Homotopy and link homotopy

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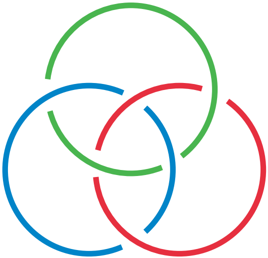
Joint work with Frederick R. Cohen and Rafal Komendarczyk



Consider a *parametrized* n-component link $L_1, L_2, L_3, \ldots, L_n$, where

$$L_i : S^1 \to \mathbb{R}^3$$

have disjoint images.

The Evaluation Map

Given an n-component link $L = \{L_1, \ldots, L_n\}$, there is a natural *evaluation map*

$$F_L : \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}} \longrightarrow \mathrm{Conf}(n)$$

which is just $F_L = L_1 \times \ldots \times L_n$.

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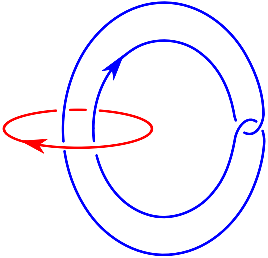
Here $\mathrm{Conf}(n)$ is the configuration space of n distinct points in $\mathbb{R}^3$:

$$\mathrm{Conf}(n) = \mathrm{Conf}_n \mathbb{R}^3 = \{(x_1, \ldots, x_n) \in (\mathbb{R}^3)^n : x_i \neq x_j \text{ for } i \neq j\}.$$

Definition

A *link homotopy* of a link L is a deformation during which each component may cross itself, but distinct components must remain disjoint.



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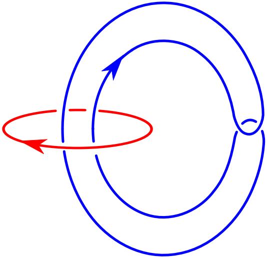

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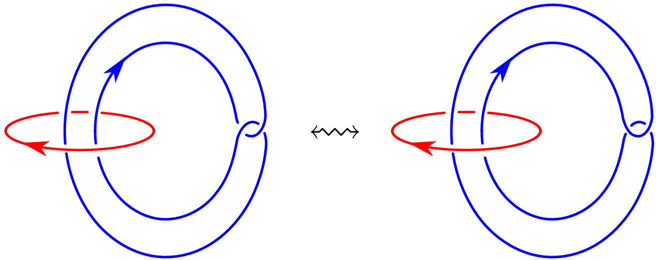

Let Link(n) denote the set of n-component links up to link homotopy.

$$F_L : (S^1)^n \longrightarrow \mathrm{Conf}(n).$$

Link homotopies of L induce homotopies of F_L.

We get an induced map

$$\begin{aligned} \kappa : \mathrm{Link}(n) &\longrightarrow [(S^1)^n, \mathrm{Conf}(n)] \\ L &\longmapsto [F_L] \end{aligned}$$

$$\begin{array}{ccccc} S^1 \times S^1 & \hookrightarrow & \mathrm{Conf}(2) & \xrightarrow[\text{equiv.}]{\text{homotopy}} & S^2 \\ (t_1, t_2) & \longmapsto & (L_1(t_1), L_2(t_2)) & \longmapsto & \dfrac{L_1(t_1) - L_2(t_2)}{|L_1(t_1) - L_2(t_2)|} \end{array}$$

Theorem (Gauss)

The degree of the composition is equal to the linking number.

$n = 2$

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The degree of the composition is equal to the linking number.

Since the linking number classifies 2-component links up to link homotopy and the degree classifies maps $S^1 \times S^1 \to S^2$ up to homotopy, κ is bijective for $n = 2$.

