THE GAUSS LINKING INTEGRAL IN S^3 AND H^3

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1. Goal

Let K_1 and K_2 be disjoint closed curves in S^3 or H^3. We want to find an explicit integral formula for the linking number of the two curves, $Lk(K_1, K_2)$.

Next week, Ben will address Călugăreanu's famous formula, Link = Twist + Writhe and maybe talk about helicity and the spectrum of the curl operator.

2. Linking integrals

The classical linking integral, due to Gauss, in $\mathbb{R}^3$:

$$Lk(K_1, K_2) = \frac{1}{4\pi} \int_{K_1 \times K_2} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x - y}{|x - y|^3} ds dt.$$

It will be convenient for us to re-write this as

$$Lk(K_1, K_2) = \int_{K_1 \times K_2} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \nabla_y \varphi(x, y) ds dt$$

where $\varphi(\alpha) = \frac{1}{4\pi\alpha} = -\varphi_0(\alpha)$ and $\alpha(x, y) = |x - y|$. $\varphi_0(\alpha)$ is the fundamental solution of the Laplacian: $\Delta\varphi_0 = \delta$.

In order to prove a similar formula on S^3 and H^3, we need to know why this is true.

Gauss was interested in linking integrals in the context of astronomy; he was interested in knowing how many times the orbits of Earth and an asteroid were linked. It is believed that Gauss essentially counted how many times the vector from Earth to the asteroid "covered the heavenly sphere"; a degree-of-map argument. Unfortunately, this approach doesn't work on S^3, although it can be modified for H^3.

An alternative proof, which Gauss was surely also aware of, is roughly the following: run a current through the first loop (i.e. K_1). Then, by Ampere's Law, the circulation of the resulting magnetic field about the second loop is equal to the amount of current enclosed by the second loop. But this is just the current on the first loop multiplied by the linking number of the loops.

Specifically, the magnetic field on $\mathbb{R}^3$ due to a compactly supported current distribution V is given by the convolution formula of Biot and Savart:

$$BS(V)(y) = \int_{\mathbb{R}^3} V(x) \times \nabla_y \varphi_0(x, y) dx.$$

Hence, if $V(x) = \frac{dx}{ds}$ and $x(s)$ is parametrized by arc length, then the circulation of $BS(V)$ around K_2 is given by

$$\begin{aligned}
\int_{K_2} BS(V) \cdot \frac{dy}{dt} dt &= \int_{K_2} \left(\int_{K_1} V(x) \times \nabla_y \varphi_0(x,y) ds \right) \cdot \frac{dy}{dt} dt \\
&= \int_{K_1 \times K_2} \frac{dx}{ds} \times \nabla_y \varphi_0(x,y) \cdot \frac{dy}{dt} ds dt \\
&= -\int_{K_1 \times K_2} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \nabla_y \varphi_0(x,y) ds dt.
\end{aligned}$$

By Ampere's Law, since $|V(x)| = 1$, this should be equal to $Lk(K_1, K_2)$, which it is.

This approach generalizes and is the one we will use to get the following linking integrals. Note, though, that in Gauss' formula, $\frac{dx}{ds}$ and $\frac{dy}{dt}$ lie in different tangent spaces; this isn't a problem in $\mathbb{R}^3$, but in other spaces we must be careful about how we carry one to the other in order to take their cross product. In S^3, we can do this either by left-translation or by parallel transport, whereas in H^3 we can only do this by parallel transport. Parallel transport has the advantages of being much more general and of giving formulas quite similar to that in $\mathbb{R}^3$; left-translation yields nastier integrals but is much easier to prove.

Theorem 2.1. (1) *On S^3 in left-translation format:*

$$\begin{aligned}
Lk(K_1, K_2) &= \int_{K_1 \times K_2} L_{yx^{-1}} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \nabla_y \varphi(x,y) ds dt \\
&\quad - \frac{1}{4\pi^2} \int_{K_1 \times K_2} L_{yx^{-1}} \frac{dx}{ds} \cdot \frac{dy}{dt} ds dt,
\end{aligned}$$

where $\varphi(\alpha) = \frac{1}{4\pi^2}(\pi - \alpha)\cot\alpha = -\varphi_0(\alpha)$ and $\Delta\varphi_0 = \delta - [\delta] = \delta - \frac{1}{2\pi^2}$.

(2) *On S^3 in parallel transport format:*

$$Lk(K_1, K_2) = \int_{K_1 \times K_2} P_{yx} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \nabla_y \varphi(x,y) ds dt,$$

where $\varphi(\alpha) = \frac{1}{4\pi^2}(\pi - \alpha)\csc\alpha = -\varphi_0(\alpha)$ where $\Delta\varphi_0 - \varphi_0 = \delta$.

(3) *On H^3 in parallel transport format:*

$$Lk(K_1, K_2) = \int_{K_1 \times K_2} P_{yx} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \nabla_y \varphi(x,y) ds dt,$$

where $\varphi(x,y) = \frac{1}{4\pi}\operatorname{csch}\alpha = -\varphi_0(\alpha)$ and $\Delta\varphi_0 + \varphi_0 = \delta$.

3. Magnetic fields

If we start with a smooth, compactly supported V, then its magnetic field $BS(V)$, given above, has the following properties:

(1) It is divergence-free: $\nabla \cdot BS(V) = 0$.

(2) It satisfies Maxwell's equation

$$\nabla_y \times BS(V)(y) = V(y) + \nabla_y \int_{\mathbb{R}^3} V(x) \cdot \nabla_x \varphi_0(x, y) dx,$$

where φ_0 is the fundamental solution of the Laplacian. Note that, using integration by parts, we can re-write the right hand side as $V(y) - \nabla_y \int_{\mathbb{R}^3} (\nabla_x \cdot V(x)) \varphi_0(x, y) dx$. When we think of $V(x)$ as a steady current, its negative divergence is the time rate of accumulation of charge at x, so the integral here is the time rate of increase of electric field E at y, so this is just Maxwell's equation $\nabla \times B = V + \frac{\partial E}{\partial t}$.

(3) $BS(v)(y) \to 0$ as $y \to \infty$.

Ampere's Law states that given a divergence-free current flow, the circulation of the resulting magnetic field around a loop is equal to the flux of the current through any surface bounded by that loop. This is an immediate consequence of (2) above, since if the current flow V is divergence-free, then (2) says that $\nabla \times BS(V) = V$ and then Ampere's Law is just the curl theorem.

Now, in $\mathbb{R}^3$, conditions (1), (2) and (3) completely characterize the Biot-Savart operator. In S^3, (1) and (2) alone suffice to characterize the Biot-Savart operator, since there are no non-zero vector fields on S^3 which are simultaneously divergence-free and curl-free. In H^3, it's not yet clear how to characterize the Biot-Savart operator; even strengthening (3) to require that $BS(V)(y)$ goes to zero exponentially fast is not enough.

Our goal, therefore, is to find an operator which satisfies (1), (2) and (3), stated here:

Theorem 3.1. *Biot-Savart operators exist in S^3 and H^3 and are given by the following formulas, where V is a smooth, compactly supported vector field:*

(1) *On S^3 in parallel transport format:*

$$BS(V)(y) = \int_{S^3} L_{yx^{-1}} V(x) \times \nabla_y \varphi_0(x, y) dx - \frac{1}{4\pi^2} \int_{S^3} L_{yx^{-1}} V(x) dx + 2\nabla_y \int_{S^3} L_{yx^{-1}} V(x) \cdot \nabla_y \varphi_1(x, y) dx,$$

where $\varphi_0(\alpha) = \frac{-1}{4\pi^2}(\pi - \alpha) \cot \alpha$ *and* $\varphi_1(\alpha) = \frac{-1}{16\pi^2} \alpha(2\pi - \alpha)$.

(2) *On S^3 in parallel transport format:*

$$BS(V)(y) = \int_{S^3} P_{yx} V(x) \times \nabla_y \varphi_0(x, y) dx,$$

where $\varphi_0(\alpha) = \frac{-1}{4\pi^2}(\pi - \alpha) \csc \alpha$.

(3) *On H^3 in parallel transport format:*

$$BS(V)(y) = \int_{H^3} P_{yx} V(x) \times \nabla_y \varphi_0(x, y) dx,$$

where $\varphi_0(\alpha) = \frac{-1}{4\pi} \text{csch} \alpha$.

4. The Green's operator

One way to find the Biot-Savart operator is to first figure out the Green's operator (defined as the inverse of the Laplacian) and then

$$BS(V) = -\nabla \times Gr(V)$$

is easily seen to satisfy the 2 properties of the Biot-Savart operator required in the previous section. Hence, it will be useful to determine the Green's operator on S^3 and H^3.

The **vector Laplacian** on $\mathbb{R}^3$, S^3 or H^3 is defined by

$$\Delta V = -\nabla \times \nabla \times V + \nabla(\nabla \cdot V),$$

which corresponds to the Laplacian on forms:

$$\Delta\beta = (d\delta + \delta d)\beta.$$

In $\mathbb{R}^3$, the **Green's operator** for a smooth, compactly supported vector field V is given by

$$Gr(V)(y) = \int_{\mathbb{R}^3} V(x)\varphi_0(x,y)dx,$$

where $\varphi_0(\alpha) = \frac{-1}{4\pi\alpha}$. It has the properties:

$$(1)\Delta Gr(V) = V \quad \text{and} \quad (2) Gr(V)(y) \to 0 \text{ as } y \to \infty.$$

In $\mathbb{R}^3$, S^3 and H^3, an operator satisfying (1) and (2) will be referred to as a **vector-valued Green's operator**. In $\mathbb{R}^3$, (1) and (2) characterize the Green's operator, and in S^3 (1) characterizes it. It is not entirly clear how to characterize the Green's operator in H^3.

Theorem 4.1. *Vector-valued Green's operators exist in S^3 and H^3, and are given by the following formulas, in which V is a smooth, compactly supported vector field:*

(1) *On S^3 in left-translation format:*

$$Gr(V)(y) = \int_{S^3} L_{yx^{-1}}V(x)\varphi_0(x,y)dx + 2\int_{S^3} L_{yx^{-1}}V(x)\times\nabla_y\varphi_1(x,y)dx + 4\nabla_y\int_{S^3} L_{yx^{-1}}V(x)\cdot\nabla_y\varphi_2(x,y)$$

where

$$\begin{aligned}
\varphi_0(\alpha) &= \frac{-1}{4\pi^2}(\pi-\alpha)\cot\alpha \\
\varphi_1(\alpha) &= \frac{-1}{16\pi^2}\alpha(2\pi-\alpha) \\
\varphi_2(\alpha) &= \frac{-1}{192\pi^2}(3\alpha(2\pi-\alpha) + 2\alpha(\pi-\alpha)(2\pi-\alpha)\cot\alpha),
\end{aligned}$$

and these three kernel functions satisfy

$$\varphi_2 \xrightarrow{\Delta} \varphi_1 - [\varphi_1] \xrightarrow{\Delta} \varphi_0 - [\varphi_0] \xrightarrow{\Delta} \delta - [\delta] \tag{1}$$

(2) *On S^3 in parallel transport format:*

$$Gr(V)(y) = \int_{S^3} P_{yx}V(x)\varphi_2(x,y)dx + \nabla_y \int_{S^3} P_{yx}V(x)\cdot\nabla_y\varphi_3(x,y)dx,$$

where

$$\varphi_2(\alpha) = \frac{-1}{4\pi^2}(\pi-\alpha)\csc\alpha + \frac{-1}{8\pi^2}\frac{(\pi-\alpha)^2}{1+\cos\alpha}$$

and

$$\varphi_3(\alpha) = \frac{-1}{24}(\pi-\alpha)\cot\alpha - \frac{1}{16\pi^2}\alpha(2\pi-\alpha) + \frac{1}{8\pi^2}\int_\alpha^\pi \left(\frac{(\pi-\alpha)^3}{3\sin^2\alpha} + \frac{(\pi-\alpha)^2}{\sin\alpha}\right)d\alpha.$$

