The triple linking number is an ambiguous Hopf invariant

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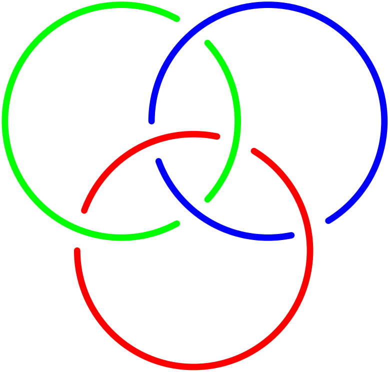
Let $L = L_1 \cup L_2 \cup L_3$ be a smooth 3-component link in S^3, where

$$L_1 = \{x(s)\}$$
$$L_2 = \{y(t)\}$$
$$L_3 = \{z(u)\}$$



Theorem (Milnor)

3-component links $L = L_1 \cup L_2 \cup L_3$ *are determined up to link-homotopy by the pairwise linking numbers* $p_L, q_L, r_L \in \mathbb{Z}$ *and the triple linking number* $\overline{\mu}_L \in \mathbb{Z}/\gcd(p, q, r)$.

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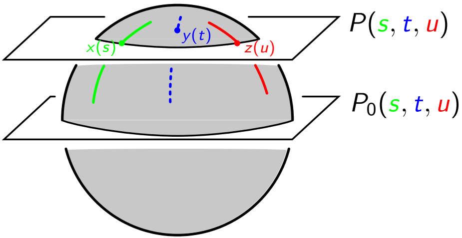
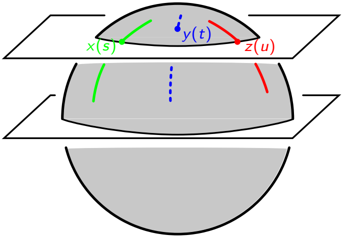
Theorem (Pontryagin)

Maps $f : S^1 \times S^1 \times S^1 \to S^2$ are determined up to homotopy by the degrees $p_f, q_f, r_f \in \mathbb{Z}$ of the restrictions of f to the 2-dimensional subtori and by an ambiguous Hopf invariant $\nu_f \in \mathbb{Z}/2\gcd(p, q, r)$.



$$g_L : S^1 \times S^1 \times S^1 \longrightarrow G_2\mathbb{R}^4$$
$$(s, \quad t, \quad u) \longmapsto P_0(s, t, u)$$



$$\begin{aligned} g_L : S^1 \times S^1 \times S^1 &\to G_2\mathbb{R}^4 \\ (s, \quad t, \quad u) &\mapsto P_0(s, t, u) \end{aligned}$$

$$S^1 \times S^1 \times S^1 \xrightarrow{g_L} G_2\mathbb{R}^4 \simeq S^2 \times S^2 \xrightarrow{\pi_1} S^2.$$

Define $f_L := \pi_1 \circ g_L$.

f_L in coordinates

Let

$$\widetilde{f}(s,t,u) = (x(s)\cdot iy(t) + y(t)\cdot iz(u) + z(u)\cdot ix(s),\\ x(s)\cdot jy(t) + y(t)\cdot jz(u) + z(u)\cdot jx(s),\\ x(s)\cdot ky(t) + y(t)\cdot kz(u) + z(u)\cdot kx(s)).$$

Then

$$f_L(s,t,u) = \frac{\widetilde{f}(s,t,u)}{\|\widetilde{f}(s,t,u)\|}.$$

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Note

$f_L(s,t,u)$ is invariant under even permutations of the coordinates.

Theorem

Given a 3-component link L and the associated map $f_L : S^1 \times S^1 \times S^1 \to S^2$,

- *The pairwise linking numbers* p_L, q_L, r_L *of L are the same as the degrees* $p_{f_L}, q_{f_L}, r_{f_L}$ *of* f_L *restricted to the 2-dimensional subtori.*
- *Moreover,* $\overline{\mu}_L$ *is equal to 1/2 the Pontryagin invariant* ν_{f_L}.

Pairwise linking numbers are degrees

$$S^1 \times S^1 \times S^1 \xrightarrow{C} \mathrm{Conf}_3(S^3) \simeq S^3 \times S^2 \xrightarrow{\pi_2} S^2$$

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Define $F_L := \pi_2 \circ C$; in coordinates,

$$F_L(s,t,u) = \frac{\mathrm{Stereo}_1\left(x(s)^{-1}y(t) - x(s)^{-1}z(u)\right)}{\left\|\mathrm{Stereo}_1\left(x(s)^{-1}y(t) - x(s)^{-1}z(u)\right)\right\|},$$

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Facts

- F_L is homotopic to f_L.
- F_L restricted to the subtorus $\{s_0\} \times S^1 \times S^1$ is just the familiar Gauss map, so its degree is equal to the pairwise linking number
$$r_L = Lk(L_2, L_3).$$

$$[[S^1 \cup S^1 \cup S^1, S^3]] = \{\text{3-component links up to link-homotopy}\}$$

$$\Big\downarrow \mathbb{F} \qquad \text{map taking } L \text{ to } f_L$$

$$[S^1 \times S^1 \times S^1, S^2] = \{\text{maps } S^1 \times S^1 \times S^1 \longrightarrow S^2 \text{ up to homotopy}\}$$

The Situation

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\begin{array}{c}
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Theorem A says $\mathbb{F}$ is one-to-one and describes it.

