Rulings and augmentations for bordered Legendrian knots

Clayton Shonkwiler

Department of Mathematics
University of Georgia

September 25, 2011

Joint work with Joshua M. Sabloff and David Shea Vela-Vick

Legendrian knots

A *Legendrian knot* is an embedded S^1 in a contact manifold (M^3, ξ) which is tangent to the contact planes.

Legendrian knots

A *Legendrian knot* is an embedded S^1 in a contact manifold (M^3, ξ) which is tangent to the contact planes.

- Focus on the standard contact structure on $\mathbb{R}^3$, $\xi = \ker dz - ydx$.

Legendrian knots

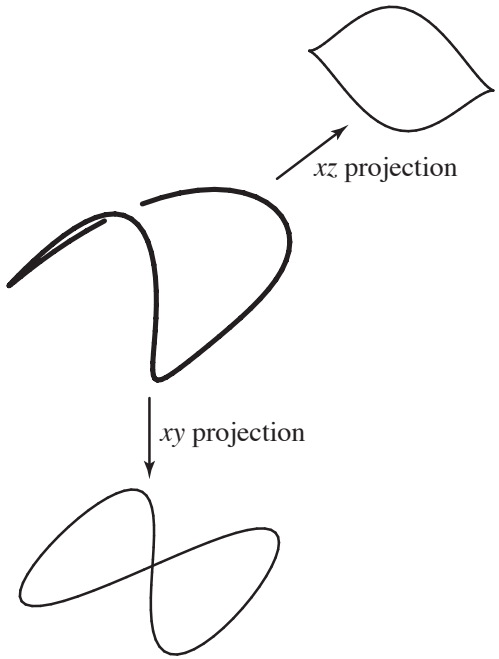
A *Legendrian knot* is an embedded S^1 in a contact manifold (M^3, ξ) which is tangent to the contact planes.

- Focus on the standard contact structure on $\mathbb{R}^3$, $\xi = \ker dz - ydx$.
- *Lagrangian* and *front* projections

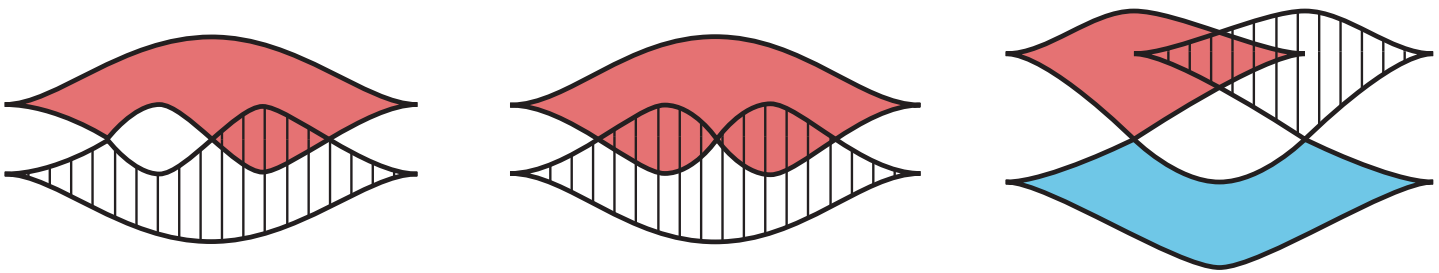
Legendrian knots

A *Legendrian knot* is an embedded S^1 in a contact manifold (M^3, ξ) which is tangent to the contact planes.

- Focus on the standard contact structure on $\mathbb{R}^3$, $\xi = \ker dz - ydx$.
- *Lagrangian* and *front* projections

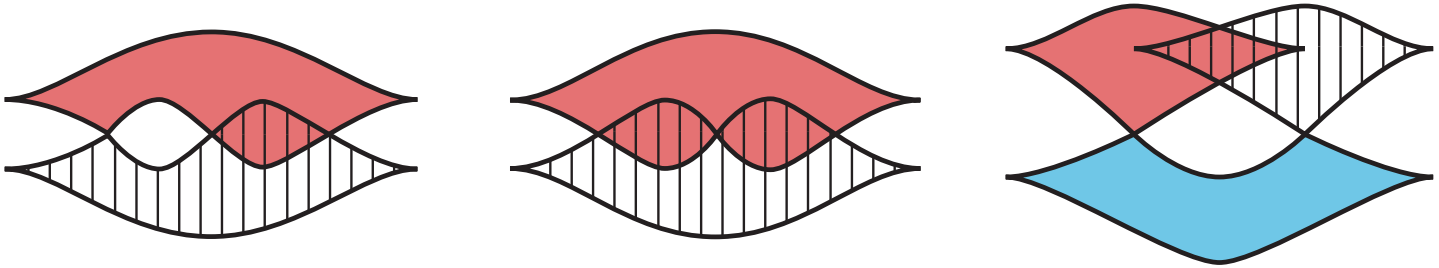


Rulings



A *ruling* of a front Σ is a one-to-one correspondence between left and right cusps along with paths joining the cusps such that

