MATH 104 FINAL EXAM SOLUTIONS

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(1) Solve the initial value problem

$$x\frac{dy}{dx} + \frac{y}{\ln x} = 2, \quad x > 1, \qquad y(e) = 3.$$

Answer: First, re-write in standard form:

$$\frac{dy}{dx} + \frac{1}{x \ln x} y = \frac{2}{x}.$$

Then $P(x) = \frac{1}{x \ln x}$ and $Q(x) = \frac{2}{x}$. Hence,

$$\int P(x) dx = \int \frac{dx}{x \ln x}.$$

Letting $u = \ln x$, $du = \frac{1}{x} dx$, so this integral becomes

$$\int \frac{du}{u} = \ln |u| = \ln(\ln x),$$

where we can discard the absolute values since $x > 1$. Thus,

$$v(x) = e^{\int P(x)dx} = e^{\ln(\ln x)} = \ln x.$$

Therefore,

$$y = \frac{1}{\ln x} \int \frac{2 \ln x}{x} dx.$$

Letting $u = \ln x$, $du = \frac{1}{x} dx$, so this becomes

$$\frac{1}{\ln x} \int 2u du = \frac{1}{\ln x} \left[2\frac{u^2}{2} + C \right] = \frac{1}{\ln x} \left[(\ln x)^2 + C \right] = \ln x + \frac{C}{\ln x}.$$

Now, using the initial condition,

$$3 = y(e) = \ln e + \frac{C}{\ln e} = 1 + C,$$

so $C = 2$. Therefore, $y = \ln x + \frac{2}{\ln x}$.

(2) Consider the sequence $\{a_n\}$ where

$$a_n = \frac{e^n \cos n}{3^n}.$$

Determine whether the sequence converges or diverges. If it converges, find the limit of the sequence.

Answer: Since $|\cos n| \le 1$,

$$\left|\frac{e^n \cos n}{3^n}\right| \le \frac{e^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{e}{3}\right)^n.$$

Since $\frac{e}{3} < 1$, $\left(\frac{e}{3}\right)^n \to 0$ as $n \to \infty$. Therefore, by the Sandwich Theorem, $a_n \to 0$.

(3) Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n}.$$

Answer: Using the nth root test,

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{3^n}\right|} = \lim_{n\to\infty} \left|\frac{x}{3}\right|.$$

The series converges absolutely by the nth root test when $\left|\frac{x}{3}\right| < 1$, which is to say that $|x| < 3$ or $-3 < x < 3$. Now we need only check the endpoints. When $x = -3$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=0} \infty \left(\frac{-3}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n,$$

which diverges. When $x = 3$, the series becomes

$$\sum_{n=0}^{\infty} \frac{3^n}{3^n} = \sum_{n=0}^{\infty} 1,$$

which also diverges. Therefore, the interval of convergence of the power series is $-3 < x < 3$.

(4) Evaluate the integral

$$\int 2x \csc^2 x dx.$$

Answer: We integrate by parts, letting $u = 2x$ and $dv = \csc^2 x dx$. Then $du = 2dx$ and $v = -\cot x dx$. Hence, the integral is equal to

$$-2x \cot x + 2 \int \cot x dx = -2x \cot x + 2 \int \frac{\cos x}{\sin x}.$$

Now, letting $u = \sin x$, $du = \cos x dx$, so the integral on the right becomes

$$2 \int \frac{du}{u} = 2 \ln |u| + C = 2 \ln |\sin x| + C.$$

Therefore,

$$\int 2x \csc^2 x dx = -2x \cot x + 2 \ln |\sin x| + C.$$

(5) Find the length of the curve $y = \frac{2}{3}x^{3/2}$ between $x = 0$ and $x = 3$.

