} = \left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n.$$

Since $\left(\frac{2}{3}\right)^n$ and $\left(\frac{1}{3}\right)^n$ both converge to zero, $\{a_n\}$ also converges to zero.
49. Is it true that a sequence $\{a_n\}$ of positive numbers must converge if it is bounded from above? Give reasons for your answer.
Answer: No, it's not true that a bounded sequence of positive numbers must converge. For example, consider the sequence

$$1, 2, 1, 2, 1, 2, 1, 2, \ldots$$

This sequence is bounded by 2 (that is, every term in the sequence is less than or equal to 2), but the sequence does not converge.

§8.2

2. Let

$$a_n = \frac{n + (-1)^n}{n}.$$

Does $\{a_n\}$ converge or diverge?
Answer: Note that

$$a_n = \frac{n + (-1)^n}{n} = 1 + \frac{(-1)^n}{n}.$$

Hence, the question boils down to whether $\left\{\frac{(-1)^n}{n}\right\}$ converges or diverges. Now,

$$\left|\frac{(-1)^n}{n}\right| \leq \frac{1}{n}.$$

Since $\frac{1}{n} \to 0$, the Sandwich Theorem tells us that $\frac{(-1)^n}{n} \to 0$. Hence,

$$\frac{n + (-1)^n}{n} \to 1 + 0 = 1.$$

6. Let

$$a_n = \frac{n+3}{n^2 + 5n + 6}.$$

Does $\{a_n\}$ converge or diverge?

Answer: By L'Hôpital's Rule,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+3}{n^2+5n+6} = \lim_{n\to\infty} \frac{1}{2n+5} = 0.$$

14. Let

$$a_n = \left(-\frac{1}{2}\right)^n.$$

Does $\{a_n\}$ converge or diverge?

Answer: Note that

$$\left|-\frac{1}{2}\right|^n \le \left(\frac{1}{2}\right)^n = \frac{1}{2^n}.$$

Now, $\frac{1}{2^n} \to 0$, so, by the sandwich theorem, $a_n \to 0$.

18. Let

$$a_n = n\pi\cos(n\pi).$$

Does $\{a_n\}$ converge or diverge?

Answer: Note that $\cos(n\pi) = (-1)^n$. Hence,

$$a_n = n\pi\cos(n\pi) = (-1)^n n\pi.$$

As $n \to \infty$, $n\pi \to \infty$, so $\{a_n\}$ diverges.

33. Let

$$a_n = \frac{\ln n}{n^{1/n}}.$$

Does $\{a_n\}$ converge or diverge?

Answer: Note that $n^{1/n} = \sqrt[n]{n} \to 1$ as $n \to \infty$ (see table page 625). Since $\ln n \to \infty$ as $n \to \infty$, this means that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n^{1/n}} = \infty,$$

so the sequence diverges.

§8.3

10. Write the first few terms of the following series, then find the sum of the series:

$$\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}.$$

Answer: If $\{a_n\}$ is the sequence of terms, then

$$\begin{aligned}
a_0 &= (-1)^0 \frac{5}{4^0} = 5 \\
a_1 &= (-1)^1 \frac{5}{4^1} = \frac{-5}{4} \\
a_2 &= (-1)^2 \frac{5}{4^2} = \frac{5}{16} \\
a_3 &= (-1)^3 \frac{5}{4^3} = \frac{-5}{64} \\
a_4 &= (-1)^4 \frac{5}{4^4} = \frac{5}{256}
\end{aligned}$$

To find the sum of the series, we re-write:

$$\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{5}{4^{n-1}} = \sum_{n=1}^{\infty} 5 \cdot \left(\frac{-1}{4}\right)^{n-1}.$$

Hence, this is a geometric series with $a = 5$ and $b = \frac{-1}{4}$, so

$$\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n} = \frac{5}{1 - \left(-\frac{1}{4}\right)} = \frac{5}{\frac{5}{4}} = 4.$$

12. Write the first few terms of the following series, then find the sum of the series:

$$\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right).$$

Answer: Just computing directly, we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right) &= \sum_{n=0}^{\infty} \frac{5}{2^n} - \sum_{n=0}^{\infty} \frac{1}{3^n} \\
&= \sum_{n=1}^{\infty} \frac{5}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{3^{n-1}} \\
&= \sum_{n=1}^{\infty} 5 \cdot \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} 1 \cdot \left(\frac{1}{3}\right)^{n-1} \\
&= \frac{5}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{3}} \\
&= \frac{5}{\frac{1}{2}} - \frac{1}{\frac{2}{3}} \\
&= 10 - \frac{3}{2} \\
&= \frac{17}{2}.
\end{aligned}$$

16. Use partial fractions to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}.$$

Answer: First, let $\frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$ and solve for A and B:

$$6 = A(2n+1) + B(2n-1).$$

Letting $x = -\frac{1}{2}$, we see that

$$6 = A(0) + B(-2) = -2B,$$

so $B = -3$. Letting $x = \frac{1}{2}$, we see that

$$6 = A(2) + B(0) = 2A,$$

so $A = 3$. Hence,

$$\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \left(\frac{3}{2n-1} - \frac{3}{2n+1} \right)$$

Hence, the nth partial sum is given by

$$\begin{aligned} s_n &= (3-1) + \left(1 - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{7}\right) + \ldots + \left(\frac{3}{2n-1} - \frac{3}{2n+1}\right) \\ &= 3 - \frac{3}{2n+1}. \end{aligned}$$

Thus, as $n \to \infty$, $s_n \to 3$, so

$$\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = 3.$$

23. Does the series converge or diverge?

$$\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n.$$

Answer: Note that

$$\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^{n-1} = \frac{1}{1 - \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1},$$

since $r = \frac{1}{\sqrt{2}} < 1$.

