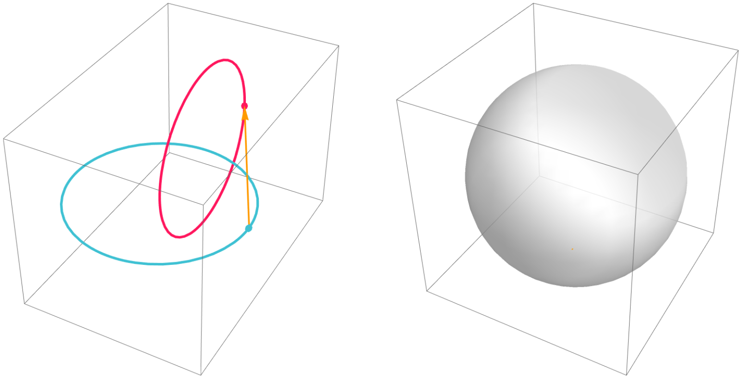
Homotopy String Links and the κ-Invariant

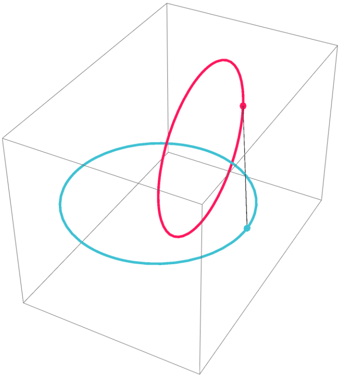
Clayton Shonkwiler

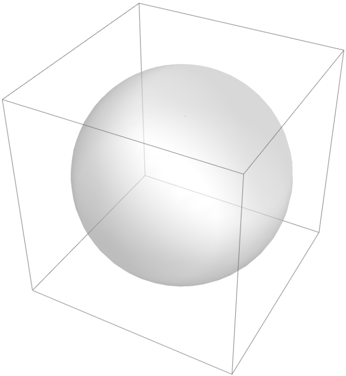
Colorado State University

Helicity in Venice
April 11, 2016



Linking number $=$ Degree





Linking number = Degree

Higher linking invariants $=$ Other homotopy invariants?

Higher linking invariants = Other homotopy invariants?
Higher helicities? (...maybe not...)

Theorem (with DeTurck, Gluck, Komendarczyk, Melvin, Nuchi, and Vela-Vick)

The triple linking number is the Hopf invariant of an associated map $S^1 \times S^1 \times S^1 \to S^2$.

Theorem (with DeTurck, Gluck, Komendarczyk, Melvin, Nuchi, and Vela-Vick)

The triple linking number is the Hopf invariant of an associated map $S^1 \times S^1 \times S^1 \to S^2$.

Definition

A parametrized link $L_1 \sqcup \ldots \sqcup L_n : S^1 \sqcup \ldots \sqcup S^1 \to \mathbb{R}^3$ has an *associated map*

$$L_1 \times \ldots \times L_n : S^1 \times \ldots \times S^1 \to \mathrm{Conf}(\mathbb{R}^3, n),$$

where $\mathrm{Conf}(\mathbb{R}^3, n)$ is the configuration space of n distinct points in $\mathbb{R}^3$.

Theorem (with DeTurck, Gluck, Komendarczyk, Melvin, Nuchi, and Vela-Vick)

The triple linking number is the Hopf invariant of an associated map $S^1 \times S^1 \times S^1 \to S^2$.

Definition

A parametrized link $L_1 \sqcup \ldots \sqcup L_n : S^1 \sqcup \ldots \sqcup S^1 \to \mathbb{R}^3$ has an *associated map*

$$L_1 \times \ldots \times L_n : S^1 \times \ldots \times S^1 \to \mathrm{Conf}(\mathbb{R}^3, n),$$

where $\mathrm{Conf}(\mathbb{R}^3, n)$ is the configuration space of n distinct points in $\mathbb{R}^3$.

$\mathrm{Conf}(\mathbb{R}^3, 3)$ is an $S^2 \vee S^2$ bundle over S^2.

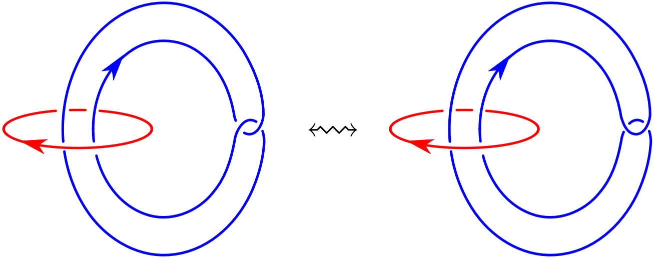
The κ-invariant

Define the map $\kappa : \mathrm{Link}(n) \to [\mathbb{T}^n, \mathrm{Conf}(\mathbb{R}^3, n)]$ by

$$\kappa([L]) = [L_1 \times \ldots \times L_n].$$

Define the map $\kappa : \mathrm{Link}(n) \to [\mathbb{T}^n, \mathrm{Conf}(\mathbb{R}^3, n)]$ by

$$\kappa([L]) = [L_1 \times \ldots \times L_n].$$



A link homotopy

Define the map $\kappa : \mathrm{Link}(n) \to [\mathbb{T}^n, \mathrm{Conf}(\mathbb{R}^3, n)]$ by

$$\kappa([L]) = [L_1 \times \ldots \times L_n].$$

Theorem (Koschorke; with Cohen and Komendarczyk)

The map κ separates Brunnian links.