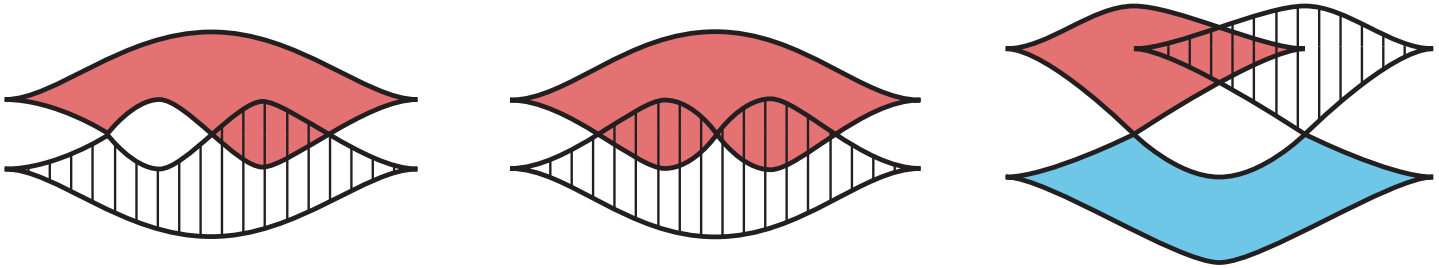
Rulings



A *ruling* of a front Σ is a one-to-one correspondence between left and right cusps along with paths joining the cusps such that

- The two paths corresponding to paired cusps meet *only* at the cusps.

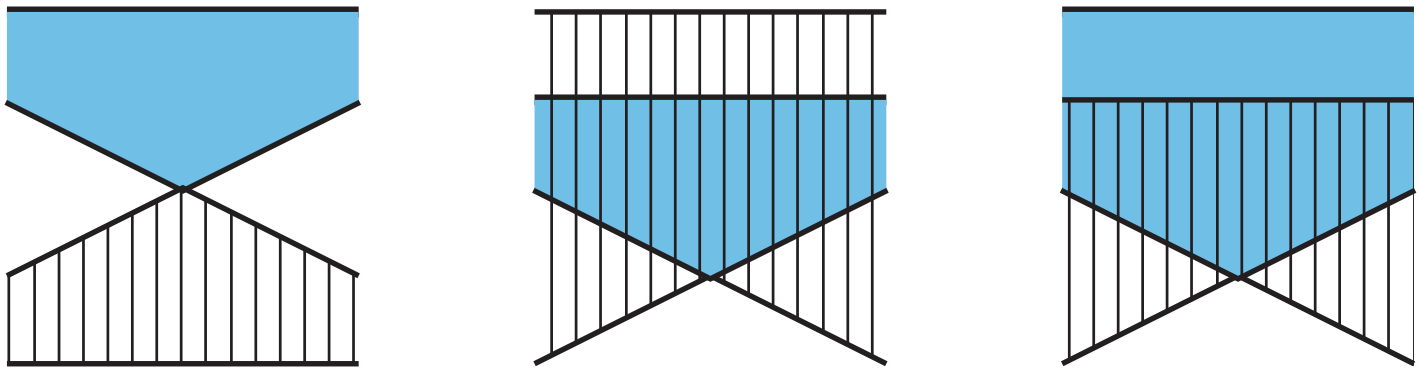
Rulings



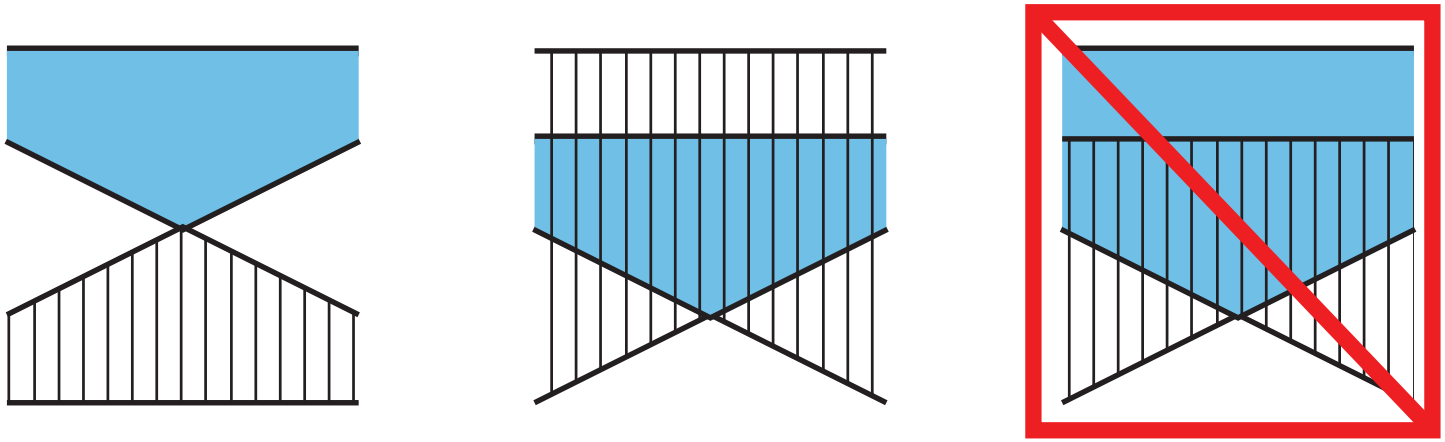
A *ruling* of a front Σ is a one-to-one correspondence between left and right cusps along with paths joining the cusps such that

- The two paths corresponding to paired cusps meet *only* at the cusps.
- Any two paths meet only at crossings or cusps.