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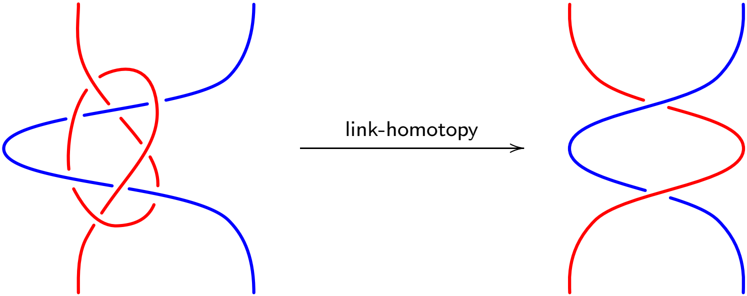
Algebra?

A *string link* is a pure tangle:



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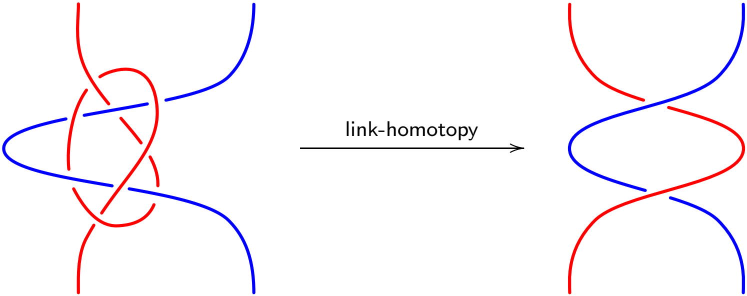
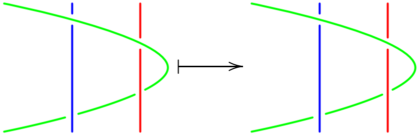
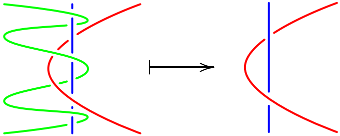


n-stranded string links up to link homotopy form a group, $\mathscr{H}(n)$, with the group operation given by stacking string links.

$$1 \longrightarrow RF(n-1) \longrightarrow \mathscr{H}(n) \longrightarrow \mathscr{H}(n-1) \longrightarrow 1$$





The Structure of $\mathscr{H}(n)$

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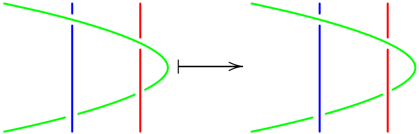

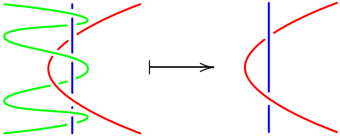

Proposition (Habegger-Lin)

This is a split exact sequence.

Theorem (Habegger-Lin)

Two string links $\sigma, \sigma' \in \mathscr{H}(n) \simeq \mathscr{H}(n-1) \ltimes RF(n-1)$ *close up to link-homotopic n-component links if and only if* σ *and* σ' *are related by a sequence of conjugacies and "partial conjugacies" in* $\mathscr{H}(n)$.

The connection between string links and links

Theorem (Habegger-Lin)

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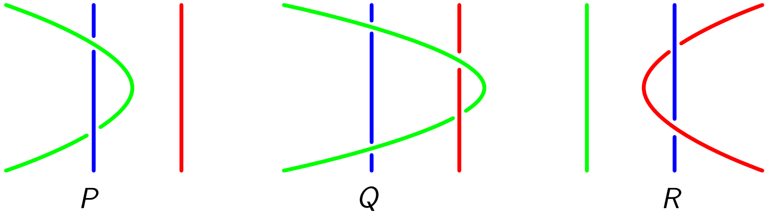
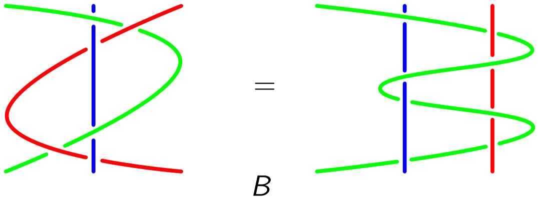
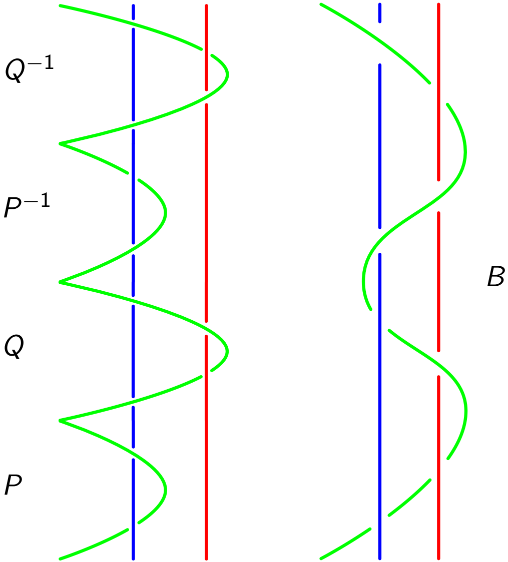
Remark

When $n = 3$, there are no "partial conjugacies", so two elements of $\mathscr{H}(3)$ close up to the same 3-component link (up to link-homotopy) if and only if they are conjugate.


