MATH 104 SAMPLE FINAL SOLUTIONS

CLAY SHONKWILER

(1) Evaluate the integral

$$\int e^{-x/2} \cos x dx.$$

Answer: We integrate by parts. Let $u = e^{-x/2}$ and $dv = \cos x dx$. Then $du = -\frac{1}{2}e^{-x/2}dx$ and $v = \sin x$. Then the above integral is equal to

$$e^{-x/2} \sin x + \frac{1}{2}\int e^{-x/2} \sin x dx.$$

To compute this integral, again let $u = e^{-x/2}$ and let $dv = \sin x dx$. Then $du = -\frac{1}{2}e^{-x/2}$ and $v = -\cos x$. Hence

$$\begin{aligned}\int e^{-x/2} \cos x dx &= e^{-x/2} \sin x + \frac{1}{2}\int e^{-x/2} \sin x dx \\ &= e^{-x/2} \sin x + \frac{1}{2}\left[-e^{-x/2} \cos x - \frac{1}{2}\int e^{-x/2} \cos x dx\right] \\ &= e^{-x/2} \sin x - \frac{1}{2}e^{-x/2} \cos x - \frac{1}{4}\int e^{-x/2} \cos x dx.\end{aligned}$$

Therefore,

$$\frac{5}{4}\int e^{-x/2} \cos x dx = e^{-x/2} \sin x - \frac{1}{2}e^{-x/2} \cos x,$$

so

$$\int e^{-x/2} \cos x dx = \frac{4}{5}e^{-x/2} \sin x - \frac{2}{5}e^{-x/2} \cos x.$$

(2) Consider the series

$$\sum_{n=1}^{\infty} \frac{12n}{6n^2 + \ln n}.$$

Does this series converge or diverge?

Answer: Since $\ln n$ grows significantly slower than both n and n^2, we suspect that it won't much affect whether the series converges or diverges. Therefore, let's do a limit comparison with the series $\sum \frac{12n}{6n^2} = \sum \frac{2}{n}$, which we know diverges:

$$\lim_{n\to\infty} \frac{\frac{12n}{6n^2+\ln n}}{\frac{2}{n}} = \lim_{n\to\infty} \frac{12n^2}{12n^2 + 2\ln n}.$$

By two applications of L'Hôpital's Rule, this limit is equal to

$$\lim_{n\to\infty} \frac{24n}{24n + \frac{2}{n}} = \lim_{n\to\infty} \frac{24}{24 - \frac{2}{n^2}} = 1.$$

Therefore, since $\sum \frac{2}{n}$ diverges, the series $\sum \frac{12n}{6n^2+\ln n}$ also diverges.

(3) Consider the sequence $\{a_n\}$ with

$$a_n = \frac{(-1)^{\frac{n(n+1)}{2}} 5^n}{\pi^{2n}}.$$

Does this sequence converge or diverge? If it converges, what does it converge to?

Answer: Note that

$$\left| \frac{(-1)^{\frac{n(n+1)}{2}} 5^n}{\pi^{2n}} \right| = \frac{5^n}{\pi^{2n}} = \left(\frac{5}{\pi^2}\right)^n.$$

Since $\frac{5}{\pi^2} < 1$, the sequence $\{\left(\frac{5}{\pi^2}\right)^n\}$ converges to zero. Therefore, by the sandwich theorem, the given sequence converges to zero.

(4) Consider the curve $y = \sqrt{x}$. Find the surface area of the surface obtained by rotating the portion of the curve between $x = 3/4$ and $x = 15/4$ about the x-axis.

Answer: Recall the surface area formula

$$SA = \int_{3/4}^{15/4} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Now, $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, so $\left(\frac{dy}{dx}\right)^2 = \frac{1}{4x}$. Hence,

$$SA = 2\pi \int_{3/4}^{15/4} \sqrt{x}\sqrt{1 + \frac{1}{4x}} dx = 2\pi \int_{3/4}^{15/4} \sqrt{x + \frac{1}{4}} dx.$$

Letting $u = x + \frac{1}{4}$, $du = dx$ and so the above integral becomes

$$2\pi \int_1^4 \sqrt{u} du = 2\pi \left[\frac{2}{3}u^{3/2}\right]_1^4 = 2\pi \left[\frac{2}{3}(8) - \frac{2}{3}(1)\right] = \frac{28\pi}{3}.$$

(5) Suppose you invest \$1000 in an account earning 8% interest compounded continuously. After how many years will you quadruple your money (use the approximation $\ln 4 \approx 1.6$).

Answer Remember that the amount of money in your account is modeled by

$$A(t) = Pe^{rt}.$$

Since the initial amount in the account is \$1000, $P = 1000$. Also, $r = 0.08$. You will quadruple your money when

$$4000 = A(t) = 1000e^{0.08t},$$

so

$$4 = e^{0.08t}.$$

Solving for t,

$$\ln 4 = 0.08t,$$

so

$$t = \frac{\ln 4}{0.08} \approx \frac{1.6}{0.08} = \frac{160}{8} = 20.$$

Therefore, you will quadruple your money in approximately 20 years.