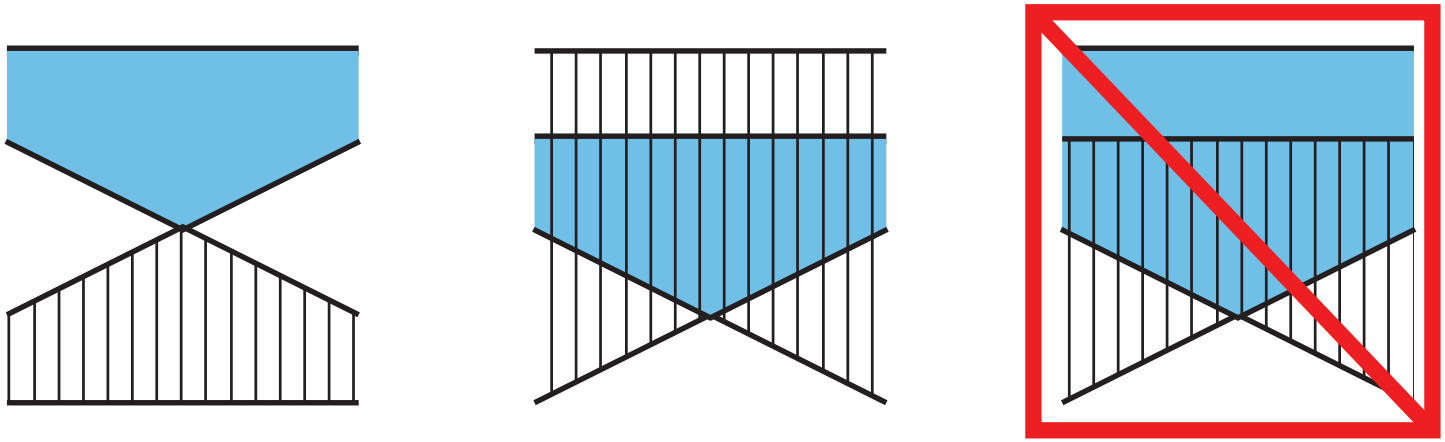
Only certain types of "switches" are legal.



Only certain types of "switches" are legal.



Only certain types of "switches" are legal.



Rulings with only the first two kinds of switches are called *normal rulings* of the front.

The Ruling Polynomial

Define

$$\theta(R) = \text{cusps}(\Sigma) - \text{switches}(R)$$
$$\Theta_\Sigma(k) = \{\text{rulings } R : \theta(R) = k\}$$

The Ruling Polynomial

Define

$$\theta(R) = \text{cusps}(\Sigma) - \text{switches}(R)$$
$$\Theta_\Sigma(k) = \{\text{rulings } R : \theta(R) = k\}$$

The *ruling polynomial* of Σ is:

$$R_K(z) = \sum \#\Theta_\Sigma(1-k)z^k.$$

Define

$$\theta(R) = \text{cusps}(\Sigma) - \text{switches}(R)$$
$$\Theta_\Sigma(k) = \{\text{rulings } R : \theta(R) = k\}$$

The *ruling polynomial* of Σ is:

$$R_K(z) = \sum \#\Theta_\Sigma(1-k)z^k.$$

Theorem (Chekanov–Pushkar)

The ruling polynomial is a Legendrian invariant.

Define

$$\theta(R) = \text{cusps}(\Sigma) - \text{switches}(R)$$
$$\Theta_\Sigma(k) = \{\text{rulings } R : \theta(R) = k\}$$

The *ruling polynomial* of Σ is:

$$R_K(z) = \sum \#\Theta_\Sigma(1-k)z^k.$$

Theorem (Chekanov–Pushkar)

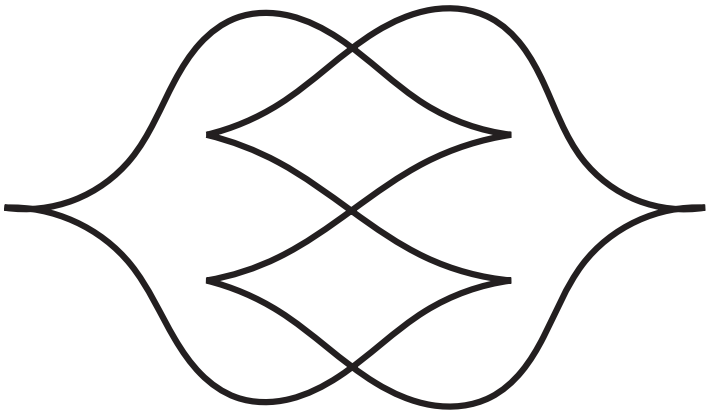
The ruling polynomial is a Legendrian invariant.