P
Q
R


$=$
B


Q^{-1}
P^{-1}
Q
P
B

Lemma

1.

$$\mathscr{H}(3) = \langle P, Q, R, B | [P, Q] = [Q, R] = [R, P] = B, \\ [P, B] = [Q, B] = [R, B] = 1 \rangle$$

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2. *Elements of $\mathscr{H}(3)$ can be written in the normal form*
$$P^p Q^q R^r B^\mu$$

Lemma

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2. *Elements of $\mathscr{H}(3)$ can be written in the normal form*

$$P^p Q^q R^r B^\mu$$

3. *Two elements $P^p Q^q R^r B^\mu, P^{p'} Q^{q'} R^{r'} B^{\mu'} \in \mathscr{H}(3)$ are conjugate if and only if*

$$p = p', \quad q = q', \quad r = r', \quad \text{and } \mu \equiv \mu' \pmod{\gcd(p, q, r)}.$$

Adding some algebra

$$\begin{array}{ccc}
[[S^1 \cup S^1 \cup S^1, S^3]] & \twoheadleftarrow & \mathscr{H}(3) \\
\Big\downarrow \mathbb{F} & & \Big\downarrow ? \\
[S^1 \times S^1 \times S^1, S^2] & \twoheadleftarrow & ?
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$$\mathrm{Maps}(S^1 \times S^1 \times S^1, S^2) = \mathrm{Maps}(S^1, \mathrm{Maps}(S^1 \times S^1, S^2)).$$

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Homotopy classes:

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The bottom map sends based loops to free loops and interprets them as maps $S^1 \times S^1 \times S^1 \to S^2$. Point inverse images of the bottom map are conjugacy classes.

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$$\pi_1 \mathrm{Maps}(S^1 \times S^1, S^2) = \bigcup_{p \in \mathbb{Z}} \pi_1 \left(\mathrm{Maps}(S^1 \times S^1, S^2), \varphi_p \right)$$

where $\varphi_p : S^1 \times S^1 \to S^2$ is a map of degree p.

$$\begin{array}{ccc} [[S^1 \cup S^1 \cup S^1, S^3]] & \twoheadleftarrow & \mathscr{H}(3) \\ \downarrow \mathbb{F} & & \downarrow ? \\ [S^1 \times S^1 \times S^1, S^2] & \twoheadleftarrow & \bigcup_{p \in \mathbb{Z}} \pi_1 \left(\mathrm{Maps}(S^1 \times S^1, S^2), \varphi_p \right) \end{array}$$

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Let $\mathscr{H}_0(3) \simeq RF(2)$ be the subgroup consisting of those 3-strand string links in which the first and second strands are unlinked.

$$\mathscr{H}_0(3) = \langle Q, R, B | [Q, R] = B, [Q, B] = [R, B] = 1 \rangle$$

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Consider the coset

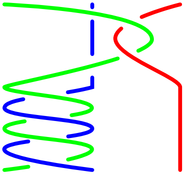
$$\mathscr{H}_p(3) := P^p \mathscr{H}_0(3)$$

and endow it with the group structure of $\mathscr{H}_0(3)$.

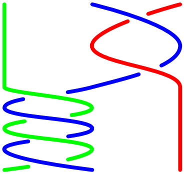
$$\begin{array}{ccc} [[S^1 \cup S^1 \cup S^1, S^3]] & \twoheadleftarrow & \bigcup_{p\in\mathbb{Z}} \mathscr{H}_p(3) \\ \downarrow \mathbb{F} & & \downarrow \widehat{\mathbb{F}} \\ [S^1 \times S^1 \times S^1, S^2] & \twoheadleftarrow & \bigcup_{p\in\mathbb{Z}} \pi_1\left(\mathrm{Maps}(S^1 \times S^1, S^2), \varphi_p\right) \end{array}$$

We want to define $\widehat{\mathbb{F}}$ as a union of group homomorphisms.

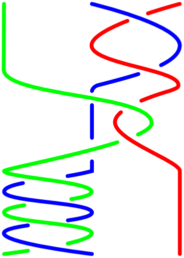
Let $Q_p := P^p Q$, $R_p := P^p R$, $B_p = P^p B$.



Q_p



R_p



$Q_p R_p$

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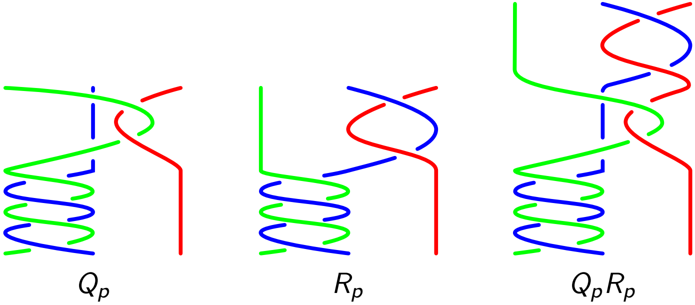


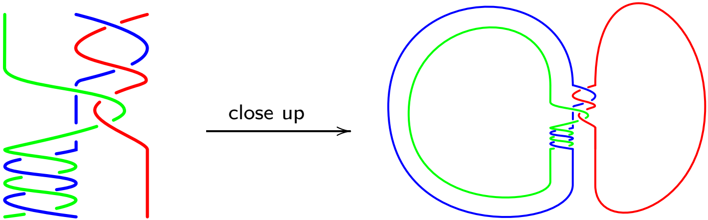
Then

$$\mathscr{H}_p(3) = \langle Q_p, R_p, B_p | [Q_p, R_p] = B_p, [Q_p, B_p] = [R_p, B_p] = 1 \rangle.$$

To define

$$\widehat{\mathbb{F}} : \mathscr{H}_p(3) \to \pi_1(\mathrm{Maps}(S^1 \times S^1, S^2), \varphi_p),$$

close up $\sigma \in \mathscr{H}_p(3)$ to get a 3-component link $\widehat{\sigma}$.



