Poincaré Duality Angles on Riemannian Manifolds with Boundary

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Let $M^n$ be a compact Riemannian manifold with non-empty boundary $\partial M$. 
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de Rham’s Theorem

Suppose $M^n$ is a compact, oriented, smooth manifold. Then

$$H^p(M; \mathbb{R}) \cong \mathcal{C}^p(M)/\mathcal{E}^p(M),$$

where $\mathcal{C}^p(M)$ is the space of closed $p$-forms on $M$ and $\mathcal{E}^p(M)$ is the space of exact $p$-forms.
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$$\langle \omega, \eta \rangle := \int_M \omega \wedge \star \eta.$$
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Kodaira called this the space of *harmonic $p$-fields* on $M$. 
Hodge’s Theorem

If $M^n$ is a closed, oriented, smooth Riemannian manifold,

$$H^p(M; \mathbb{R}) \cong \mathcal{H}^p(M).$$
Define \( i : \partial M \hookrightarrow M \) to be the natural inclusion.
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The \( L^2 \)-orthogonal complement of the exact forms inside the space of closed forms is now:

\[
\mathcal{H}_N^p(M) := \{ \omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^*\#\omega = 0 \}.
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Then

$$H^p(M; \mathbb{R}) \cong \mathcal{H}_N^p(M).$$
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\[ \mathcal{H}^p_D(M) := \{ \omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^*\omega = 0 \}. \]

If \( i^*\omega = 0 \), then \( \omega \) is said to satisfy the *Dirichlet boundary condition*. 
The concrete realizations of $H^p(M; \mathbb{R})$ and $H^p(M, \partial M; \mathbb{R})$ meet only at the origin:

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Non-orthogonality

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Interior and boundary subspaces

Interior subspace of $\mathcal{H}_N^p(M)$:

$$\ker i^* \text{ where } i^* : H^p(M; \mathbb{R}) \to H^p(\partial M; \mathbb{R})$$
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Interior subspace of $\mathcal{H}^p_D(M)$:

$$\ast E_{\partial} \mathcal{H}^{n-p}_N(M) = c E_{\partial} \mathcal{H}^p_D(M)$$

$$= \{ \eta \in \mathcal{H}^p_D(M) : i^* \ast \eta = d \psi, \psi \in \Omega^{n-p-1}(\partial M) \}.$$
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**Definition (DeTurck–Gluck)**

The *Poincaré duality angles* of the Riemannian manifold $M$ are the principal angles between the interior subspaces.
What do the Poincaré duality angles tell you?

Guess

If $M$ is “almost” closed, the Poincaré duality angles of $M$ should be small.
Consider $\mathbb{CP}^2$ with its usual Fubini-Study metric. Let $p \in \mathbb{CP}^2$. Then define

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$\partial M_r$ is a 3-sphere.
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$M_r$ has relative cohomology in dimensions 2 and 4.

Therefore, $M_r$ has a single Poincaré duality angle $\theta_r$ between $H^2_N(M_r)$ and $H^2_D(M_r)$. 
So the goal is to find closed and co-closed 2-forms on $M_r$ which satisfy Neumann and Dirichlet boundary conditions.
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Find harmonic 2-fields

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Find closed and co-closed $SU(2)$-invariant forms on $M_r$ satisfying Neumann and Dirichlet boundary conditions.
Finding isometry-invariant 2-forms

The hypersurfaces at constant distance $t$ from $\mathbb{CP}^1$ are Berger 3-spheres: $S^3(\cos t, \sin t)$. 

Diagram:
- $\mathbb{CP}^1$ circle
- Point $p$ at $\pi/2$ from $\mathbb{CP}^1$
The hypersurfaces at constant distance $t$ from $\mathbb{CP}^1$ are Berger 3-spheres:

$$S^3(\cos t)\sin t$$
The Poincaré duality angle for $M_r$

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\cos \theta_r = \frac{1 - \sin^4 r}{1 + \sin^4 r}.
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As $r \to \pi/2$, $\theta_r \to \pi/2$.

