MATH 104 MID-TERM EXAM SOLUTIONS

CLAY SHONKWILER

(1) Find the area of the region enclosed by the curves $y = \sqrt{x-1}$ and $y = \frac{x-1}{2}$.

Answer: First, we find the points of intersection by setting the two functions equal to eachother: $\sqrt{x-1} = \frac{x-1}{2}$. Squaring both sides yields $x - 1 = \frac{x^2-2x+1}{4}$, so, multiplying by 4 and moving it all to one side

$$0 = x^2 - 6x + 5 = (x-5)(x-1),$$

so $x = 1$ and $x = 5$. Therefore, the desired area is given by

$$\begin{aligned}\int_1^5 \left(\sqrt{x-1} - \frac{x-1}{2}\right) dx &= \frac{2}{3}(x-1)^{3/2} - \frac{(x-1)^2}{4}\Big]_1^5 \\ &= \left(\frac{2}{3}(8) - \frac{16}{4}\right) - (0-0) \\ &= \frac{16}{3} - 4 \\ &= \frac{4}{3}.\end{aligned}$$

(2) Which grows faster, $\log_3 x$ or $\log_{10} 10x$?

Answer: To answer this question, we consider the limit

$$\lim_{x\to\infty} \frac{\log_3 x}{\log_{10} 10x} = \lim_{x\to\infty} \frac{\frac{\ln x}{\ln 3}}{\frac{\ln 10x}{\ln 10}} = \lim_{x\to\infty} \frac{\ln 10}{\ln 3} \cdot \frac{\ln x}{\ln 10x}$$

By L'Hôpital's Rule, this, in turn, is equal to

$$\lim_{x\to\infty} \frac{\ln 10}{\ln 3} \cdot \frac{\frac{1}{x}}{\frac{1}{10x} \cdot 10} = \lim_{x\to\infty} \frac{\ln 10}{\ln 3} \frac{x}{x} = \frac{\ln 10}{\ln 3}.$$

Since this limit is finite and non-zero, $\log_3 x$ and $\log_{10} 10x$ grow at the same rate.

(3) Find the arc length of the curve $y = -\sqrt{1-x^2}$ from $x = \pi/6$ to $x = \pi/2$.

Answer: We use the arc length formula

$$L = \int_{\pi/6}^{\pi/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Now,
$$\frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{x}{\sqrt{1-x^2}},$$
so $\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{1-x^2}$. Thus,
$$\begin{aligned} L = \int_{\pi/6}^{\pi/2} \sqrt{1 + \frac{x^2}{1-x^2}} dx &= \int_{\pi/6}^{\pi/2} \sqrt{\frac{1-x^2}{1-x^2} + \frac{x^2}{1-x^2}} dx \\ &= \int_{\pi/6}^{\pi/2} \sqrt{\frac{1}{1-x^2}} dx \\ &= \int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{1-x^2}} dx \\ &= \sin^{-1} x \Big]_{\pi/6}^{\pi/2}. \end{aligned}$$

Now, I made a mistake in writing this problem; $\sin^{-1}(\pi/2)$ doesn't even make sense and $\sin^{-1}(\pi/6)$ isn't exactly easy to figure out. Anyway, I gave full credit for getting to this stage.

(4) Let $f(x) = \sec\left(\sin^{-1} \frac{x}{6}\right)$. Evaluate the integral
$$\int_0^3 f(x) dx.$$

Answer: Let $\alpha = \sin^{-1} \frac{x}{6}$. Then we can draw a reference triangle with a side of length x opposite the angle α and a hypotenuse of length 6. Then the adjacent side is $\sqrt{6^2 - x^2} = \sqrt{36 - x^2}$. Thus, $\cos\alpha = \frac{\sqrt{36-x^2}}{6}$, so
$$\sec\alpha = \sec\left(\sin^{-1} \frac{x}{6}\right) = \frac{6}{\sqrt{36 - x^2}}.$$