σ $\widehat{\sigma}$

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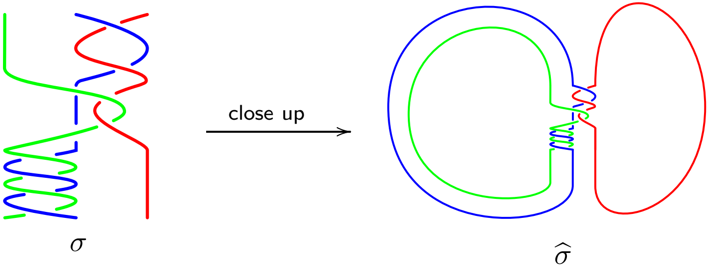


Then $f_{\widehat{\sigma}}$ represents an element $\alpha_\sigma \in \pi_1(\mathrm{Maps}(S^1 \times S^1, S^2), \varphi_p)$.
Define $\widehat{\mathbb{F}}(\sigma) := \alpha_\sigma$.

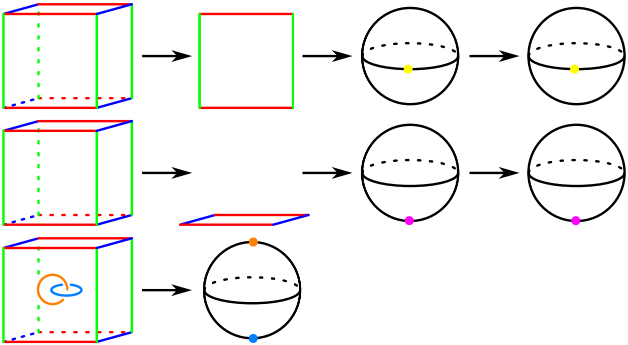
A Presentation for $\pi_1(\text{Maps}(S^1 \times S^1, S^2), \varphi_0)$

Define:

$Y :=$

$Z :=$

$W :=$



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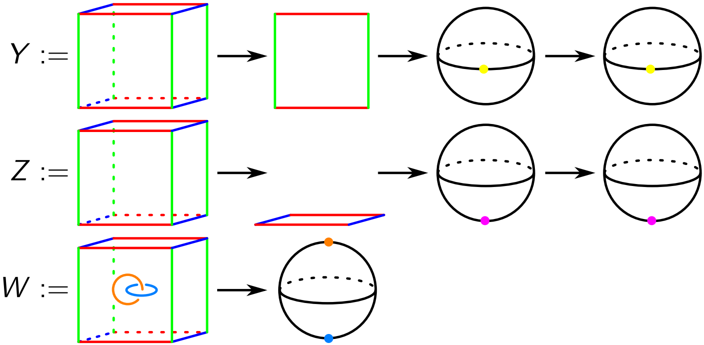
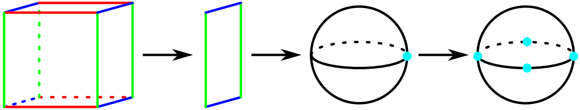


Theorem (Fox)

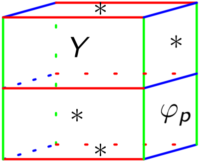
$$\pi_1(\mathrm{Maps}(S^1 \times S^1, S^2), \varphi_p) = \tau_3(S^2)$$
$$= \langle Y, Z, W \mid [Y, Z] = W^2, [Y, W] = [Z, W] = 1 \rangle$$

Define:

$\varphi_p :=$



$Y_p :=$



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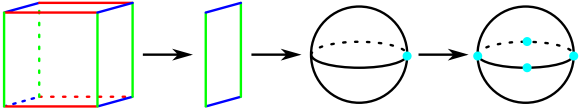

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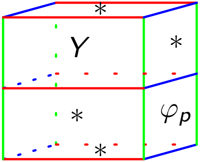

Theorem (Larmore-Thomas, Kallel)

$$\pi_1(\mathrm{Maps}(S^1 \times S^1, S^2), \varphi_p)$$
$$= \langle Y_p, Z_p, W_p | [Y_p, Z_p] = W_p^2, W_p^{2|p|} = 1, [Y_p, W_p] = [Z_p, W_p] = 1 \rangle$$

Recall that we have the presentations

$$\mathscr{H}_p(3) = \langle Q_p, R_p, B_p | [Q_p, R_p] = B_p, [Q_p, B_p] = [R_p, B_p] = 1 \rangle$$

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$\mathscr{H}_p(3) \longrightarrow \pi_1(\mathrm{Maps}_p(S^1 \times S^1, S^2), \varphi_p)$

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and that

$$Q_p^q R_p^r B_p^{\mu}, \text{ and } Q_p^{q'} R_p^{r'} B_p^{\mu'}$$

close up to the same 3-component link if and only if

$$q = q', \quad r = r', \quad \mu \equiv \mu' \pmod{\gcd(p, q, r)}.$$

Recall that we have the presentations

$$\mathscr{H}_p(3) = \langle Q_p, R_p, B_p | [Q_p, R_p] = B_p, [Q_p, B_p] = [R_p, B_p] = 1 \rangle$$

$$\pi_1(\mathrm{Maps}(S^1 \times S^1, S^2), \varphi_p)$$
$$= \langle Y_p, Z_p, W_p | [Y_p, Z_p] = W_p^2, W_p^{2|p|} = 1, [Y_p, W_p] = [Z_p, W_p] = 1 \rangle$$

and that

$$Q_p^q R_p^r B_p^\mu, \text{ and } Q_p^{q'} R_p^{r'} B_p^{\mu'}$$

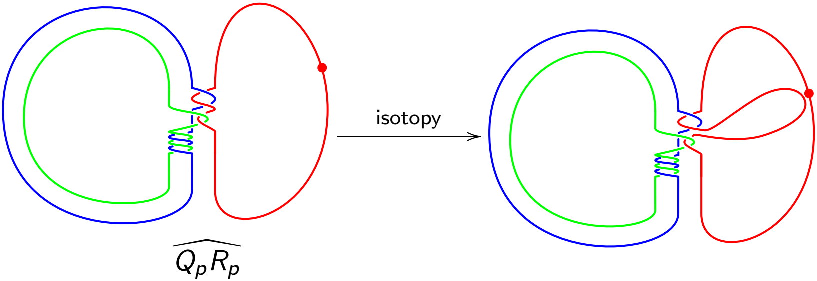
close up to the same 3-component link if and only if