(6) Consider the series

$$\sum_{n=1}^{\infty} \frac{4^n}{e^{2n+1}}.$$

Does this series converge or diverge? If it converges, to what sum does it converge?

Answer: Note that

$$\sum_{n=1}^{\infty} \frac{4^n}{e^{2n+1}} = \sum_{n=1}^{\infty} \frac{1}{e}\frac{4^n}{e^{2n}} = \sum_{n=1}^{\infty} \frac{1}{e}\left(\frac{4}{e^2}\right)^n = \sum_{n=1}^{\infty} \frac{4}{e^3}\left(\frac{4}{e^2}\right)^{n-1}.$$

This is a geometric series with $a = \frac{4}{e^3}$ and $r = \frac{4}{e^2} < 1$. Hence, the series converges to

$$\frac{\frac{4}{e^3}}{1 - \frac{4}{e^2}} = \frac{\frac{4}{e^3}}{\frac{e^2-4}{e^2}} = \frac{4}{e^3 - 4e}.$$

(7) Does the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{2^n n!}$$

converge or diverge?

Answer: Use the Ratio Test:

$$\lim_{n\to\infty} \frac{\frac{(2(n+1))!}{2^{n+1}(n+1)!}}{\frac{(2n)!}{2^n n!}} = \lim_{n\to\infty} \frac{(2n+2)!}{2^{n+1}(n+1)!}\cdot\frac{2^n n!}{(2n)!} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{2(n+1)} = \lim_{n\to\infty} \frac{4n^2+5n+2}{2n+1}.$$

By L'Hôpital's Rule, this limit is equal to

$$\lim_{n\to\infty} \frac{8n+5}{2} = \infty,$$

so the series diverges.

(8) Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2x)^n}{n^2}.$$

For what values of x does the series converge absolutely? For what values does it converge conditionally?

Answer: Using the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{\frac{(2x)^{n+1}}{(n+1)^2}}{\frac{(2x)^n}{n^2}} \right| = \lim_{n\to\infty} \left| \frac{(2x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(2x)^n} \right| = \lim_{n\to\infty} \left| \frac{2xn^2}{(n+1)^2} \right|.$$

By two applications of L'Hôpital's Rule, this limit is equal to

$$\lim_{n\to\infty}\left|\frac{4xn}{2(n+1)}\right| = \lim_{n\to\infty}\left|\frac{4x}{2}\right| = |2x|.$$

Now, $|2x| < 1$ when $|x| < \frac{1}{2}$, so the radius of convergence of the power series is $\frac{1}{2}$. Now, we need to check the endpoints, $x = \pm\frac{1}{2}$. When $x = -\frac{1}{2}$, the series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2},$$

which converges, by the alternating series test. In fact,

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2},$$

which is a convergent p-series. On the other hand, when $x = \frac{1}{2}$, the series becomes

$$\sum_{n=1}^{\infty}\frac{1^n}{n^2},$$

which is a convergent p-series. Therefore, the interval of convergence of the power series is $\left[-\frac{1}{2}, \frac{1}{2}\right]$. The series converges absolutely for $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

(9) Consider the region contained by $y = \frac{1}{1+x^2}$, the x- and y-axes and $x = 1$. What is the volume of the solid obtained by rotating this region about the y-axis?

Answer: Using the shell method, the volume is equal to

$$\int_0^1 2\pi r h dx.$$

Now, $r = x$ and $h = \frac{1}{1+x^2}$, so

$$V = \int_0^1 2\pi x \frac{1}{1+x^2}dx = \pi\int_0^1 \frac{2xdx}{1+x^2}.$$

Letting $u = 1+x^2$, $du = 2xdx$, so we can re-write this integral as

$$\pi\int_1^2 \frac{du}{u} = \pi\left[\ln|u|\right]_1^2 = \pi\left[\ln 2 - \ln 1\right] = \pi\ln 2.$$

(10) Evaluate the integral

$$\int \frac{2dx}{(2x-2)\sqrt{4x^2-8x+1}}.$$

Answer: We complete the square:

$$4x^2 - 8x + 1 = (4x^2 - 8x + 4) + 1 - 4 = (2x-2)^2 - 3.$$

Hence, letting $u = 2x - 2$, $du = 2dx$ and so we can re-write the integral as

$$\int \frac{du}{u\sqrt{u^2-3}} = \frac{1}{\sqrt{3}} \sec^{-1}\left|\frac{u}{\sqrt{3}}\right| + C = \frac{1}{\sqrt{3}} \sec^{-1}\left|\frac{2x-2}{\sqrt{3}}\right| + C.$$