(3) *On H^3 in parallel transport format:*

$$Gr(V)(y) = \int_{H^3} P_{yx}V(x)\varphi_2(x,y)dx + \nabla_y \int_{H^3} P_{yx}V(x)\cdot\nabla_y\varphi_3(x,y)dx,$$

where

$$\varphi_2(\alpha) = \frac{-1}{4\pi}\text{csch}\alpha + \frac{1}{4\pi}\frac{\alpha}{1+\text{cosh}\alpha}$$

and

$$\varphi_3(\alpha) = \frac{1}{4\pi}\frac{\alpha}{e^{2\alpha}-1} + \frac{1}{4\pi}\int_0^\alpha \left(\frac{\alpha}{\text{sinh}\alpha} - \frac{\alpha^2}{2\text{sinh}^2\alpha}\right)d\alpha.$$

5. Proofs on S^3 in left-translation format

First, we prove formula (1) of Theorem 4.1. We might hope, by analogy with $\mathbb{R}^3$, that the Green's operator on S^3 would be

$$A(V,\varphi_0) = \int_{S^3} L_{yx^{-1}}V(x)\varphi_0(x,y)dx$$

but this is false. However, the fundamental solution of the Laplacian φ_0 is not the only possible kernel function and $A(v,\varphi)$ is not the only way of convolving a vector field with a kernel. Consider the following convolutions:

$$\begin{aligned}
A(V,\varphi)(y) &= \int_{S^3} L_{yx^{-1}}V(x)\varphi(x,y)dx \\
B(V,\varphi)(y) &= \int_{S^3} L_{yx^{-1}}V(x)\times\nabla_y\varphi(x,y)dx \\
g(V,\varphi)(y) &= \int_{S^3} L_{yx^{-1}}V(x)\cdot\nabla_y\varphi(x,y)dx \\
G(V,\varphi)(y) &= \nabla_y g(V,\varphi)(y) = \nabla_y\int_{S^3} L_{yx^{-1}}V(x)\cdot\nabla_y\varphi(x,y)dx.
\end{aligned}$$

Proposition 5.1. *On S^3, the vector Laplacian has the following effect on the convolutions above:*

$$\begin{aligned}
\Delta A(V,\varphi) &= A(V,\Delta\varphi) - 4A(V,\varphi) - 2B(V,\varphi) \\
\Delta B(V,\varphi) &= B(V,\Delta\varphi) + 2A(V,\Delta\varphi) - 2G(V,\varphi) \\
\Delta G(V,\varphi) &= G(V,\Delta\varphi).
\end{aligned}$$

To prove this, we need the following lemmas:

Lemma 5.2. *Left- and right-invariant vector fields on S^3 are curl eigenfields with eigenvalues -2 and +2, respectively.*

The proof of this is completely straightforward.

Lemma 5.3.

$$\begin{aligned}
\nabla \times A(V,\varphi) &= -B(V,\varphi) - 2A(V,\varphi) \\
\nabla \cdot A(V,\varphi) &= g(V,\varphi) \\
\nabla \times B(V,\varphi) &= A(V,\Delta\varphi) - G(V,\varphi) \\
\nabla \cdot B(V,\varphi) &= -2g(V,\varphi).
\end{aligned}$$

Proving this is an exercise in differentiating under the integral sign and making use of Lemma 5.2.

With Lemma 5.3 in hand, we prove one case of Proposition 5.1 and leave the rest as exercises:

Proof.

$$\begin{aligned}
\Delta A(V,\varphi) &= -\nabla \times \nabla \times A(V,\varphi) + \nabla(\nabla \cdot A(V,\varphi)) \\
&= -\nabla \times (-B(V,\varphi) - 2A(V,\varphi)) + \nabla g(V,\varphi) \\
&= \nabla \times B(V,\varphi) + 2\nabla \times A(V,\varphi) + G(V,\varphi) \\
&= A(V,\Delta\varphi) - G(V,\varphi) - 2B(V,\varphi) + G(V,\varphi) \\
&= A(V,\Delta\varphi) - 4A(V,\varphi) - 2B(V,\varphi).
\end{aligned}$$

□

Now, proving Theorem 4.1 (1) boils down to showing that

$$Gr(V)(y) = A(V,\varphi_0)(y) + 2B(V,\varphi_1)(y) + 4G(V,\varphi_2)(y).$$

However, this is straightforward: just make use of the relationship (1) and note that, e.g., $B(V,\varphi_0) = B(V,\varphi_0 - [\varphi_0])$ since we're taking gradients in the definition of B (and similarly we can replace $\varphi_1 - [\varphi_1]$ for φ_1 in G). Thus,

$$\begin{aligned}
\Delta Gr(V) &= \Delta A(V,\varphi_0) + 2\Delta B(V,\varphi_1) + 4\Delta G(V,\varphi_2) \\
&= A(V,\Delta\varphi_0) - 4A(V,\varphi_0) + 4A(V,\Delta\varphi_1) \\
&= A(V,\delta - [\delta]) - 4A(V,\varphi_0) + 4A(V,\varphi_0 - [\varphi_0]) \\
&= A(V,\delta) - A(V,[\delta]) - 4A(V,[\varphi_0]).
\end{aligned}$$

Since $[\delta] = \frac{1}{2\pi^2}$ and $[\varphi_0] = \frac{-1}{8\pi^2}$, the last two terms cancel and so we get

$$\Delta Gr(V) = A(V,\delta) = V.$$

Now, to prove Theorem 3.1, we define $BS(V)$ by

$$BS(V) = -\nabla \times Gr(V).$$

This satisfies the defining characterics of the Biot-Savart operator. To see that is satisfies $\nabla \cdot BS(V) = 0$, note that the divergence of a curl is always zero. The second property is just a direct computation. Now,

$$\begin{aligned}
BS(V) &= -\nabla \times Gr(V) \\
&= -\nabla \times A(V, \varphi_0) - 2\nabla \times B(V, \varphi_1) - 4\nabla \times G(V, \varphi_2) \\
&= B(V, \varphi_0) + 2A(V, \varphi_0) - 2A(V, \Delta\varphi_1) + 2G(V, \varphi_1) \\
&= B(V, \varphi_0) + 2A(V, \varphi_0) - 2A(V, \varphi_0 - [\varphi_0]) + 2G(V, \varphi_1) \\
&= B(V, \varphi_0) + 2A(V, [\varphi_0]) + 2G(V, \varphi_1) \\
&= \int_{S^3} L_{yx^{-1}} V(x) \times \nabla_y \varphi_0(x, y) - \frac{1}{4\pi^2} \int_{S^3} L_{yx^{-1}} V(x) dx \\
&\quad + 2\nabla_y \int_{S^3} L_{yx^{-1}} V(x) \cdot \nabla_y \varphi_1(x, y) dx,
\end{aligned}$$

as claimed.

Now we're ready to prove the linking integral on S^3 in parallel transport format. Well, almost: there is one niggling issue. We actually only proved Theorem 3.1 (and, thus, Ampere's Law) in the case that V is smooth and compactly supported, but this is not true of current running through a wire. Nonetheless, we can think of the current in a wire as the limit of smooth current distributions supported in a tubular neighborhood of the wire and it turns out that the Ampere's Law we want is just the limit of the Ampere's Laws we already know.

With that in mind, we now tackle the proof of Theorem 2.1:

Proof. **(of Theorem 2.1)** Let $K_1 = \{x(s)\}$ and $K_2 = \{y(t)\}$ be disjoint oriented smooth closed curves in S^3. For the moment, assume K_1 is simple and that its parametrization $x = x(s)$ is by arc length, so that $V(x) = \frac{dx}{ds}$ is a unit vector. We think of V as a current flow along K_1. Then

$$\begin{aligned}
BS(V)(y) &= \int_{K_1} L_{yx^{-1}} V(x) \times \nabla_y \varphi_0(x, y) ds - \frac{1}{4\pi^2} \int_{K_1} L_{yx^{-1}} V(x) ds \\
&\quad + 2\nabla_y \int_{K_1} L_{yx^{-1}} V(x) \cdot \nabla_y \varphi_1(x, y) ds.
\end{aligned}$$