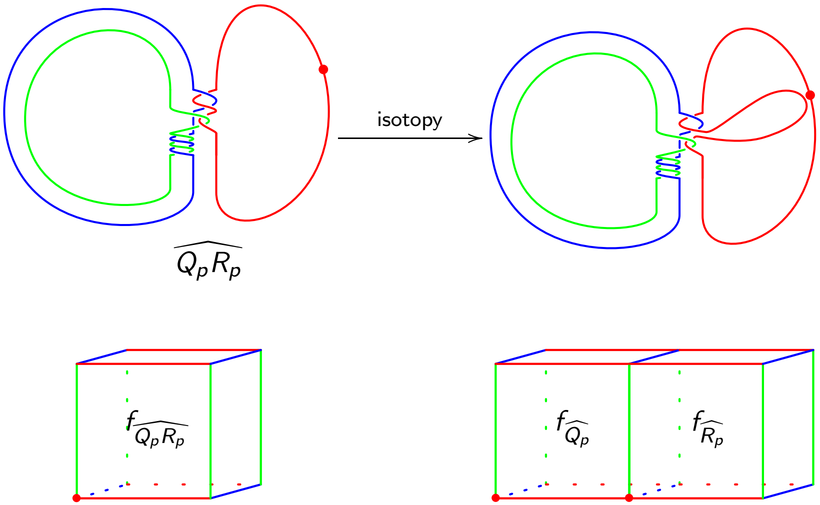
$$q = q', \quad r = r', \quad \mu \equiv \mu' \pmod{\gcd(p, q, r)}.$$

Therefore, to complete the proof of the theorem, we need only show that $\widehat{\mathbb{F}}$ respects the group operations and that

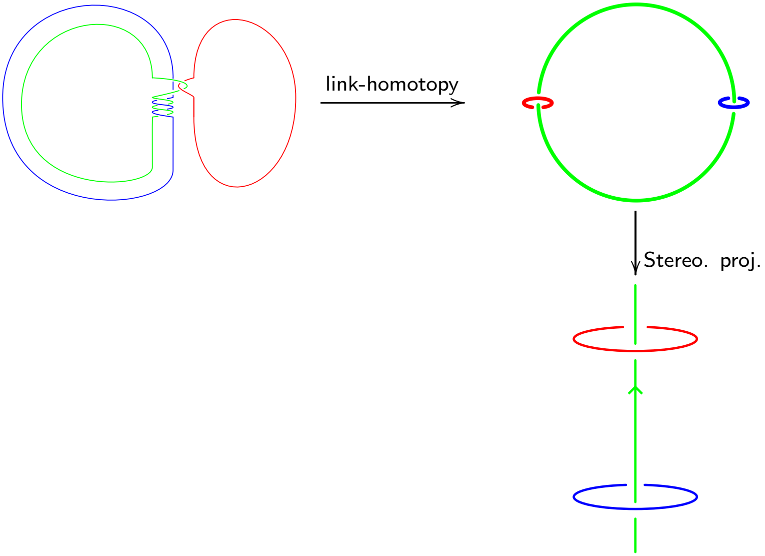
$$\widehat{\mathbb{F}}(Q_p) = Y_p$$
$$\widehat{\mathbb{F}}(R_p) = Z_p$$

Group Operation

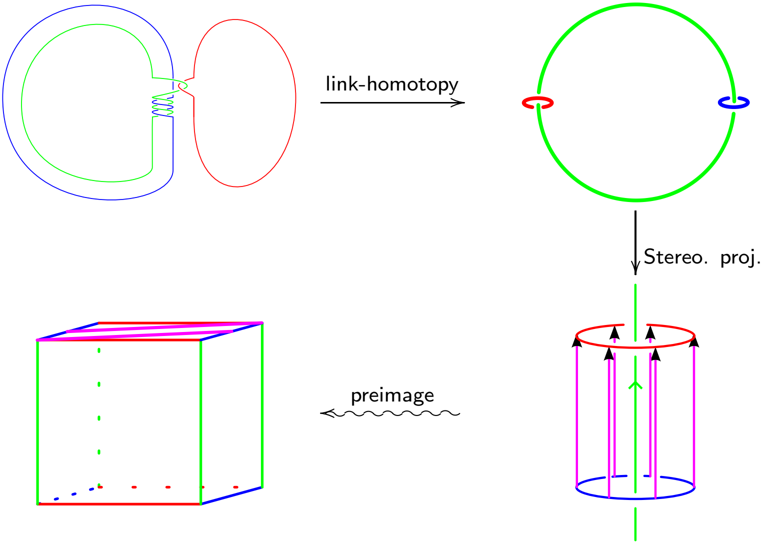



isotopy
$\widehat{Q_p R_p}$
$f_{\widehat{Q_p R_p}}$
$f_{\widehat{Q_p}}$
$f_{\widehat{R_p}}$

