Poincaré Duality Angles on Riemannian Manifolds with Boundary

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Realizing cohomology groups as spaces of differential forms

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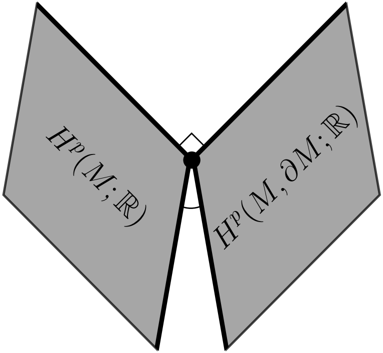

de Rham's Theorem

Suppose M^n is a compact, oriented, smooth manifold. Then

$$H^p(M;\mathbb{R}) \cong \mathcal{C}^p(M)/\mathcal{E}^p(M),$$

where $\mathcal{C}^p(M)$ is the space of closed p-forms on M and $\mathcal{E}^p(M)$ is the space of exact p-forms.

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$$\langle \omega, \eta \rangle := \int_M \omega \wedge \star \eta.$$

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Kodaira called this the space of *harmonic p-fields* on M.

Hodge's Theorem

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$$H^p(M; \mathbb{R}) \cong \mathcal{H}^p(M).$$

Define $i : \partial M \hookrightarrow M$ to be the natural inclusion.

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The L^2-orthogonal complement of the exact forms inside the space of closed forms is now:

$$\mathcal{H}^p_N(M) := \{\omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^* \star \omega = 0\}.$$

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If $i^* \star \omega = 0$, then ω is said to satisfy the *Neumann boundary condition*.

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Then

$$H^p(M; \mathbb{R}) \cong \mathcal{H}^p_N(M).$$

Hodge–Morrey–Friedrichs Decomposition (continued)

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$$\mathcal{H}^p_D(M) := \{\omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^*\omega = 0\}.$$

If $i^*\omega = 0$, then ω is said to satisfy the *Dirichlet boundary condition*.

The concrete realizations of $H^p(M;\mathbb{R})$ and $H^p(M,\partial M;\mathbb{R})$ meet only at the origin:

$$\mathcal{H}^p_N(M) \cap \mathcal{H}^p_D(M) = \{0\}$$

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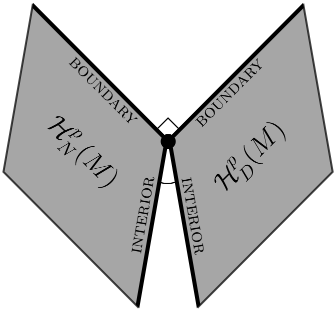

Interior subspace of $\mathcal{H}^p_N(M)$:

$$\ker i^* \text{ where } i^* : H^p(M;\mathbb{R}) \to H^p(\partial M;\mathbb{R})$$

Interior and boundary subspaces

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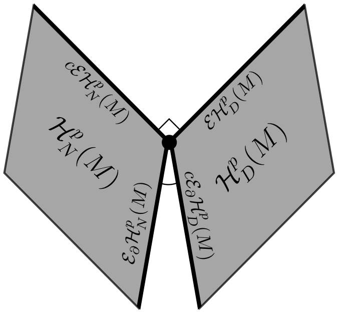
Interior subspace of $\mathcal{H}_D^p(M)$:

$$\begin{aligned}\star \mathcal{E}_\partial \mathcal{H}_N^{n-p}(M) &= c\mathcal{E}_\partial \mathcal{H}_D^p(M) \\ &= \{\eta \in \mathcal{H}_D^p(M) : i^* \star \eta = d\psi, \psi \in \Omega^{n-p-1}(\partial M)\}.\end{aligned}$$



Definition (DeTurck–Gluck)

The *Poincaré duality angles* of the Riemannian manifold M are the principal angles between the interior subspaces.

What do the Poincaré duality angles tell you?

Guess

If M is "almost" closed, the Poincaré duality angles of M should be small.