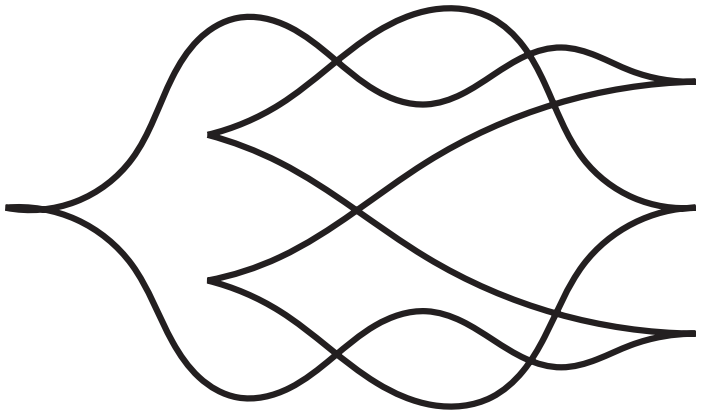
Theorem (Rutherford)

$R_K(z)$ is the coefficient of $a^{-tb(K)-1}$ in the Kauffman polynomial $F_K(a,z)$.

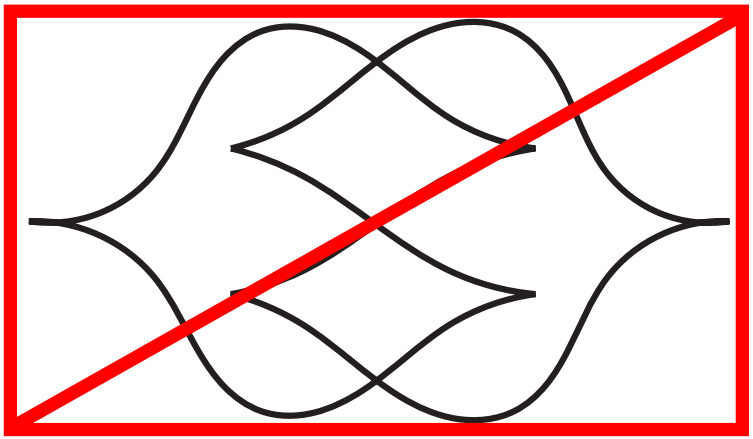
Simple Fronts

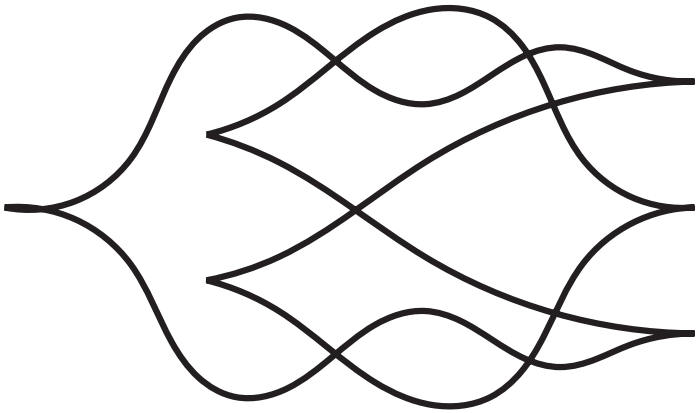
We will restrict attention to *simple* front diagrams.



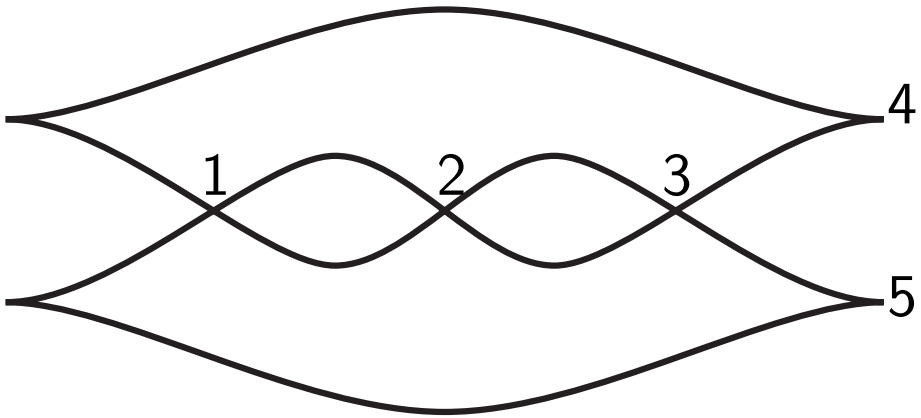


We will restrict attention to *simple* front diagrams.