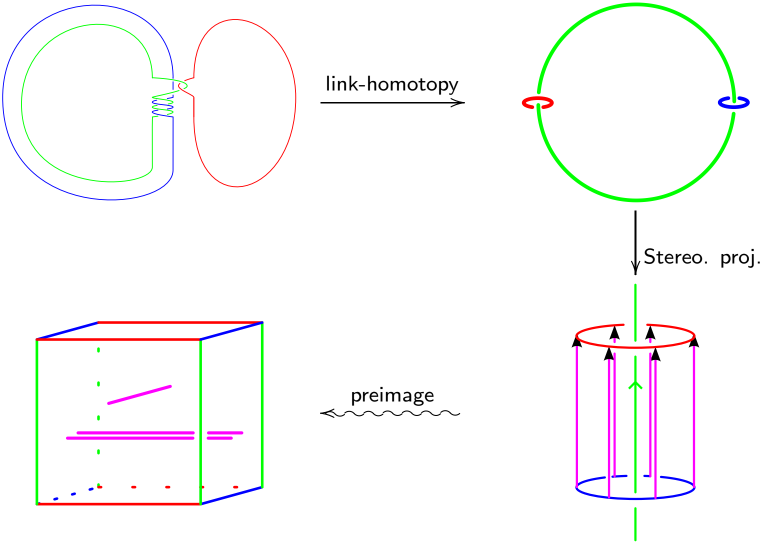
$\widehat{\mathbb{F}}(Q_p) = Y_p$

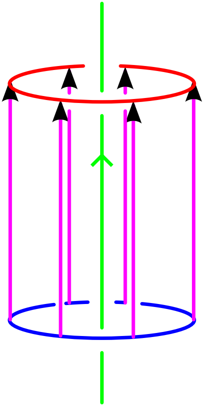


$\widehat{\mathbb{F}}(Q_p) = Y_p$

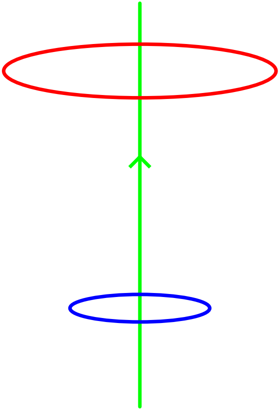


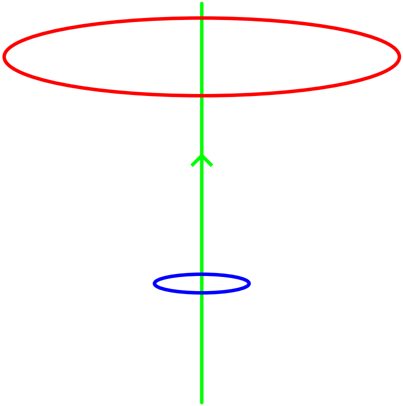
$\widehat{\mathbb{F}}(Q_p) = Y_p$





Movie





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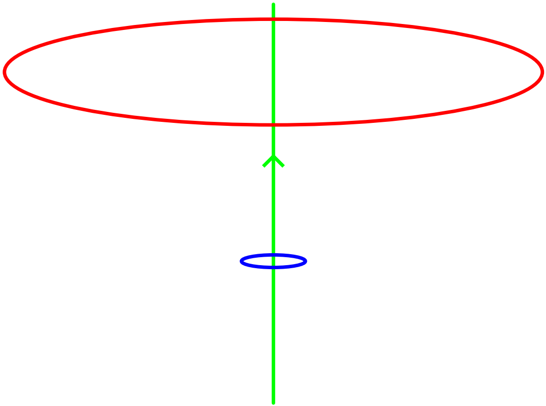

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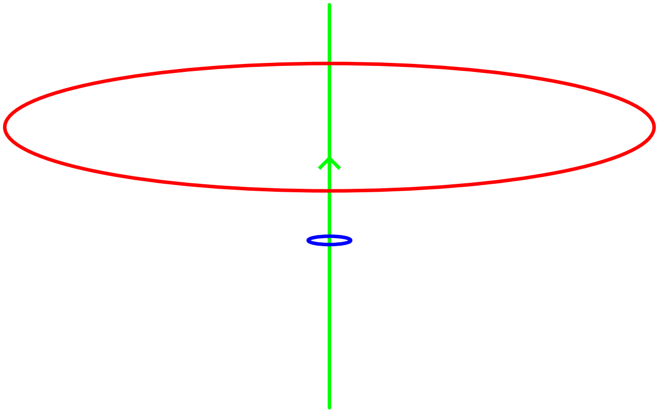

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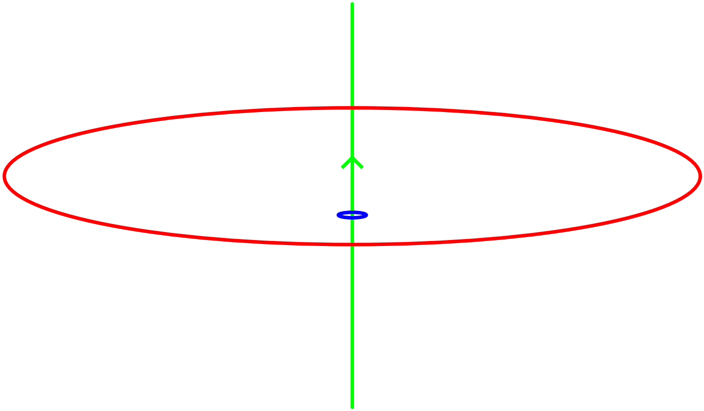