For example...

Consider $\mathbb{CP}^2$ with its usual Fubini-Study metric. Let $p \in \mathbb{CP}^2$. Then define

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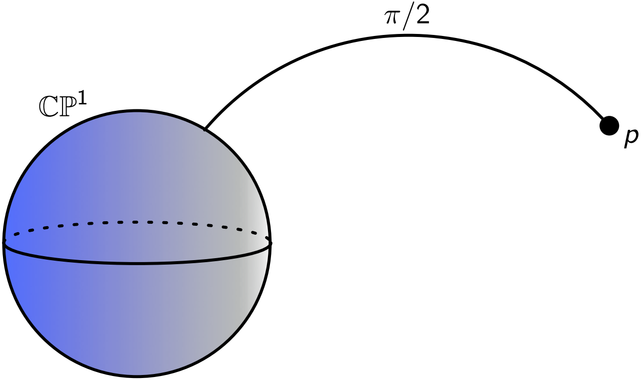

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Therefore, M_r has a single Poincaré duality angle θ_r between $\mathcal{H}^2_N(M_r)$ and $\mathcal{H}^2_D(M_r)$.

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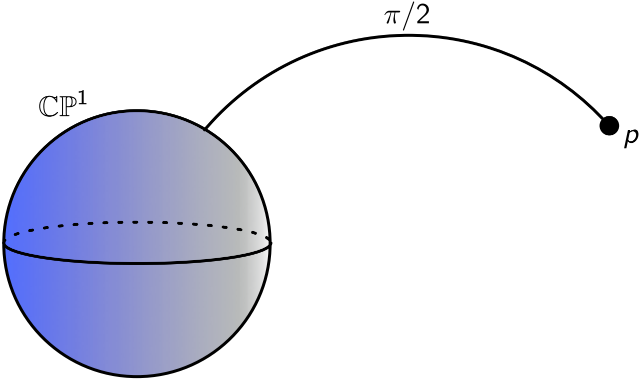
Find closed and co-closed $SU(2)$-invariant forms on M_r satisfying Neumann and Dirichlet boundary conditions.

Finding isometry-invariant 2-forms



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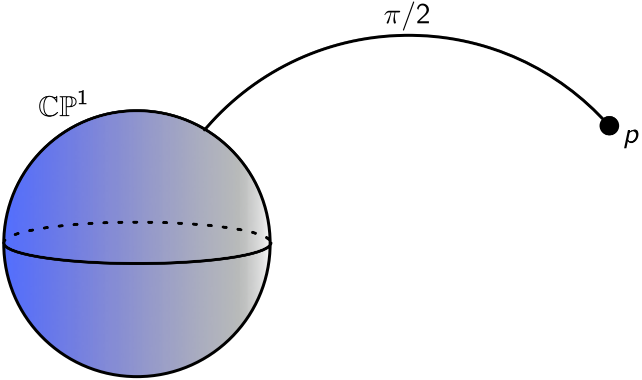

The hypersurfaces at constant distance t from $\mathbb{CP}^1$ are Berger 3-spheres:

$$S^3(\cos t)_{\sin t}$$

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Generalizes to $\mathbb{CP}^n - B_r(p)$.

Consider

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Theorem

- *As $r \to 0$, all the Poincaré duality angles of N_r go to zero.*
- *As r approaches its maximum value of $\pi/2$, all the Poincaré duality angles of N_r go to $\pi/2$.*

Conjecture

If M^n is a closed Riemannian manifold and N^k is a closed submanifold of codimension ≥ 2, the Poincaré duality angles of

$$M - \nu_r(N)$$

go to zero as $r \to 0$.

A question

What can you learn about the topology of M from knowledge of ∂M?

Electrical Impedance Tomography

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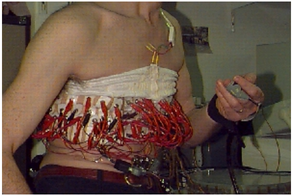

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The Voltage-to-Current map

Suppose f is a potential on the boundary of a region $M \subset \mathbb{R}^3$.