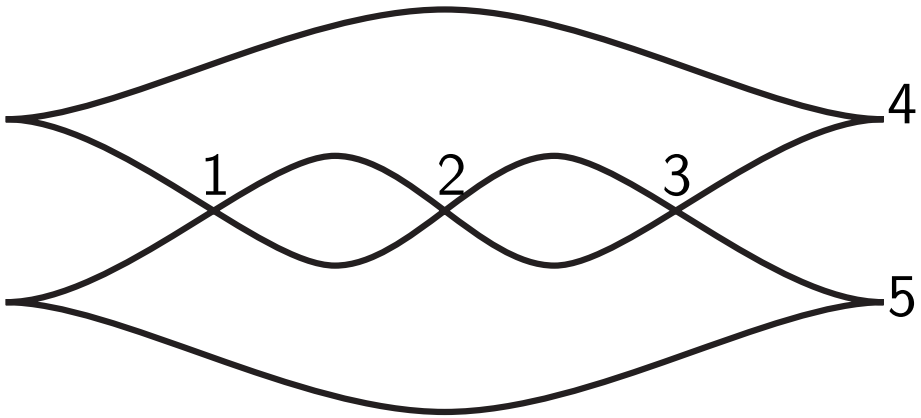


Label the crossings and right cusps of a simple front Σ with $\{1, \ldots, m\}$



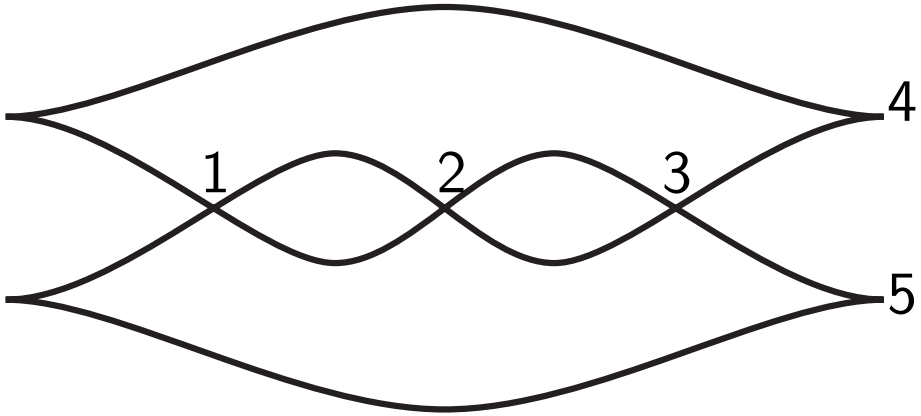
The Chekanov–Eliashberg DGA

Label the crossings and right cusps of a simple front Σ with $\{1, \ldots, m\}$

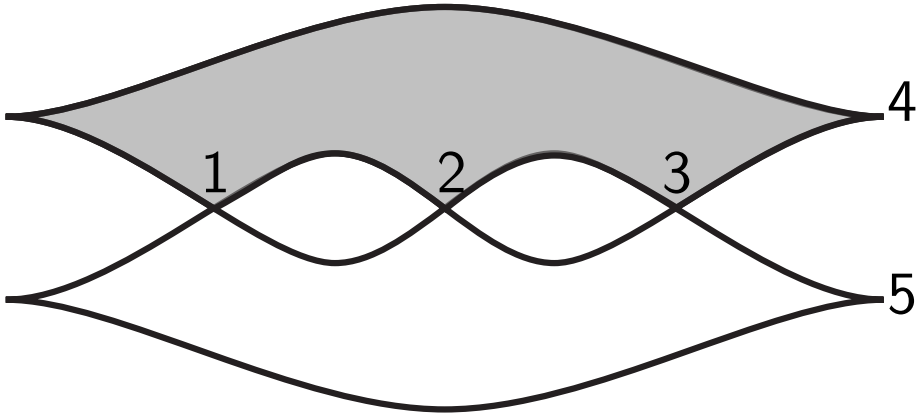


Let A be the $\mathbb{Z}_2$-vector space generated by $\{q_1, \ldots, q_m\}$, and let $\mathrm{Ch}(K) = \bigoplus_{k=0}^{\infty} A^{\otimes k}$ be the unital tensor algebra over A.

The Differential



The terms of ∂q_i are determined by immersed disks with convex corners whose boundary lie on the knot diagram with unique rightmost point at q_i and unique leftmost point at some left cusp.

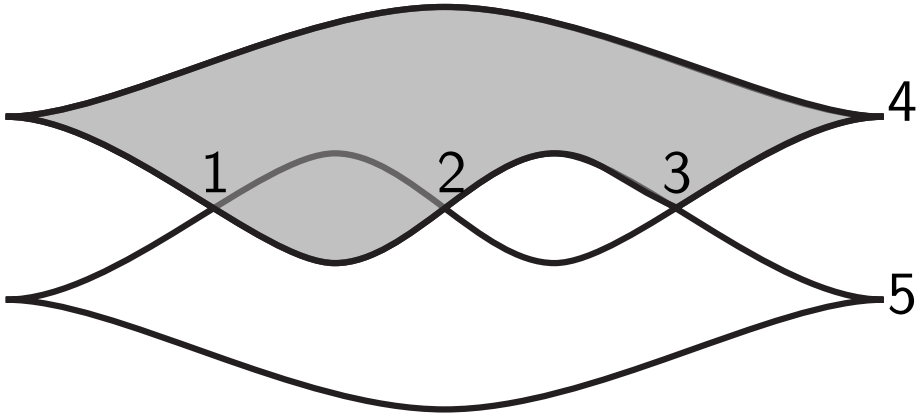


The terms of ∂q_i are determined by immersed disks with convex corners whose boundary lie on the knot diagram with unique rightmost point at q_i and unique leftmost point at some left cusp.

Record the non-q_i corners in counterclockwise order, adding 1 if q_i is a right cusp:

$$\partial q_4 = 1 + q_1 q_2 q_3$$

The Differential



The terms of ∂q_i are determined by immersed disks with convex corners whose boundary lie on the knot diagram with unique rightmost point at q_i and unique leftmost point at some left cusp.

Record the non-q_i corners in counterclockwise order, adding 1 if q_i is a right cusp:

$$\partial q_4 = 1 + q_1 q_2 q_3 + q_3 + \dots$$

Theorem (Chekanov)

$(Ch(K), \partial)$ is a DGA. Its "stable tame isomorphism" class is invariant under Legendrian isotopies.