Answer: Recall the length-of-curve formula

$$L = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Now, $\frac{dy}{dx} = \sqrt{x}$, so $\left(\frac{dy}{dx}\right)^2 = x$. Hence,

$$L = \int_0^3 \sqrt{1+x}dx = \frac{2}{3}(1+x)^{3/2}\Big]_0^3 = \frac{2}{3}\left[8-1\right] = \frac{14}{3}.$$

(6) Evaluate the integral

$$\int \frac{dx}{x^2 - 12x + 41}.$$

Answer: We complete the square in the denominator first:

$$x^2 - 12x + 41 = (x^2 - 12x + 36) + 41 - 36 = (x-6)^2 + 5.$$

Letting $u = x - 6$, $du = dx$, so the above integral becomes

$$\int \frac{du}{u^2 + \sqrt{5}^2} = \frac{1}{\sqrt{5}} \tan^{-1} \frac{u}{\sqrt{5}} + C = \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x-6}{\sqrt{5}}\right) + C.$$

(7) Evaluate the integral

$$\int \frac{3x-4}{x^2-6x+9}dx.$$

Answer: Note that $x^2 - 6x + 9 = (x-3)^2$. Therefore, we solve by partial fractions by letting $\frac{3x-4}{x^2-6x+9} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$ and solving for A and B:

$$3x - 4 = A(x-3) + B = Ax + B - 3A.$$

Hence, $A = 3$. Since $-4 = B - 3A = B - 3(3) = B - 9$, we see that $B = 5$. Therefore,

$$\begin{aligned}\int \frac{3x-4}{x^2-6x+9}dx &= \int \left[\frac{3}{x-3} + \frac{5}{(x-3)^2}\right] dx \\ &= 3\ln|x-3| - \frac{5}{x-3} + C.\end{aligned}$$

(8) Does the series

$$\sum_{n=1}^{\infty} \frac{2n+3}{8n^3 - 6n + 1}$$

converge or diverge?

Answer: As n gets large, we expect that the constant terms and lower-order terms should be relatively insignificant, so we expect this series to act like $\sum \frac{2n}{8n^3} = \sum \frac{1}{4n^2}$. Therefore, we do a limit comparison between these two series:

$$\lim_{n\to\infty} \frac{\frac{2n+3}{8n^3-6n+1}}{\frac{1}{4n^2}} = \lim_{n\to\infty} \frac{8n^3+12n^2}{8n^3-6n+1}.$$

Using three applications of L'Hôpital's Rule, this limit is equal to

$$\lim_{n\to\infty} \frac{24n^2+24n}{24n^2-6} = \lim_{n\to\infty} \frac{48n+24}{48n} = \lim_{n\to\infty} \frac{48}{48} = 1.$$

Therefore, the two series either both converge or both diverge. Since $\sum \frac{1}{4n^2} = \frac{1}{4}\sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, it converges so we conclude that

$$\sum_{n=1}^{\infty} \frac{2n+3}{8n^3-6n+1}$$

converges as well.

(9) Does the series

$$\sum_{n=1}^{\infty} \frac{n!}{3^{4n}}$$

converge or diverge?

Answer: Using the Ratio Test:

$$\lim_{n\to\infty} \frac{(n+1)!}{3^{4(n+1)}} \cdot \frac{3^n}{n!} = \lim_{n\to\infty} \frac{n+1}{3^4} = \infty.$$

Therefore, we see that, by the Ratio Test, the series diverges.

(10) Determine whether the series

$$\sum_{n=1}^{\infty} (-3)^n \frac{1}{2^n \pi^{n-1}}$$

converges or diverges. If it converges, find the sum of the series.

Answer: Re-write this series as

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n \pi^{n-1}} = \frac{-3}{2}\left(\frac{-3}{2\pi}\right)^{n-1}.$$

Hence, this is a geometric series with $a = \frac{-3}{2}$ and $r = \left|\frac{-3}{2\pi}\right| < 1$. Thus, the series converges to

$$\frac{\frac{-3}{2}}{1-\frac{-3}{2\pi}} = \frac{\frac{-3}{2}}{\frac{2\pi+3}{2\pi}} = \frac{-3\pi}{2\pi+3}.$$

(11) Consider a bacteria culture that starts with a single, isolated bacterium. Supposing that the rate of change in the population of the culture is proportional to its size and that there are 10 bacteria in the culture after 1 hour, how many bacteria should we expect in the culture after 3 hours? (Hint: the number you get should be quite simple)

Answer: Since the culture starts with a single bacterium, the population is modeled by

$$A(t) = 1 \cdot e^{kt} = e^{kt}.$$

Now,

$$10 = A(1) = e^{k(1)} = e^k,$$

so $k = \ln 10$. Therefore, after 3 hours, there should be

$$A(3) = e^{k(3)} = e^{3\ln 10} = e^{\ln 10^3} = 10^3 = 1000$$

bacteria in the culture.