$n = 3$

$$S^1 \times S^1 \times S^1 \hookrightarrow \mathrm{Conf}_3 S^3 \xrightarrow[\text{equiv.}]{\text{hom.}} S^3 \times S^2 \xrightarrow{\pi} S^2.$$

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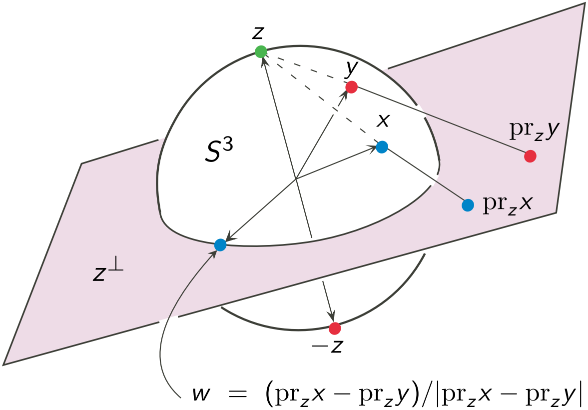

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Corollary

$\kappa : \mathrm{Link}(3) \to [(S^1)^3, \mathrm{Conf}(3)]$ *is injective (though not bijective).*

Conjecture (Koschorke)

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Theorem (Koschorke)

The restriction $\kappa : \mathrm{BLink}(n) \to [(S^1)^n, \mathrm{Conf}(n)]$ *is injective.*

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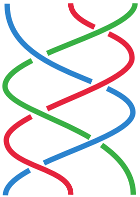
Theorem (with Cohen and Komendarczyk)

The restriction $\kappa : \mathrm{CLink}(n) \to [(S^1)^n, \mathrm{Conf}(n)]$ *is injective.*

Habegger and Lin introduced the group of homotopy string links $\mathscr{H}(n)$.



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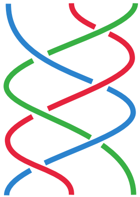

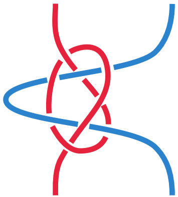

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Theorem (Habegger–Lin)

The map $\mathscr{H}(n) \to \mathsf{Link}(n)$ *induced by the Markov closure is surjective. Moreover, if two string links close up to link-homotopic links, then they are related by a sequence of conjugations and "partial conjugations".*

For each $i = 1, \ldots, n$, there is a natural projection

$$\delta_i : \mathscr{H}(n) \to \mathscr{H}(n-1)$$

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is the subgroup of *Brunnian string links* and

$$C\mathscr{H}(n) = \bigcup_i \ker \delta_i$$

is the set of string links with a trivial $(n-1)$-stranded sublink.

Key Proposition 1

$B\mathscr{H}(n)$ is the center of $\mathscr{H}(n)$.

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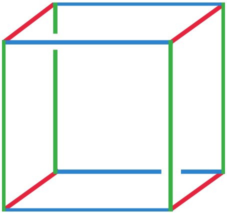
Corollary

The restriction of the Markov closure map $B\mathscr{H}(n) \to \mathrm{BLink}(n)$ is bijective.

Let $T(n)$ be the nth Fox torus homotopy group of $\mathrm{Conf}(n)$:

$$T(n) = [\Sigma(S^1)^{n-1}, \mathrm{Conf}(n)] = [(S^1)^{n-1}, \Omega\,\mathrm{Conf}(n)].$$



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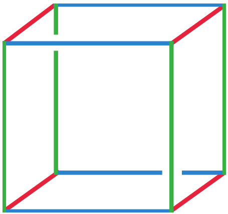

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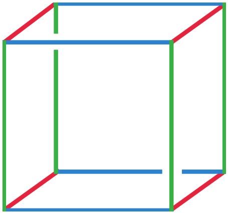

If $L \in \mathrm{BLink}(n)$, then $L - L_n$ is link homotopically trivial, so the restriction of F_L to the face $(S^1)^{n-1} \times \{*\}$ is homotopic to the constant map.

Therefore, we can interpret $\kappa(L) = [F_L]$ as an element of $T(n)$.

Key Proposition 2

The diagram

$$\begin{array}{ccc} B\mathscr{H}(n) & \xrightarrow{\phi} & T(n) \\ \big\downarrow & & \big\downarrow p^{\#} \\ \mathrm{BLink}(n) & \xrightarrow{\kappa} & [(S^1)^n, \mathrm{Conf}(n)] \end{array}$$

commutes and $p^{\#} \circ \phi$ is injective. Therefore κ is injective as well.

Thanks!