(11) Evaluate the integral

$$\int \frac{2x+3}{x^2-7x+12}dx.$$

Answer: Note that $x^2 - 7x + 12 = (x-3)(x-4)$. Thus, we set up the partial fraction $\frac{2x+3}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$ and solve for A and B:

$$2x + 3 = A(x-4) + B(x-3).$$

Letting $x = 4$,

$$2(4) + 3 = A(0) + B(1) = B,$$

so $B = 11$. Now, letting $x = 3$,

$$2(3) + 3) = A(-1) + B(0) = -A,$$

so $A = -9$. Therefore,

$$\int \frac{2x+3}{x^2-12x+7}dx = \int \left[\frac{-9}{x-3} + \frac{11}{x-4}\right] dx = -9\ln|x-3| + 11\ln|x-4| + C.$$

(12) Solve the initial value problem

$$x\frac{dy}{dx} + y = \sin x + \cos x, \quad x > 0, \qquad y(\pi) = \frac{2}{\pi}.$$

Answer: Re-write in standard form:

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\sin x + \cos x}{x},$$

so $P(x) = \frac{1}{x}$ and $Q(x) = \frac{\sin x + \cos x}{x}$. Hence,

$$\int P(x)dx = \int \frac{1}{x}dx = \ln|x| = \ln x,$$

since $x > 0$. Thus,

$$v(x) = e^{\int P(x)dx} = e^{\ln x} = x.$$

Therefore,

$$\begin{aligned} y = \frac{1}{v(x)} \int v(x)Q(x)dx = \frac{1}{x}\int x\frac{\sin x + \cos x}{x}dx &= \frac{1}{x}\int [\sin x + \cos x]\, dx \\ &= \frac{1}{x}[-\cos x + \sin x + C] \\ &= \frac{\sin x - \cos x + C}{x}. \end{aligned}$$

Using the initial value,

$$\frac{2}{\pi} = y(\pi) = \frac{\sin\pi - \cos\pi + C}{\pi} = \frac{1+C}{\pi},$$

so $C = 1$. Therefore,

$$y = \frac{\sin x - \cos x + 1}{x}.$$

(13) Evaluate the integral

$$\int \frac{x}{(x^2+3)^{3/2}} dx.$$

Answer: Let $x = \sqrt{3}\tan\theta$. Then $dx = \sqrt{3}\sec^2\theta d\theta$, so we can re-write the integral as

$$\begin{aligned}
\int \frac{\sqrt{3}\tan\theta}{(3\tan^2\theta+3)^{3/2}}\sqrt{3}\sec^2\theta d\theta = \int \frac{3\tan\theta\sec^2\theta d\theta}{(3\sec^2\theta)^{3/2}} \\
&= \int \frac{3\tan\theta\sec^2\theta d\theta}{3\sqrt{3}\sec^3\theta} \\
&= \frac{1}{\sqrt{3}}\int \frac{\tan\theta d\theta}{\sec\theta} \\
&= \frac{1}{\sqrt{3}}\int \sin\theta d\theta \\
&= -\frac{1}{\sqrt{3}}\cos\theta + C
\end{aligned}$$

Now, $\theta = \tan^{-1}\frac{x}{\sqrt{3}}$, so $\cos\theta = \frac{\sqrt{3}}{\sqrt{3+x^2}}$. Hence, the integral is equal to

$$-\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3+x^2}} = -\frac{1}{\sqrt{3+x^2}}.$$

(14) Consider the sequence $\{a_n\}$ where

$$a_n = \frac{\sqrt[3]{n}}{\ln(n+1)}.$$

Does the sequence converge or diverge? If it converges, what does it converge to?

Answer: Using L'Hôpital's Rule,

$$\lim_{n\to\infty}\frac{\sqrt[3]{n}}{\ln(n+1)} = \lim_{n\to\infty}\frac{\frac{1}{3n^{2/3}}}{\frac{1}{n+1}} = \lim_{n\to\infty}\frac{n+1}{3n^{2/3}}.$$

Using L'Hôpital again, this is equal to

$$\lim_{n\to\infty}\frac{1}{2n^{-1/3}} = \lim_{n\to\infty}\frac{\sqrt[3]{n}}{2} = \infty.$$

Therefore, the sequence diverges.

(15) Does the series
$$\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$$
converge or diverge?

Answer: Using the Integral Test,
$$\int_1^\infty \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \int_1^b \frac{dx}{x(\ln x)^2}.$$
Now, letting $u = \ln x$, $du = \frac{1}{x}dx$, so
$$\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\ln x} + C.$$
Hence,
$$\lim_{b\to\infty} \int_1^b \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \left[-\frac{1}{\ln x}\right]_1^b = \lim_{b\to\infty} \left[-\frac{1}{\ln b} + \frac{1}{\ln 1}\right] = \infty$$
since $\ln 1 = 0$. Therefore, by the Integral Test, $\sum \frac{1}{n(\ln n)^2}$ diverges.

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu