Homotopy periods of link maps and Milnor's invariants

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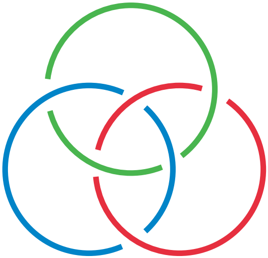
Joint work with Frederick R. Cohen and Rafal Komendarczyk

Links



Consider a *parametrized* n-component link $L_1, L_2, L_3, \ldots, L_n$, where

$$L_i : S^1 \to \mathbb{R}^3$$

have disjoint images.

Given an n-component link $L = \{L_1, \ldots, L_n\}$, there is a natural *evaluation map*

$$F_L : \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}} \longrightarrow \mathrm{Conf}(n)$$

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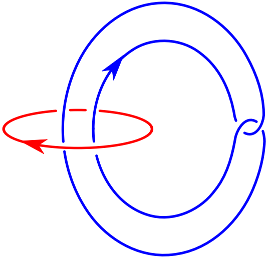
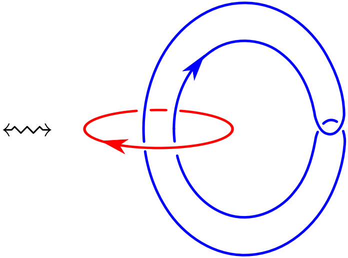
Here $\mathrm{Conf}(n)$ is the configuration space of n distinct points in $\mathbb{R}^3$:

$$\mathrm{Conf}(n) = \mathrm{Conf}_3\mathbb{R}^3 = \{(x_1, \ldots, x_n) \in (\mathbb{R}^3)^n : x_i \neq x_j \text{ for } i \neq j\}.$$

Definition

A *link homotopy* of a link L is a deformation during which each component may cross itself, but distinct components must remain disjoint.





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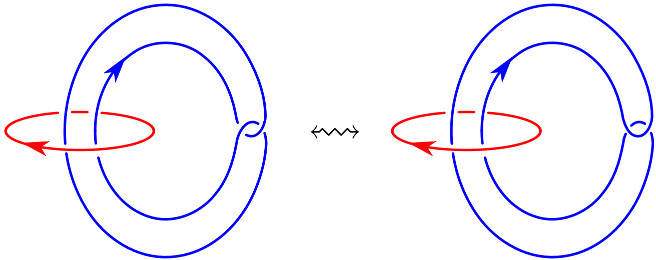

Let $LM(n)$ denote the set of n-component links up to link homotopy.

$$F_L : (S^1)^n \longrightarrow \mathrm{Conf}(n).$$

Link homotopies of L induce homotopies of F_L.

We get an induced map

$$\begin{array}{rcl} \kappa : LM(n) & \longrightarrow & [(S^1)^n, \mathrm{Conf}(n)] \\ L & \longmapsto & [F_L] \end{array}$$

$$\begin{array}{ccccc} S^1 \times S^1 & \hookrightarrow & \mathrm{Conf}_2\mathbb{R}^3 & \xrightarrow[\text{equiv.}]{\text{homotopy}} & S^2 \\ (t_1, t_2) & \longmapsto & (L_1(t_1), L_2(t_2)) & \longmapsto & \dfrac{L_1(t_1) - L_2(t_2)}{|L_1(t_1) - L_2(t_2)|} \end{array}$$

Theorem (Gauss)

The degree of the composition is equal to the linking number.

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Since the linking number classifies 2-component links up to link homotopy and the degree classifies maps $S^1 \times S^1 \to S^2$ up to homotopy, κ is bijective for $n = 2$.

$n = 3$

$$S^1 \times S^1 \times S^1 \hookrightarrow \mathrm{Conf}_3 S^3 \xrightarrow[\text{equiv.}]{\text{hom.}} S^3 \times S^2 \xrightarrow{\pi} S^2.$$

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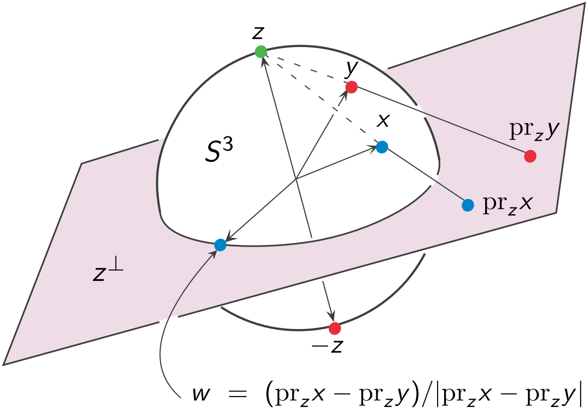

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Corollary

$\kappa : LM(3) \to [(S^1)^3, \mathrm{Conf}(3)]$ *is injective (though not bijective).*

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Theorem (Koschorke)

The restriction $\kappa : BLM(n) \to [(S^1)^n, \mathrm{Conf}(n)]$ *is injective.*

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Milnor defined invariants $\overline{\mu}(I; j)$, which are integers modulo the greatest common divisor of the lower-order invariants $\overline{\mu}(J; j)$ with J a proper subset of I. These are link homotopy invariants when the i_k are all distinct from each other and from j.