The κ-invariant

Define the map $\kappa : \mathrm{Link}(n) \to [\mathbb{T}^n, \mathrm{Conf}(\mathbb{R}^3, n)]$ by

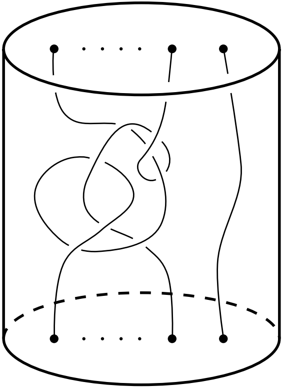
$$\kappa([L]) = [L_1 \times \ldots \times L_n].$$

Theorem (Koschorke; with Cohen and Komendarczyk)

The map κ separates Brunnian links.

Conjecture (Koschorke '97)

The map κ separates links. Hence, the homotopy periods of κ form a complete set of link-homotopy invariants.



A *string link* is a map $\sigma_1 \sqcup \ldots \sqcup \sigma_n : I_1 \sqcup \ldots \sqcup I_n \to \mathcal{C}$ satisfying appropriate boundary conditions.

A string link $\sigma_1 \sqcup \ldots \sqcup \sigma_n : I_1 \sqcup \ldots \sqcup I_n \to \mathcal{C}$ has an associated map

$$\sigma_1 \times \ldots \times \sigma_n : I^n \to \mathrm{Conf}(\mathcal{C}, n)$$

satisfying appropriate boundary conditions.

The String Link κ Map

A string link $\sigma_1 \sqcup \ldots \sqcup \sigma_n : I_1 \sqcup \ldots \sqcup I_n \to \mathcal{C}$ has an associated map

$$\sigma_1 \times \ldots \times \sigma_n : I^n \to \mathrm{Conf}(\mathcal{C}, n)$$

satisfying appropriate boundary conditions.

Define the map $\check{\kappa} : \mathrm{Link}(n) \to [I^n, \mathrm{Conf}(\mathcal{C}, n)]_\partial$ by

$$\check{\kappa}([\sigma]) = [\sigma_1 \times \ldots \times \sigma_n].$$

Our Theorem

$$
\begin{array}{ccc}
\mathcal{H}(n) & \xrightarrow{\quad \check{\kappa} \quad} & [I^n, \mathrm{Conf}(\mathcal{C}, n)]_\partial \\
\downarrow & & \downarrow \\
\mathrm{Link}(n) & \xrightarrow{\quad \kappa \quad} & [\mathbb{T}^n, \mathrm{Conf}(\mathbb{R}^3, n)]
\end{array}
$$

Theorem (with Cohen, Komendarczyk, and Koytcheff)

$\check{\kappa}$ *separates string links.*

Questions

- Is the map $\check{\kappa} \mapsto \kappa$ compatible with Habegger–Lin's classification of links up to link homotopy?
- Are there nice integral expressions for homotopy periods of κ?

Thank you for listening!

Thank you for listening!

- *Homotopy Brunnian links and the κ-invariant*
  Frederick R. Cohen, Rafał Komendarczyk, and Clayton Shonkwiler
  Proceedings of the American Mathematical Society **143** (2015), 1347–1362.

- *Homotopy string links and the κ-invariant*
  Frederick R. Cohen, Rafał Komendarczyk, Robin Koytcheff, and Clayton Shonkwiler
  `arXiv:1504.03233`

`http://shonkwiler.org`

Idea of the Proof

Idea of the Proof

$\mathcal{H}(n)$ is a group and $[I^n, \mathrm{Conf}(\mathcal{C}, n)]_\partial$ is a monoid with identity. Use induction, the fact that $\check{\kappa}$ is a monoid homomorphism, the commutative diagram:

$$\begin{array}{ccc}
\mathcal{BH}(n) & \xrightarrow{\check{\kappa}|_{\mathcal{BH}(n)}} & [I^n, \mathrm{Conf}(\mathcal{C}, n)]_\partial \\
\downarrow \iota & & \downarrow \jmath \\
\mathcal{H}(n) & \xrightarrow{\check{\kappa}} & [I^n, \mathrm{Conf}(\mathcal{C}, n)]_\partial \\
\downarrow \delta & & \downarrow \psi \\
\prod_{i=1}^{n} \mathcal{H}_i(n-1) & \xrightarrow{\check{\kappa}^n} & \prod_{i=1}^{n} [I_i^{n-1}, \mathrm{Conf}(\mathcal{C}, n-1)]_\partial
\end{array}$$