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If γ is the conductivity on M, the current flux through ∂M is given by

$$(\gamma \nabla u) \cdot \nu = -\gamma \frac{\partial u}{\partial \nu}$$

The map $\Lambda_{\mathrm{cl}} : C^{\infty}(\partial M) \to C^{\infty}(\partial M)$ defined by

$$f \mapsto \frac{\partial u}{\partial \nu}$$

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Theorem (Lee-Uhlmann)

If M^n is a compact, analytic Riemannian manifold with boundary, then M is determined up to isometry by Λ_{cl}.

Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

$$\Lambda : \Omega^p(\partial M) \to \Omega^{n-p-1}(\partial M)$$

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If $f \in \Omega^0(\partial M)$,

$$\Lambda f = i^* \star du = \frac{\partial u}{\partial \nu} \mathrm{dvol}_{\partial M} = (\Lambda_{\mathrm{cl}} f)\, \mathrm{dvol}_{\partial M}$$

Theorem (Belishev–Sharafutdinov)

The data $(\partial M, \Lambda)$ *completely determines the cohomology groups of* M*.*

Define the *Hilbert transform* $T := d\Lambda^{-1}$.

Connection to Poincaré duality angles

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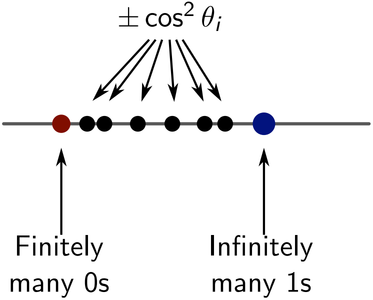
Theorem

If $\theta_1, \dots, \theta_k$ are the Poincaré duality angles of M in dimension p, then the quantities

$$(-1)^{np+n+p} \cos^2 \theta_i$$

are the non-zero eigenvalues of an appropriate restriction of T^2.

The Spectrum of T^2



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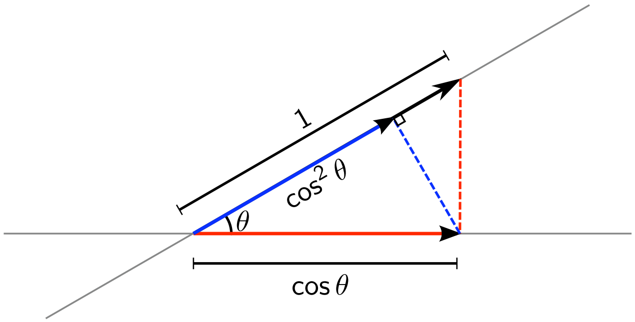

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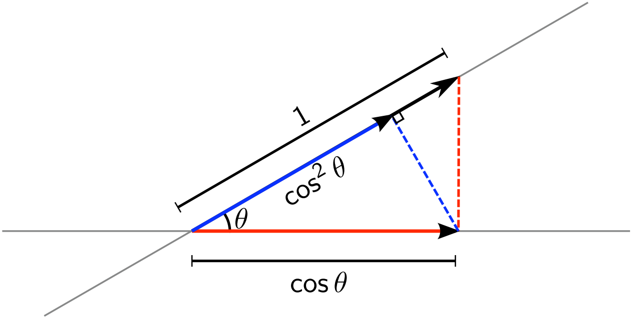

The Hilbert transform T recaptures the orthogonal projection $\mathcal{H}^p_N(M) \to \mathcal{H}^p_D(M)$.

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Theorem

The mixed cup product

$$\cup : H^p(M;\mathbb{R}) \times H^q(M,\partial M;\mathbb{R}) \to H^{p+q}(M,\partial M;\mathbb{R})$$

is completely determined by the data $(\partial M, \Lambda)$ when the relative class is restricted to come from the boundary subspace.

Thanks!