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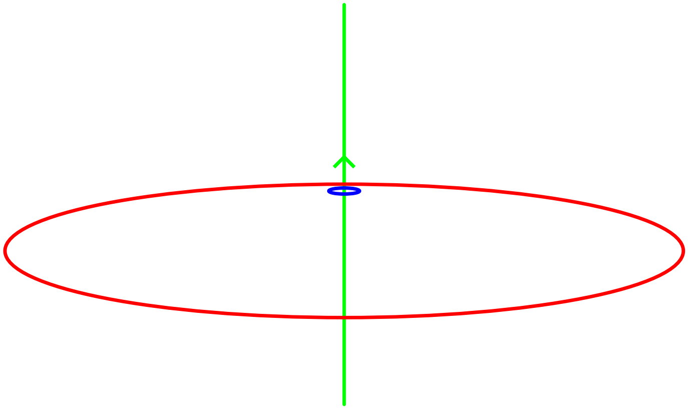

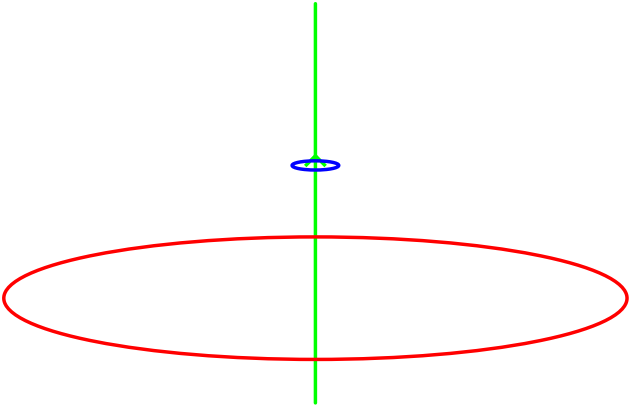

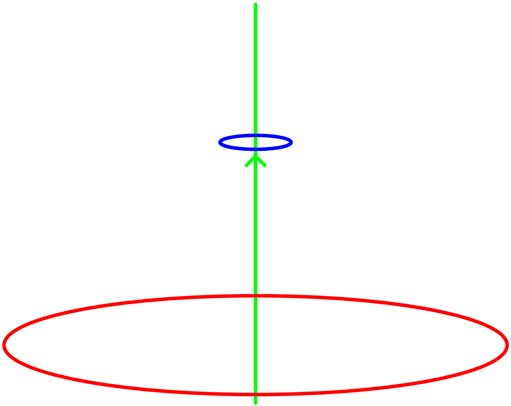

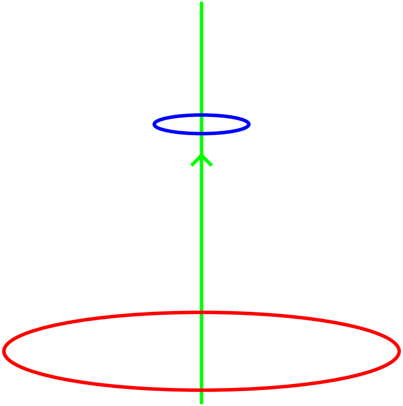

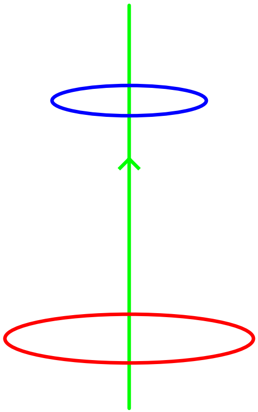

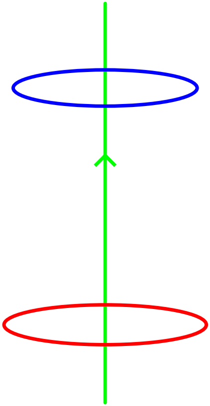

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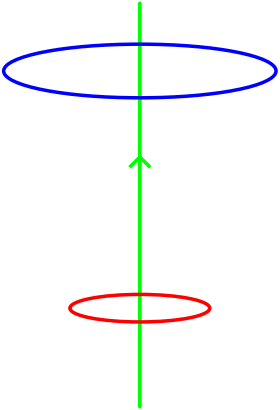

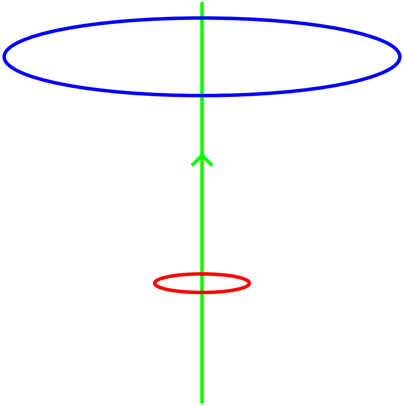

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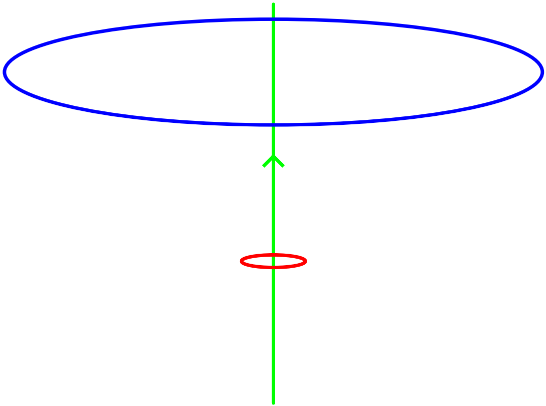