Generalizes to $\mathbb{CP}^n - B_r(p)$.  

Poincaré duality angles of Grassmannians

Consider

\[ N_r := G_2 \mathbb{R}^n - \nu_r(G_1 \mathbb{R}^{n-1}). \]
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**Theorem**

- As \( r \to 0 \), all the Poincaré duality angles of \( N_r \) go to zero.
- As \( r \) approaches its maximum value of \( \pi/2 \), all the Poincaré duality angles of \( N_r \) go to \( \pi/2 \).
Conjecture

If \( M^n \) is a closed Riemannian manifold and \( N^k \) is a closed submanifold of codimension \( \geq 2 \), the Poincaré duality angles of

\[
M - \nu_r(N)
\]

go to zero as \( r \to 0 \).
What can you learn about the topology of $M$ from knowledge of $\partial M$?
Induce potentials on the boundary of a region and determine the conductivity inside the region by measuring the current flux through the boundary.
Electrical Impedance Tomography

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Suppose $f$ is a potential on the boundary of a region $M \subset \mathbb{R}^3$. 
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Then \( f \) extends to a potential \( u \) on \( M \), where

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Then $f$ extends to a potential $u$ on $M$, where

$$\Delta u = 0, \quad u|_{\partial M} = f.$$ 

If $\gamma$ is the conductivity on $M$, the current flux through $\partial M$ is given by

$$(\gamma \nabla u) \cdot \nu = -\gamma \frac{\partial u}{\partial \nu}$$
The map $\Lambda_{\text{cl}} : C^\infty(\partial M) \to C^\infty(\partial M)$ defined by

$$f \mapsto \frac{\partial u}{\partial \nu}$$

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**Theorem (Lee-Uhlmann)**

*If $M^n$ is a compact, analytic Riemannian manifold with boundary, then $M$ is determined up to isometry by $\Lambda_{cl}$.***
Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

$$\Lambda : \Omega^p(\partial M) \to \Omega^{n-p-1}(\partial M)$$
Generalization to differential forms

Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

\[ \Lambda : \Omega^p(\partial M) \rightarrow \Omega^{n-p-1}(\partial M) \]

If \( \varphi \in \Omega^p(\partial M) \), then let \( \omega \) solve the boundary value problem

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Define

\[ \Lambda \varphi := i^* \ast d\omega. \]
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If $f \in \Omega^0(\partial M)$,

$$\Lambda f = i^* \star du = \frac{\partial u}{\partial \nu} d\text{vol}_{\partial M} = (\Lambda_{\text{cl}} f) d\text{vol}_{\partial M}$$
Theorem (Belishev–Sharafutdinov)

The data \((\partial M, \Lambda)\) completely determines the cohomology groups of \(M\).
Define the *Hilbert transform* \( T := d\Lambda^{-1} \).
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**Theorem**

If $\theta_1, \ldots, \theta_k$ are the Poincaré duality angles of $M$ in dimension $p$, then the quantities

$$(-1)^{np+n+p} \cos^2 \theta_i$$

are the non-zero eigenvalues of an appropriate restriction of $T^2$. 

The Spectrum of $T^2$

\[ \pm \cos^2 \theta_i; \]

Finitely many 0s

Infinitely many 1s
Theorem

\{\text{Interesting eigenvalues of } T^2\} \iff (-1)^{np+n+p} \cos^2 \theta;
Idea of the Proof

**Theorem**

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\{ \text{Interesting eigenvalues of } T^2 \} \leftrightarrow (-1)^{np+p} \cos^2 \theta_i
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The Hilbert transform \( T \) recaptures the orthogonal projection \( \mathcal{H}_N^p(M) \to \mathcal{H}_D^p(M) \).
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**Theorem**

*The mixed cup product*

\[ \cup : H^p(M; \mathbb{R}) \times H^q(M, \partial M; \mathbb{R}) \rightarrow H^{p+q}(M, \partial M; \mathbb{R}) \]

*is completely determined by the data \((\partial M, \Lambda)\) when the relative class is restricted to come from the boundary subspace.*
Thanks!