(12) Consider the region enclosed by $y = \ln x$, the x-axis and $x = e$. Find the volume of the solid obtained by rotating this region about the x-axis (Hint: there are two different methods for computing the volume of a solid of revolution, so if the first you try doesn't work, try the other).

Answer: It turns out that the disc method is no good, so let's try the shell method instead. Recall that, using the shell method, we need to re-express this function in terms of y, since we're rotating about the x-axis. Now, the inverse of $\ln x$ is e^x, so $y = \ln x$ and $x = e^y$ describe the same curve. Now, $\ln e = 1$ and the x-axis describes $y = 0$, so

$$V = \int_0^1 2\pi r h dy.$$

In this case, $r = y$ and $h = e^y$, so

$$V = \int_0^1 2\pi y e^y dy = 2\pi \int_0^1 y e^y dy.$$

Let $u = y$ and $dv = e^y dy$; then $du = dy$ and $v = e^y$, so this integral becomes

$$V = 2\pi \left[y e^y - \int_0^1 e^y dy \right] = 2\pi \left[y e^y - e^y \right]_0^1 = 2\pi \left[(e - e) - (0 - 1) \right] = 2\pi.$$

(13) Evaluate the integral

$$\int \frac{\sqrt{1 - x^2}}{x^4} dx.$$

Answer: Use the trigonometric substitution $x = \sin\theta$. Then $dx = \cos\theta d\theta$, so the integral becomes

$$\int \frac{\sqrt{1 - \sin^2\theta}}{\sin^4\theta} \cos\theta d\theta = \int \frac{\sqrt{\cos^2\theta}}{\sin^4\theta} \cos\theta d\theta = \int \frac{\cos^2\theta}{\sin^4\theta} d\theta = \int \cot^2\theta \csc^2\theta d\theta.$$

Let $u = \cot\theta$. Then $du = -\csc^2\theta$, so this integral becomes

$$-\int u^2 du = -\frac{u^3}{3} + C = -\frac{\cot^3\theta}{3} + C.$$

Now, since $\theta = \sin^{-1} x$, $\cot\theta = \frac{\sqrt{1-x^2}}{x}$, and so we conclude that

$$\int \frac{\sqrt{1-x^2}}{x^4} dx = -\frac{(1-x^2)^{3/2}}{3x^3} + C.$$

(14) Does the series

$$\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$$

converge or diverge?

Answer: Compute the integral:

$$\int_1^{\infty} \frac{\tan^{-1} x}{1+x^2} dx = \lim_{b\to\infty} \int_1^b \frac{\tan^{-1} x}{1+x^2} dx.$$

Now, if we let $u = \tan^{-1} x$, then $du = \frac{dx}{1+x^2}$, so

$$\int \frac{\tan^{-1} x}{1+x^2} dx = \int u du = \frac{u^2}{2} + C = \frac{\left(\tan^{-1} x\right)^2}{2} + C.$$

Therefore,

$$\begin{aligned}
\lim_{b\to\infty} \int_1^b \frac{\tan^{-1} x}{1+x^2} dx &= \lim_{b\to\infty} \left[\frac{\left(\tan^{-1} x\right)^2}{2} \right]_1^b \\
&= \lim_{b\to\infty} \left[\frac{\left(\tan^{-1} b\right)^2}{2} - \frac{\left(\tan^{-1} 1\right)^2}{2} \right] \\
&= \frac{\pi^2}{8} - \frac{\pi^2}{32} \\
&= \frac{3\pi^2}{32}.
\end{aligned}$$

(15) Consider the sequence $\{a_n\}$ where

$$a_n = \frac{\ln \frac{1}{n}}{2\sqrt{n}}.$$

Does the sequence converge or diverge? If it converges, what is the limit of the sequence?

Answer: Using L'Hôpital's Rule,

$$\lim_{n\to\infty} \frac{\ln \frac{1}{n}}{2\sqrt{n}} = \lim_{n\to\infty} \frac{n \cdot \frac{-1}{n^2}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{-\sqrt{n}}{n} = \lim_{n\to\infty} \frac{-1}{\sqrt{n}} = 0.$$

Therefore, the sequence converges to 0.

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