Theorem (Chekanov)

$(Ch(K), \partial)$ is a DGA. Its "stable tame isomorphism" class is invariant under Legendrian isotopies.

Corollary

The homology of $(Ch(K), \partial)$, called Legendrian contact homology, is a Legendrian invariant.

Augmentations

An *augmentation* of $(\mathrm{Ch}(K), \partial)$ is an algebra map $\epsilon : \mathrm{Ch}(K) \to \mathbb{Z}_2$ such that:

1. $\epsilon(1) = 1$

Augmentations

An *augmentation* of $(\mathrm{Ch}(K), \partial)$ is an algebra map $\epsilon : \mathrm{Ch}(K) \to \mathbb{Z}_2$ such that:

1. $\epsilon(1) = 1$
2. $\epsilon \circ \partial = 0$

Augmentations

An *augmentation* of $(\mathrm{Ch}(K), \partial)$ is an algebra map $\epsilon : \mathrm{Ch}(K) \to \mathbb{Z}_2$ such that:

1. $\epsilon(1) = 1$
2. $\epsilon \circ \partial = 0$

Theorem (Fuchs, Fuchs–Ishkhanov, Sabloff)

The Chekanov–Eliashberg DGA of a Legendrian knot K has an augmentation if and only if a front diagram of K has a normal ruling.

An *augmentation* of $(\mathrm{Ch}(K), \partial)$ is an algebra map $\epsilon : \mathrm{Ch}(K) \to \mathbb{Z}_2$ such that:

1. $\epsilon(1) = 1$
2. $\epsilon \circ \partial = 0$

Theorem (Fuchs, Fuchs–Ishkhanov, Sabloff)

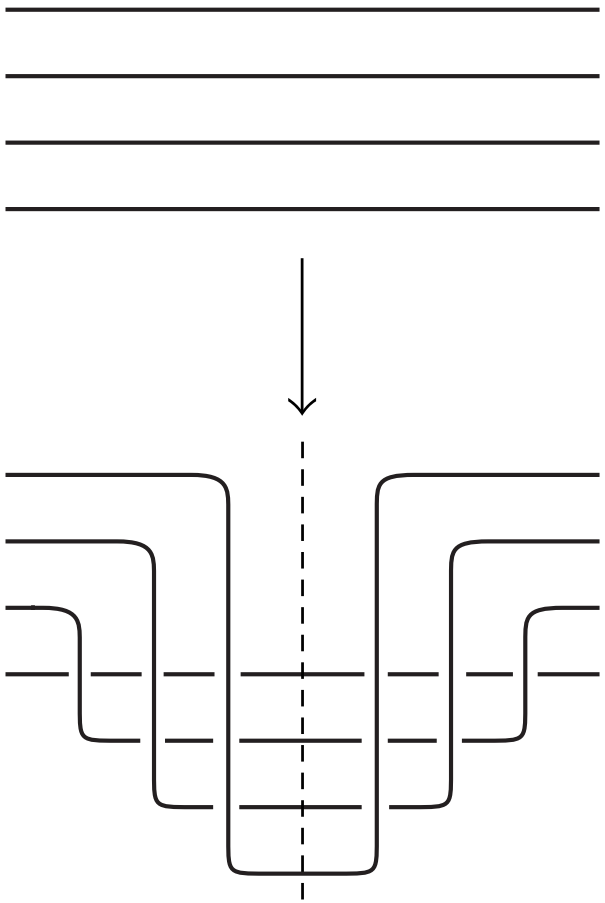
The Chekanov–Eliashberg DGA of a Legendrian knot K has an augmentation if and only if a front diagram of K has a normal ruling.

Theorem (Ng–Sabloff)

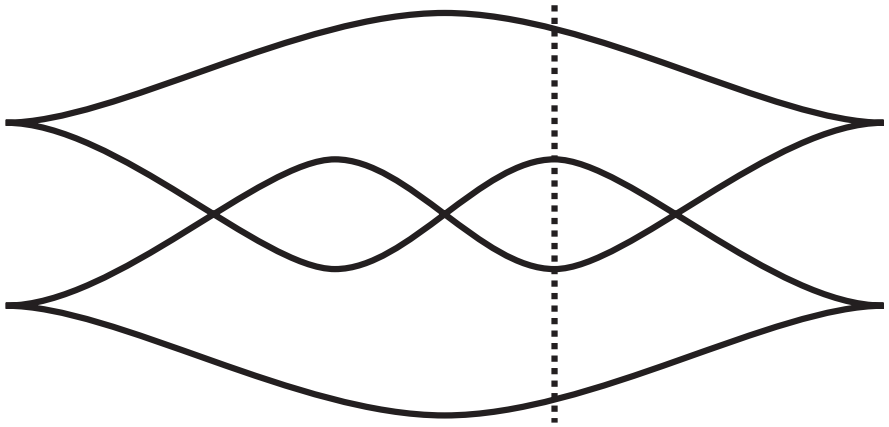
The "augmentation number" of K is given by $R_K(2^{-1/2})$.

Proof Technique

"Localize" the differential using dipped diagrams.