Hence,
$$\begin{aligned} \int_0^3 f(x) &= \int_0^3 \frac{6}{36 - x^2} dx \\ &= 6 \int_0^3 \frac{dx}{\sqrt{6^2 - x^2}} \\ &= 6 \sin^{-1} \frac{x}{6}\Big]_0^3 \\ &= 6 \sin^{-1} \frac{1}{2} - 6 \sin^{-1} 0 \\ &= 6 \frac{\pi}{6} \\ &= \pi. \end{aligned}$$

(5) Find
$$\lim_{x \to 0^+} x^2 \csc^2 x$$

Answer:

$$\lim_{x\to 0^+} x^2\csc^2 x = \lim_{x\to 0^+}\frac{x^2}{\sin^2 x}.$$

Now, $\sin 0 = 0$, so we can apply L'Hôpital's Rule to see that this limit is equal to

$$\lim_{x\to 0^+}\frac{2x}{2\sin x\cos x}.$$

Again, we have $\frac{0}{0}$, so we apply L'Hôpital's Rule again to get

$$\lim_{x\to 0^+}\frac{2}{2\cos^2 x - 2\sin^2 x} = \lim_{x\to 0^+}\frac{1}{\cos^2 x - \sin^2 x} = 1,$$

since $\cos 0 = 1$.

(6) Solve the initial value problem

$$\frac{dy}{dx} = \frac{2xy}{\ln y}, \quad x \geq 1 \quad y(1) = 1.$$

Answer: This is a separable differential equation; we multiply both sides by $\frac{\ln y}{y}dx$ to separate the variables:

$$\frac{\ln y}{y}dy = 2xdx.$$

Now, we integrate: $\int 2xdx = x^2 + C_1$. On the other hand, to compute $\int \frac{\ln y}{y}dy$, let $u = \ln y$. Then $du = \frac{1}{y}dy$, so

$$\int \frac{\ln y}{y}dy = \int udu = \frac{u^2}{2} + C_2 = \frac{(\ln x)^2}{2} + C_2.$$

Thus, if we let $C = C_1 - C_2$ and equate both sides, we see that

$$\frac{(\ln y)^2}{2} = x^2 + C.$$

Then $(\ln y)^2 = 2x^2 + 2C$, so $\ln y = \sqrt{2x^2 + 2C}$, meaning that

$$y = e^{\sqrt{2x^2+2C}}.$$

Now, using our initial condition

$$1 = y(1) = e^{\sqrt{2(1)^2+2C}} = e^{\sqrt{2+2C}},$$

so, since $1 = e^0$, $2 - 2C = 0$, meaning that $C = -1$. Therefore,

$$y = e^{\sqrt{2x^2-2}}.$$

(7) Solve the initial value problem

$$x^2\frac{dy}{dx} + xy = 12x^4, \quad x > 0 \quad y(1) = 4.$$

Answer: This is a linear first order differential equation; we divide by x^2 to put it into standard form:

$$\frac{dy}{dx} + \frac{1}{x}y = 12x^2.$$

Here $P(x) = \frac{1}{x}$ and $Q(x) = 12x^2$. Hence,

$$\int P(x)dx = \int \frac{1}{x}dx = \ln |x|.$$

Since $x > 0$, we can leave off the absolute values, so we have

$$v(x) = e^{\int P(x)dx} = e^{\ln x} = x.$$

Hence,

$$\begin{aligned} y = \frac{1}{v(x)} \int v(x)Q(x)dx &= \frac{1}{x} \int x \cdot 12x^2 dx \\ &= \frac{1}{x} \int 12x^3 dx \\ &= \frac{1}{x} \left[3x^4 + C\right] \\ &= 3x^3 + \frac{C}{x} \end{aligned}$$

Now, using our initial condition,

$$4 = y(1) = 3(1)^3 + \frac{C}{1} = 3 + C,$$

so $C = 1$. Therefore,

$$y = 3x^3 + \frac{1}{x}.$$

(8) Find the volume of the solid obtained by rotating the region bounded by $y = \frac{1}{\sqrt{1+x^2}}$, $x = 1$ and the x- and y-axes about the x-axis.