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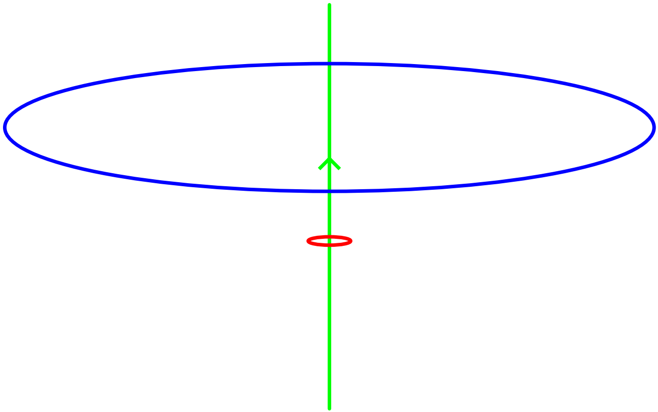

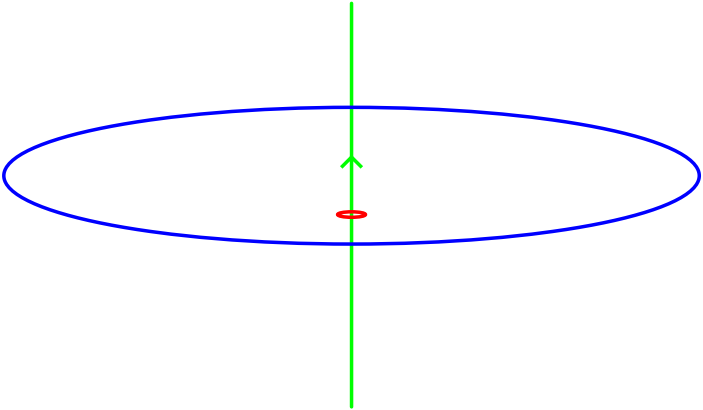

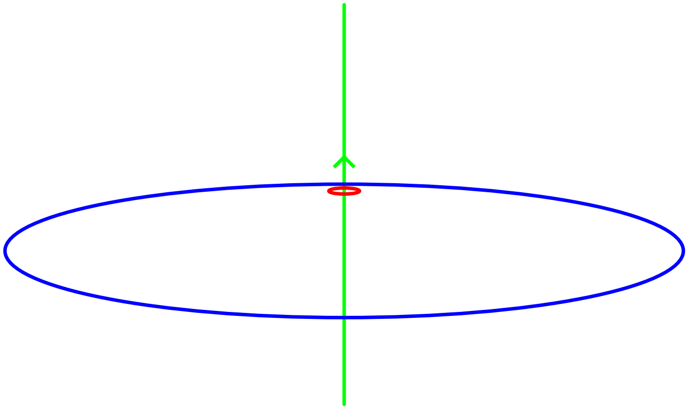

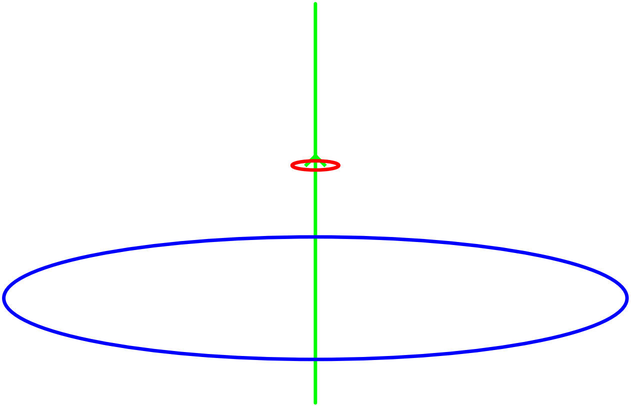

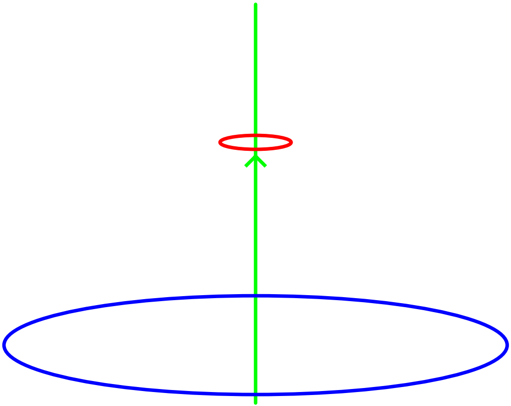

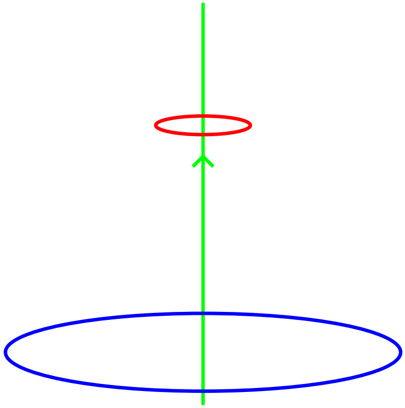

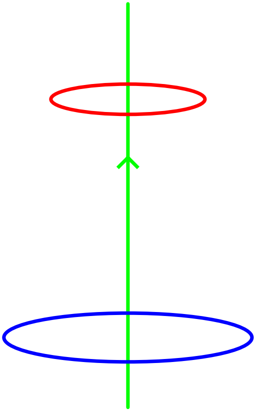

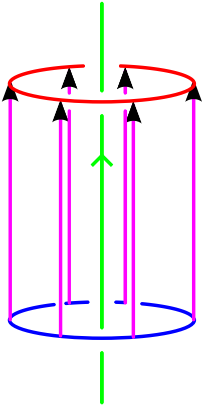

Thanks!