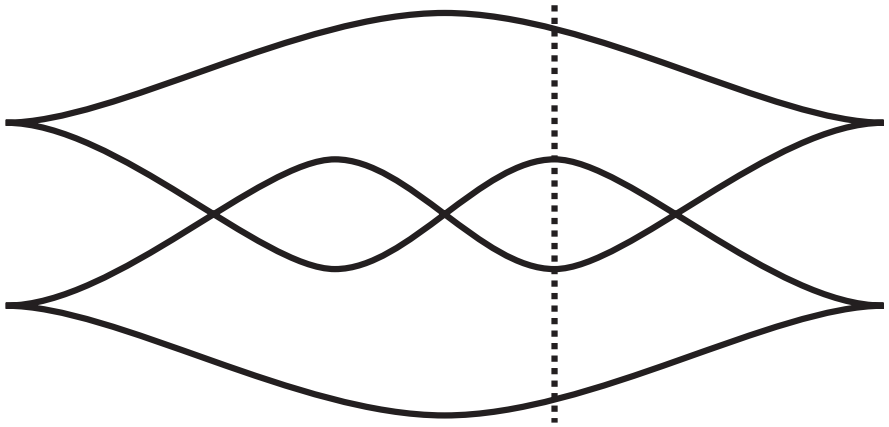


A Bordered Chekanov–Eliashberg DGA



Given a simple front Σ and a vertical line dividing it into a left half Σ^A and a right half Σ^D, Sivek defined a bordered version of the Chekanov–Eliashberg DGA.

A Bordered Chekanov–Eliashberg DGA



Given a simple front Σ and a vertical line dividing it into a left half Σ^A and a right half Σ^D, Sivek defined a bordered version of the Chekanov–Eliashberg DGA.

There are three ingredients: the interval algebra I_n, the A-algebra $A(\Sigma^A)$ and the D-algebra $D(\Sigma^D)$.

Interval Algebra and A-algebra

The interval algebra I_n is freely generated by

$$\{\rho_{ij} : 1 \leq i < j \leq n\}.$$

The differential is given by

$$\partial \rho_{ij} = \sum_{i<k<j} \rho_{ik}\rho_{kj}$$

Interval Algebra and A-algebra

The interval algebra I_n is freely generated by

$$\{\rho_{ij} : 1 \leq i < j \leq n\}.$$

The differential is given by

$$\partial \rho_{ij} = \sum_{i<k<j} \rho_{ik}\rho_{kj}$$

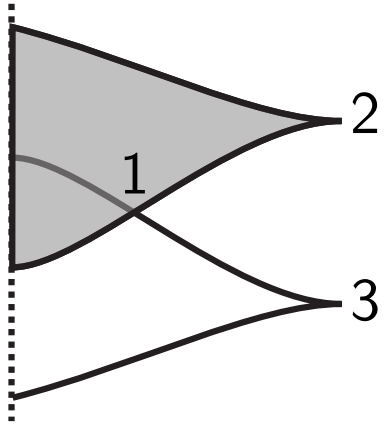
The A-algebra $A(\Sigma^A)$ is generated by the crossings and right cusps of Σ^A, with differential just as in the usual Chekanov–Eliashberg DGA.

The D-algebra

The D-algebra $D(\Sigma^D)$ is generated by the crossings and right cusps of Σ^D as well as the ρ_{ij}.

The D-algebra

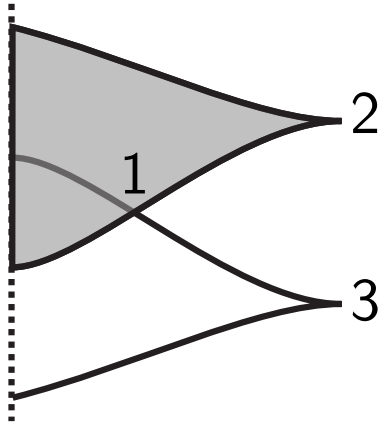
The D-algebra $D(\Sigma^D)$ is generated by the crossings and right cusps of Σ^D as well as the ρ_{ij}.



The differential is given by counting disks and *half-disks*.

The D-algebra

The D-algebra $D(\Sigma^D)$ is generated by the crossings and right cusps of Σ^D as well as the ρ_{ij}.

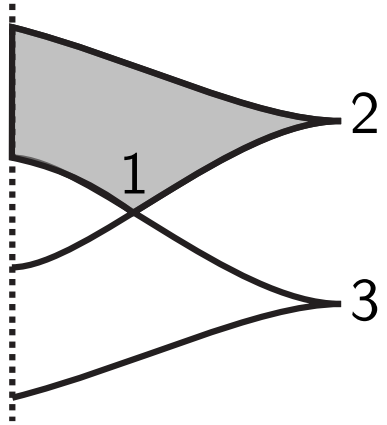


The differential is given by counting disks and *half-disks*.

$$\partial q_2 = 1 + \rho_{13}$$

The D-algebra

The D-algebra $D(\Sigma^D)$ is generated by the crossings and right cusps of Σ^D as well as the ρ_{ij}.



The differential is given by counting disks and *half-disks*.

$$\partial q_2 = 1 + \rho_{13} + \rho_{12} q_1 + \dots$$

Theorem (Sivek)

The commutative diagram

$$\begin{tikzcd} I_n \arrow[r] \arrow[d] & D(\Sigma^D) \arrow[d] \\ A(\Sigma^A) \arrow[r] & \mathrm{Ch}(K) \end{tikzcd}$$

is a pushout square in the category of DGAs.