Note that

$$\begin{aligned}
L_{yx^{-1}} V(x) \cdot \nabla_y \varphi_1(x, y) &= -V(x) \cdot \nabla_x \varphi_1(x, y) \\
&= -\frac{dx}{ds} \cdot \nabla_x \varphi_1(x(s), y) \\
&= -\frac{d}{ds} \varphi_1(x(s), y),
\end{aligned}$$

whose integral around K_1 is zero. Thus we can drop the third term in the above. Now, the circulation of $BS(V)$ around K_2 will be as follows:

$$\begin{aligned}
\int_{K_2} BS(V)(y)\cdot\frac{dy}{dt}dt &= \int_{K_2}\left(\int_{K_1} L_{yx^{-1}}V(x)\times\nabla_y\varphi_0(x,y)ds - \frac{1}{4\pi^2}\int_{K_1} L_{yx^{-1}}V(x)ds\right)\cdot\frac{dy}{dt}dt\\
&= \int_{K_1\times K_2} L_{yx^{-1}}\frac{dx}{ds}\times\nabla_y\varphi_0(x,y)\cdot\frac{dy}{dt}dsdt - \frac{1}{4\pi^2}\int_{K_1\times K_2} L_{yx^{-1}}\frac{dx}{ds}\cdot\frac{dy}{dt}dsdt\\
&= -\int_{K_1\times K_2} L_{yx^{-1}}\frac{dx}{ds}\times\frac{dy}{dt}\cdot\nabla_y\varphi_0(x,y)dsdt - \frac{-1}{4\pi^2}\int_{K_1\times K_2} L_{yx^{-1}}\frac{dx}{ds}\cdot\frac{dy}{dt}dsdt\\
&= \int_{K_1\times K_2} L_{yx^{-1}}\frac{dx}{ds}\times\frac{dy}{dt}\cdot\nabla_y\varphi(x,y)dsdt - \frac{-1}{4\pi^2}\int_{K_1\times K_2} L_{yx^{-1}}\frac{dx}{ds}\cdot\frac{dy}{dt}dsdt
\end{aligned}$$

where $\varphi(x,y) = \frac{1}{4\pi^2}(\pi-\alpha)\cot\alpha$.

Applying Ampere's Law yields, since $|V| = 1$,

$$Lk(K_1,K_2) = \int_{K_1\times K_2} L_{yx^{-1}}\frac{dx}{ds}\times\frac{dy}{dt}\cdot\nabla_y\varphi(x,y)dsdt - \frac{-1}{4\pi^2}\int_{K_1\times K_2} L_{yx^{-1}}\frac{dx}{ds}\cdot\frac{dy}{dt}dsdt.$$

We can now drop the assumption that K_1 is simple, since the linking number does not change if a curve crosses itself. Moreover, we can also drop the assumption that K_1 is parametrized by arc length, since the above integrals are independent of parametrization of either curve, provided those parametrizations respect the given orientation. This completes the proof of Theorem 2.1 formula (1). □

6. Proofs in parallel transport format

The proofs on S^3 and H^3 in parallel transport format are somewhat different. We still want to get an explicit formula for the Biot-Savart operator and then the linking number will follow as before. However, rather than computing the Green's operator first and using that to solve for $BS(V)$, we actually determine $BS(V)$ directly.

The proofs in S^3 and H^3 are essentially same, because we will consider the hyperboloid model of H^3; that is, we consider $\mathbb{R}^{1,3}$ with the indefinite inner product

$$\langle x,y\rangle = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$$

and then

$$H^3 = \{x\in\mathbb{R}^{1,3} | \langle x,x\rangle = 1 \text{ and } x_0 > 0\}.$$

Using this model, the proofs in H^3 are the same as those in S^3 except using hyperbolic trig functions.

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