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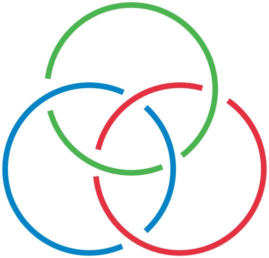
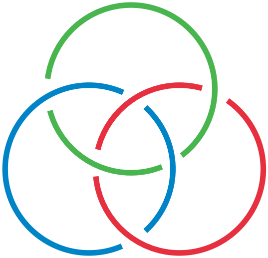
Theorem (Milnor)

Elements of $BLM(n)$ are completely classified by the $\overline{\mu}(I; j)$ with $|I| = n - 1$ and $I \cap \{j\} = \emptyset$. Moreover, all lower-order $\overline{\mu}$ invariants vanish for Brunnian links, so these invariants are integers.





- $$\overline{\mu}(i;j) = \mathrm{Lk}(L_i, L_j) = 0 \text{ for all } i \neq j.$$

- $$\overline{\mu}(1,2;3) = \overline{\mu}(2,3;1) = \overline{\mu}(3,1;2) = 1$$
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Let $\{B_{j,i} : 1 \leq i < j \leq n\}$ *be the generators of the Lie algebra* $\pi_*(\Omega\mathrm{Conf}(n)) \otimes \mathbb{Q}$.

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1. $\kappa(BLM(n)) \subset \pi_n(\mathrm{Conf}(n)) \subset [(S^1)^n, \mathrm{Conf}(n)]$ *and is a Lie* $\mathbb{Z}$*-module generated by the iterated Samelson products*

$$B(n, \sigma) = [B_{n,1}, B_{n,\sigma(2)}, \ldots, B_{n,\sigma(n-1)}]$$

for $\sigma \in \Sigma(2, \ldots, n-1)$*.*

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2. *For any* $L \in BLM(n)$,

$$\kappa([L]) = \sum_{\sigma \in \Sigma(2,\ldots,n-1)} \overline{\mu}(1, \sigma(2), \ldots, \sigma(n-1); n) B(n,\sigma)$$

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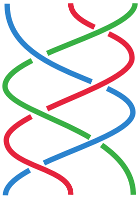
In other words, the homotopy periods of F_L are given by the $(n-2)!$ invariants $\{\overline{\mu}(1, \sigma(2), \ldots, \sigma(n-1); n)\}_\sigma$.

Habegger and Lin introduced the group of homotopy string links $\mathscr{H}(n)$.



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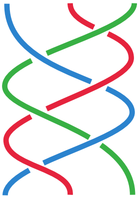

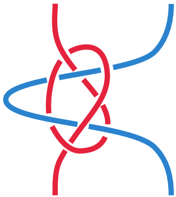

String Links

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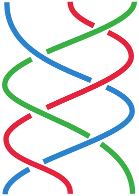

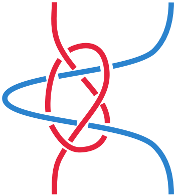

Theorem (Habegger–Lin)

The map $\mathscr{H}(n) \to LM(n)$ induced by the Markov closure is surjective. Moreover, if two string links close up to link-homotopic links, then they are related by a sequence of conjugations and "partial conjugations".

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Key Proposition 1

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Key Proposition 1

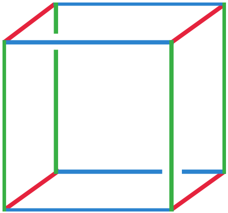
$B\mathscr{H}(n)$ is the center of $\mathscr{H}(n)$.

Corollary

The restriction of the Markov closure map $B\mathscr{H}(n) \to BLM(n)$ is bijective.

Let $T(n)$ be the nth torus homotopy group [Fox] of $\mathrm{Conf}(n)$:

$$T(n) = [\Sigma(S^1)^{n-1}, \mathrm{Conf}(n)] = [(S^1)^{n-1}, \Omega\mathrm{Conf}(n)].$$



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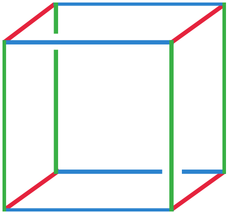

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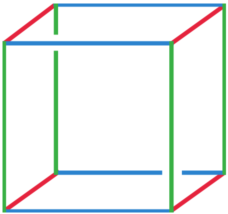

If $L \in BLM(n)$, then $L - L_n$ is link homotopically trivial, so the restriction of F_L to the face $(S^1)^{n-1} \times \{*\}$ is homotopic to the constant map.

Therefore, we can interpret $\kappa(L) = [F_L]$ as an element of $T(n)$.

Injectivity

Key Proposition 2

The diagram

$$\begin{array}{ccc} B\mathscr{H}(n) & \xrightarrow{\phi} & T(n) \\ \downarrow & & \downarrow p^{\#} \\ BLM(n) & \xrightarrow{\kappa} & [(S^1)^n, \mathrm{Conf}(n)] \end{array}$$

commutes and $p^{\#} \circ \phi$ *is injective. Therefore* κ *is injective as well.*

Thanks!