Answer: Using the disc method,

$$V = \int_0^1 \pi r^2 dx.$$

Now, the radius of the disc is given by $y = \frac{1}{\sqrt{1+x^2}}$, so

$$\begin{aligned} V = \int_0^1 \pi \left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx &= \pi \int_0^1 \frac{1}{1+x^2} dx \\ &= \pi \tan^{-1} x\Big]_0^1 \\ &= \pi \left(\frac{\pi}{4} - 0\right) \\ &= \frac{\pi^2}{4}. \end{aligned}$$

(9) Find the volume obtained by rotating the region described in problem 8 about the y-axis.

Answer: Here, we use the shell method

$$V = \int_0^1 2\pi r h dx.$$

Now, $r = x$ and $h = y = \frac{1}{\sqrt{1+x^2}}$, so

$$V = \int_0^1 2\pi x \frac{1}{\sqrt{1+x^2}} dx = \pi \int_0^1 \frac{2x}{\sqrt{1+x^2}} dx$$

Now, let $u = 1 + x^2$. Then $du = 2xdx$, so

$$V = \pi \int_1^2 \frac{du}{\sqrt{u}} = \pi \left[2\sqrt{u}\right]_1^2 = \pi(2\sqrt{2} - 2) = 2\pi(\sqrt{2} - 1).$$

(10) Let $f(x) = \tan^{-1}\left(\frac{x+1}{2}\right)$. Find

$$\left.\frac{df^{-1}}{dx}\right|_{x=\pi/3}.$$

Answer: The easiest way to see this is probably just to solve for $f^{-1}(x)$, which we do by switching the variables and solving for y:

$$x = \tan^{-1}\left(\frac{y+1}{2}\right).$$

Then $\tan x = \frac{y+1}{2}$, so

$$y = 2\tan x - 1.$$

Hence, $f^{-1}(x) = 2\tan x - 1$, so

$$\frac{df^{-1}}{dx} = 2\sec^2 x.$$

Hence,

$$\left.\frac{df^{-1}}{dx}\right|_{x=\pi/3} = 2\left(\sec \pi/3\right)^2 = 2(2)^2 = 8.$$

Alternatively, we could use the fact that

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(a)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=a}}.$$

Now,

$$\frac{df}{dx} = \frac{1}{1+\left(\frac{x+1}{2}\right)^2} \cdot \frac{1}{2} = \frac{1}{2 + \frac{(x+1)^2}{2}}.$$

Now, note that $\tan^{-1}(\sqrt{3}) = \pi/3$. Hence, if $\frac{a+1}{2} = \sqrt{3}$, then $f(a) = \pi/3$. Solving for a, we see that $a = 2\sqrt{3} - 1$. Therefore,

$$\left.\frac{df}{dx}\right|_{x=2\sqrt{3}-1} = \frac{1}{2 + \frac{((2\sqrt{3}-1)+1)^2}{2}} = \frac{1}{2 + \frac{(2\sqrt{3})^2}{2}} = \frac{1}{8}.$$

Thus,

$$\left.\frac{df^{-1}}{dx}\right|_{x=\pi/3} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2\sqrt{3}-1}} = \frac{1}{\frac{1}{8}} = 8.$$

(11) Prove that

$$\int \sec\theta \tan\theta d\theta = \sec\theta + C.$$

Proof. Note that

$$\int \sec\theta \tan\theta d\theta = \int \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} d\theta = \int \frac{\sin\theta}{\cos^2\theta} d\theta.$$

If we let $u = \cos\theta$, then $du = -\sin\theta d\theta$, so

$$\int \frac{\sin\theta}{\cos^2\theta} d\theta = -\int \frac{du}{u^2} = -\frac{-1}{u} + C = \frac{1}{\cos\theta} + C = \sec\theta + C.$$

□

(12) Let S be a solid whose base is the ellipse $\frac{x^2}{4} + y^2 = 1$ and whose cross-sections perpindicular to the x-axis are right isosceles triangles with one of the equal (i.e. non-hypotenuse) sides determined by the ellipse (meaning the other equal side is vertical). Find the volume of S.