Theorem (Sivek)

The commutative diagram

$$\begin{tikzcd} I_n \arrow[r] \arrow[d] & D(\Sigma^D) \arrow[d] \\ A(\Sigma^A) \arrow[r] & \mathrm{Ch}(K) \end{tikzcd}$$

is a pushout square in the category of DGAs.

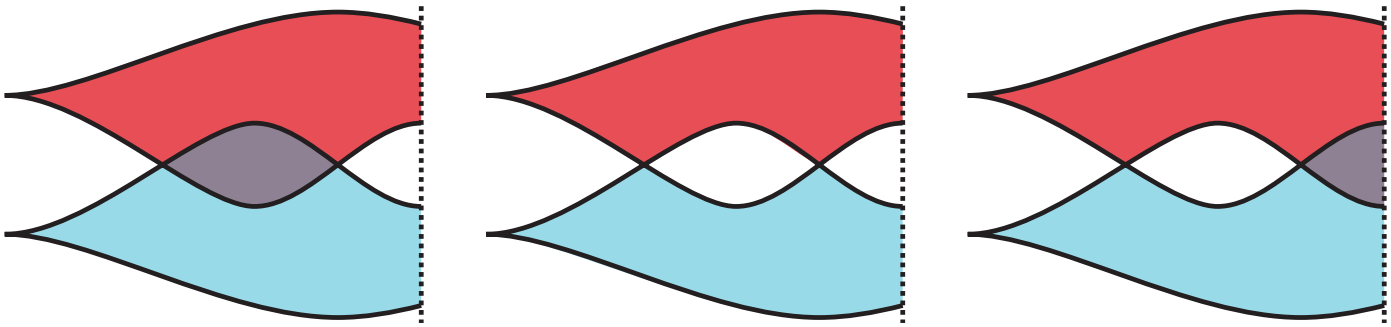
Corollary (Sivek)

An augmentation $\epsilon : \mathrm{Ch}(K) \to \mathbb{Z}_2$ *is equivalent to a commutative diagram*

$$\begin{tikzcd} I_n \arrow[r] \arrow[d] & D(\Sigma^D) \arrow[d, "\epsilon_D"] \\ A(\Sigma^A) \arrow[r, "\epsilon_A"] & \mathbb{Z}_2 \end{tikzcd}$$

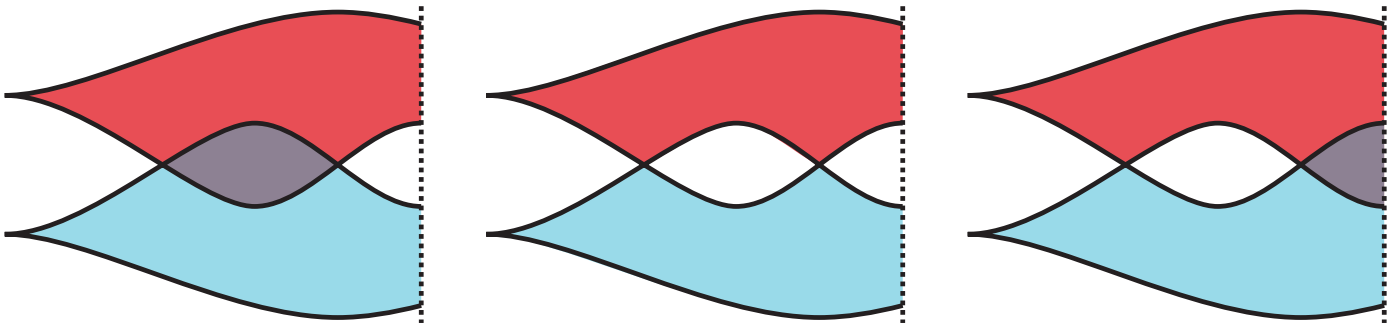
Definition

A bordered (normal) ruling for the left half-diagram Σ^A is a pairing between each left cusp of Σ^A and either a right cusp or a sub-segment of the dividing line (corresponding to some ρ_{ij}) satisfying the (normal) ruling conditions.



Definition

A bordered (normal) ruling for the left half-diagram Σ^A is a pairing between each left cusp of Σ^A and either a right cusp or a sub-segment of the dividing line (corresponding to some ρ_{ij}) satisfying the (normal) ruling conditions.



Similarly for bordered rulings of the right half-diagram Σ^D.

Theorem (with Sabloff and Vela-Vick)

The A-algebra $A(\Sigma^A)$ has an augmentation if and only if Σ^A has a bordered normal ruling.

Likewise, $D(\Sigma^D)$ has an augmentation if and only if Σ^D has a bordered normal ruling.

Theorem (with Sabloff and Vela-Vick)

The A-algebra $A(\Sigma^A)$ has an augmentation if and only if Σ^A has a bordered normal ruling.

Likewise, $D(\Sigma^D)$ has an augmentation if and only if Σ^D has a bordered normal ruling.

The proof is in the spirit of Henry's *Morse complex sequences*.

Theorem (with Sabloff and Vela-Vick)

If Σ is a front diagram for a Legendrian knot K with left half-diagram Σ^A and right half-diagram Σ^D, then

$$R_K = \langle R_{\Sigma^A} | R_{\Sigma^D} \rangle.$$

Thanks!