24. Does the series converge or diverge?

$$\sum_{n=0}^{\infty} \left(\sqrt{2}\right)^n.$$

Answer: Note that

$$\sum_{n=0}^{\infty} \left(\sqrt{2}\right)^n = \sum_{n=1}^{\infty} \left(\sqrt{2}\right)^{n-1}$$

which is a divergent geometric series, since $r = \sqrt{2} > 1$.

25. Does the series converge or diverge?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}.$$

Answer: Note that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n} = \sum_{n=1}^{\infty}(-1)^2(-1)^{n-1}\cdot\frac{3}{2}\cdot\frac{1}{2^{n-1}} = \sum_{n=1}^{\infty}\frac{3}{2}\left(\frac{-1}{2}\right)^{n-1} = \frac{\frac{3}{2}}{1+\frac{1}{2}} = 1$$

since this is a geometric series with $r = \frac{-1}{2}$, which has absolute value < 1.

46. Find the values of x for which the series

$$\sum_{n=0}^{\infty}(-1)^n x^{-2n}$$

converges. Also, find the sum of the series (as a function of x) for those values of x.

Answer: Note that

$$\sum_{n=0}^{\infty}(-1)^n x^{-2n} = \sum_{n=1}^{\infty}(-1)^{n-1}x^{-2(n-1)} = \sum_{n=1}^{\infty}x^{-2}\left(-x\right)^{n-1}.$$

This is a geometric series with $a = x^{-2}$ and $r = -x$, so it converges if $|r| = |-x| < 1$, which is to say if $|x| < 1$. For such values, we know, moreover, that the series converges to

$$\frac{x^{-2}}{1-(-x)} = \frac{\frac{1}{x^2}}{1+x} = \frac{1}{x^2+x^3}.$$

77. Helga von Koch's snowflake is a curve of infinite length that encloses a region of finite area. To see why this is so, suppose the curve is generated by starting with an equilateral triangle whose sides have length 1.

(a): Find the length L_n of the nth curve C_n and show that $\lim_{n\to\infty} L_n = \infty$.

Answer: Note that $L_1 = 1+1+1 = 3$. Now, the length of each side of C_2 is $\frac{1}{3}$, and there are $4\cdot 3 = 12$ of them, so

$$L_2 = 4\cdot 3\cdot\frac{1}{3} = 4.$$

The length of each side of C_3 is $\frac{1}{3^2}$ and there are $4\cdot(4\cdot 3) = 4^2\cdot 3 = 48$ of them, so

$$L_3 = 4^2\cdot 3\cdot\frac{1}{3^2} = \frac{4^2}{3}.$$

In general,

$$L_n = 4^{n-1}\cdot 3\cdot\frac{1}{3^{n-1}} = 3\cdot\left(\frac{4}{3}\right)^{n-1}.$$

As $n\to\infty$, $\left(\frac{4}{3}\right)^{n-1}\to\infty$ since $\frac{4}{3} > 1$, so we see that $L_n\to\infty$ as $n\to\infty$.

(b): Find the area A_n of the region enclosed by C_n and calculate $\lim_{n\to\infty} A_n$.

Answer: Recall that if T is an equilateral triangle with sides of length a, then the area of T is given by $\frac{\sqrt{3}}{4}a^2$. Hence,

$$A_1 = \frac{\sqrt{3}}{4}.$$

Now, the area of C_2 is given by the area of A_1 plus the area of three equilateral triangles of side length $\frac{1}{3}$. That is,

$$A_2 = A_1 + 3 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3}\right)^2 = A_1 + \frac{1}{3} \cdot \frac{\sqrt{3}}{4}.$$

Now, C_3 has area given by the area of C_2 plus the area of $12 = 4 \cdot 3$ equilateral triangles of side length $\frac{1}{3^2}$. That is,

$$A_3 = A_2 + 2^2 \cdot 3 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3^2}\right)^2 = A_2 + \frac{1}{3}\left(\frac{4}{9}\right)\frac{\sqrt{3}}{3}.$$

In turn, to get C_4 we add the areas of $4 \cdot 12 = 4 \cdot (4 \cdot 3)$ equilateral triangles of side length $\frac{1}{3^3}$; i.e.

$$A_4 = A_3 + 4^2 \cdot 3 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3^3}\right)^2 = A_3 + \frac{1}{3}\left(\frac{4}{9}\right)^2 \frac{\sqrt{3}4}{.}$$

Then

$$A_5 = A_4 + 4^3 \cdot 3 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3^4}\right) = A_4 + \frac{1}{3}\left(\frac{4}{9}\right)^3 \frac{\sqrt{3}4}{.}$$

Hence, the area of the infinite snowflake is given by

$$\begin{aligned}
\sum_{n=1}^{\infty} A_n &= \frac{\sqrt{3}}{4} + \sum_{n=2}^{\infty} \frac{\sqrt{3}}{12}\left(\frac{4}{9}\right)^{n-2} \\
&= \frac{\sqrt{3}}{4} + \sum_{n=1}^{\infty} \frac{\sqrt{3}}{12}\left(\frac{4}{9}\right)^{n-1} \\
&= \frac{\sqrt{3}}{4} + \frac{\frac{\sqrt{3}}{12}}{1 - \frac{4}{9}} \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20} \\
&= \frac{2\sqrt{3}}{5}
\end{aligned}$$

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