Answer: First, let's re-write the equation for the ellipse; solving for y, we see that

$$y = \pm\sqrt{1 - \frac{x^2}{4}}.$$

Hence, the length of the non-hypotenuse sides of the cross-sectional triangles is $2\sqrt{1 - \frac{x^2}{4}}$. Therefore, the area of each cross-sectional triangle is given by

$$A(x) = \frac{1}{2}\left(2\sqrt{1 - \frac{x^2}{4}}\right)^2 = 2\left(1 - \frac{x^2}{4}\right) = 2 - \frac{x^2}{2}.$$

The volume of the solid is thus given by

$$\begin{aligned} V = \int_{-2}^{2} A(x)dx &= \int_{-2}^{2} \left[2 - \frac{x^2}{2}\right] dx \\ &= 2x - \frac{x^3}{6}\Big]_{-2}^{2} \\ &= \left(4 - \frac{8}{6}\right) - \left(-4 + \frac{8}{6}\right) \\ &= 8 - \frac{16}{6} \\ &= 8 - \frac{8}{3} \\ &= \frac{16}{3}. \end{aligned}$$

(13) Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $x = 0$ and $y = 4$ about the line $x = 2$.

Answer: Using the shell method,

$$V = \int_0^2 2\pi r h dx.$$

Now, since the axis of rotation is $x = 2$, $r = 2 - x$, while $h = 4 - x^2$ (since we want the height of the region between $y = 4$ and $y = x^2$). Thus,

$$\begin{aligned} V = \int_0^2 2\pi(2 - x)(4 - x^2)dx &= 2\pi \int_0^2 \left[8 - 4x - 2x^2 + x^3\right] dx \\ &= 2\pi \left[8x - 2x^2 - \frac{2}{3}x^3 + \frac{x^4}{4}\right]_0^2 \\ &= 2\pi \left[16 - 8 - \frac{16}{3} + 4\right] \\ &= 2\pi \left[12 - \frac{16}{3}\right] \\ &= 2\pi \left[\frac{20}{3}\right] \\ &= \frac{40\pi}{3}. \end{aligned}$$

(14) Suppose you invested \$1000 in an account in which interest is continuously compounded. After 1 year, the amount now in your account is \$1100. Assuming you never withdraw any money from it, in how many years will you have accumulated \$10,000 in the account?

Answer: We know that continuously compounding interest is modeled by the formula

$$A(t) = Pe^{rt}.$$

Now, since you start the account with \$1000, $P = 1000$. Now, after 1 year, the scenario is that

$$1100 = A(t) = 1000e^{r(1)} = 1000e^r.$$

Dividing both sides by 1000, we see that

$$\frac{11}{10} = e^r.$$

Taking the natural log of both sides tells us that

$$r = \ln\frac{11}{10} = \ln 11 - \ln 10.$$

Now we're ready to answer the question: the account will have \$10,000 in it when

$$10,000 = A(t) = 1000e^{(\ln 11 - \ln 10)t}.$$

Dividing by 1000,

$$10 = e^{(\ln 11 - \ln 10)t}.$$

Taking the natural log of both sides, we see that

$$\ln 10 = (\ln 11 - \ln 10)t,$$

so

$$t = \frac{\ln 10}{\ln 11 - \ln 10} \approx 24.16 \text{ years},$$

which is option (d).

(15) Find the surface area of the solid generated by rotating the region enclosed by the curves $y = 2\sqrt{x}$, $y = 0$ and $x = 3$ about the x-axis.

Answer: Note that if $0 = y = 2\sqrt{x}$, then $x = 0$, so the surface area is given by

$$SA = \int_0^3 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx.$$

Now,

$$\frac{dy}{dx} = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}},$$

so $\left(\frac{dy}{dx}\right)^2 = \frac{1}{x}$. Hence,

$$SA = \int_0^3 2\pi\left(2\sqrt{x}\right)\sqrt{1 + \frac{1}{x}}dx = 4\pi\int_0^3 \sqrt{x}\sqrt{1 + \frac{1}{x}}dx$$
$$= 4\pi\int_0^3 \sqrt{x+1}dx.$$

Now, if we let $u = x + 1$, then $du = dx$, so we can re-write the integral as

$$SA = 4\pi \int_1^4 \sqrt{u}du = 4\pi \left. \frac{2}{3}u^{3/2} \right]_1^4 = \frac{8\pi}{3}\left[8 - 1\right] = \frac{56\pi}{3}.$$

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu