The Search for Higher Helicities

Clayton Shonkwiler

Department of Mathematics
Haverford College

May 8, 2011

Joint work with: Dennis DeTurck, Herman Gluck, Rafal Komendarczyk, Paul Melvin, and David Shea Vela-Vick

Let V be a vector field on a compact domain $\Omega \subset \mathbb{R}^3$.

The *energy* of V is

$$E(V) = \int_\Omega (V \cdot V) \mathrm{dvol}_\Omega.$$

Let V be a vector field on a compact domain $\Omega \subset \mathbb{R}^3$.

The *energy* of V is

$$E(V) = \int_\Omega (V \cdot V) \mathrm{dvol}_\Omega.$$

Suppose V is divergence-free. Then

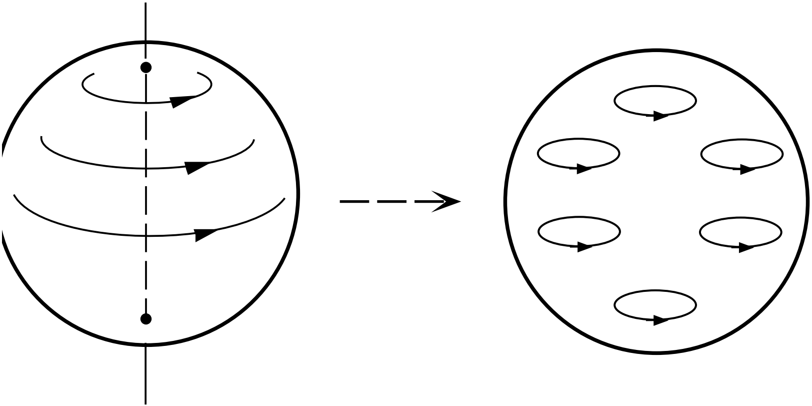
$$V = \nabla \times W,$$

where W is the vector potential for V.

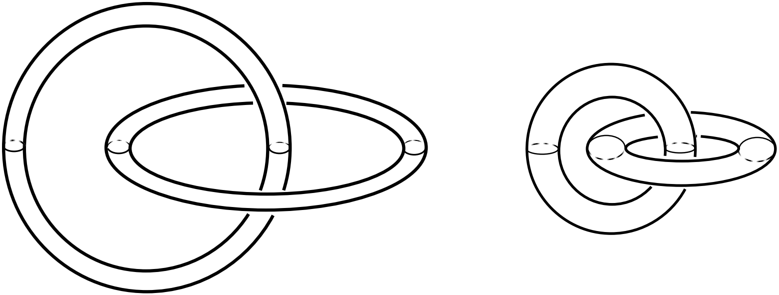
If V is a plasma flow, is there any obstruction to V relaxing to have arbitrarily small energy?

If V is a plasma flow, is there any obstruction to V relaxing to have arbitrarily small energy?

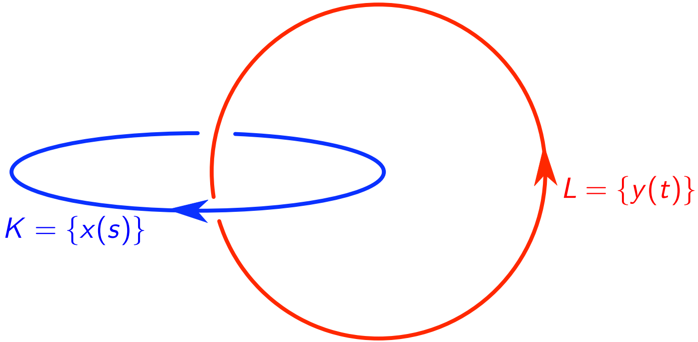
Mathematically, are there obstructions to making the field energy arbitrarily small via volume-preserving diffeomorphisms homotopic to the identity?



A field which can be relaxed to have arbitrarily small energy (Freedman).

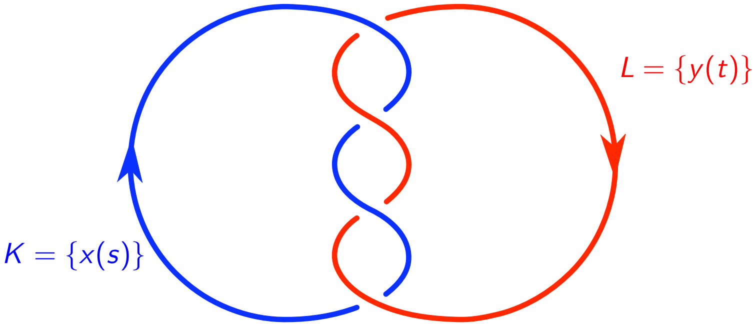


Linked orbits prevent the energy from getting arbitrarily small.



$$1 = Lk(K, L) = \frac{1}{4\pi} \int_{K \times L} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x - y}{|x - y|^3} \, ds \, dt$$

Linking number



$$2 = Lk(K, L) = \frac{1}{4\pi} \int_{K \times L} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x - y}{|x - y|^3} \, ds \, dt$$

The Gauss Linking Integral:

$$Lk(K, L) = \frac{1}{4\pi} \int_{K \times L} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x - y}{|x - y|^3} \, ds \, dt$$

The Gauss Linking Integral:

$$Lk(K, L) = \frac{1}{4\pi} \int_{K \times L} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x - y}{|x - y|^3} \, ds \, dt$$

Important: the integrand is isometry-invariant.

The Gauss Linking Integral:

$$Lk(K, L) = \frac{1}{4\pi} \int_{K \times L} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x - y}{|x - y|^3} \, ds \, dt$$

Important: the integrand is isometry-invariant.

Definition (Woltjer)

The *helicity* of a divergence-free vector field V on a compact domain $\Omega \subset \mathbb{R}^3$ is

$$H(V) := \frac{1}{4\pi} \int_{\Omega \times \Omega} V(x) \times V(y) \cdot \frac{x - y}{|x - y|^3} \, \mathrm{dvol}_x \, \mathrm{dvol}_y .$$

Theorem (Woltjer, Moffatt, Arnol′d, ...)

$H(V)$ is invariant under any volume-preserving diffeomorphism which is homotopic to the identity.

Theorem (Arnol′d)

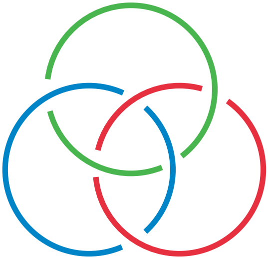
$$|H(V)| \leq R(\Omega)E(V)$$

where $R(\Omega)$ is a positive constant depending on the shape and size of Ω.

Arnol′d and Khesin:

> *The dream is to define such a hierarchy of invariants for generic vector fields such that, whereas all the invariants of order* $\leq k$ *have zero value for a given field and there exists a nonzero invariant of order* $k+1$*, this nonzero invariant provides a lower bound for the field energy.*

Idea: Higher-order linking invariants

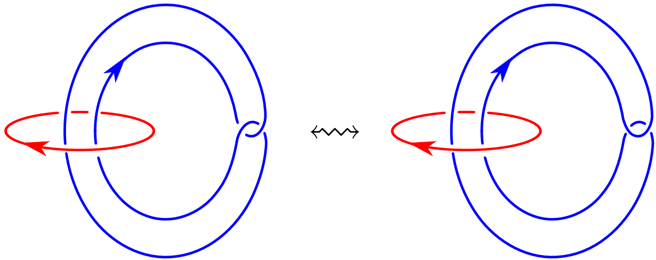


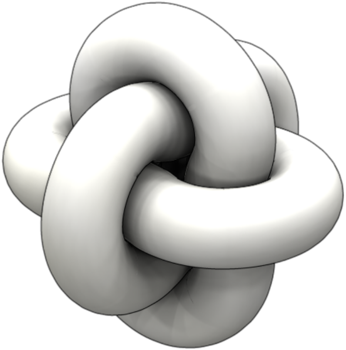
Three-component links were classified up to *link-homotopy* by Milnor:

- The pairwise linking numbers p, q, r.
- The *triple linking number* $\mu = \overline{\mu}_{123}$, which is an integer modulo $\gcd(p, q, r)$.

Definition

A *link homotopy* of a link L is a deformation during which each component may cross itself, but distinct components must remain disjoint.





If V is supported on flux tubes, then μ provides a lower bound for the field energy (Monastyrsky–Retakh, Berger, etc.; see also Freedman–He).

Where does the Gauss Linking Integral come from?

Where does the Gauss Linking Integral come from?

$$S^1 \times S^1 \hookrightarrow \mathrm{Conf}_2\mathbb{R}^3 \xrightarrow[\text{equiv.}]{\text{homotopy}} S^2$$

$$(s,t) \longmapsto (x(s), y(t)) \longmapsto \frac{x(s)-y(t)}{|x(s)-y(t)|}$$

Where does the Gauss Linking Integral come from?

$$S^1 \times S^1 \hookrightarrow \mathrm{Conf}_2\mathbb{R}^3 \xrightarrow[\text{equiv.}]{\text{homotopy}} S^2$$

$$(s,t) \longmapsto (x(s), y(t)) \longmapsto \frac{x(s) - y(t)}{|x(s) - y(t)|}$$

If f is the composition of the above maps

$$Lk(K, L) = \int_{S^1 \times S^1} f^*\omega,$$

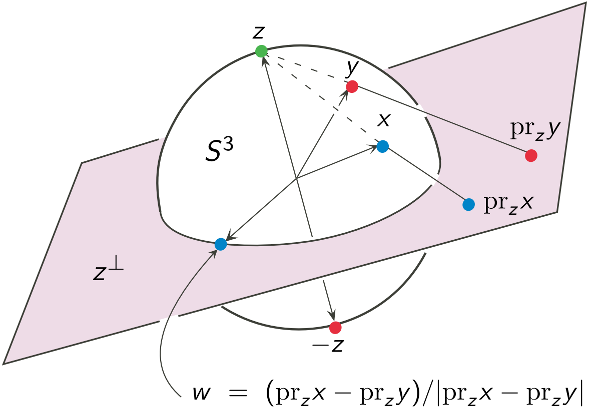
where ω is the standard area form on S^2, scaled to have area 1 instead of 4π.

Consider a three-component link in S^3:

$$S^1 \times S^1 \times S^1 \hookrightarrow \mathrm{Conf}_3 S^3 \xrightarrow[\text{equiv.}]{\text{hom.}} S^3 \times S^2 \xrightarrow{\quad \pi \quad} S^2.$$

Consider a three-component link in S^3:

$$S^1 \times S^1 \times S^1 \hookrightarrow \mathrm{Conf}_3 S^3 \xrightarrow[\text{equiv.}]{\text{hom.}} S^3 \times S^2 \xrightarrow{\pi} S^2.$$



Consider a three-component link in S^3:

$$S^1 \times S^1 \times S^1 \hookrightarrow \mathrm{Conf}_3 S^3 \xrightarrow[\text{equiv.}]{\text{hom.}} S^3 \times S^2 \xrightarrow{\pi} S^2.$$

This determines a map

$$\begin{aligned} g: \quad \mathcal{L}(3) &\longrightarrow [S^1 \times S^1 \times S^1, S^2] \\ L &\longmapsto [g_L] \end{aligned}$$

from link-homotopy classes of 3-component links to homotopy classes of maps from the 3-torus to the 2-sphere.

Consider a three-component link in S^3:

$$S^1 \times S^1 \times S^1 \hookrightarrow \mathrm{Conf}_3 S^3 \xrightarrow[\text{equiv.}]{\text{hom.}} S^3 \times S^2 \xrightarrow{\pi} S^2.$$

This determines a map

$$\begin{aligned} g: \quad \mathcal{L}(3) &\longrightarrow [S^1 \times S^1 \times S^1, S^2] \\ L &\longmapsto [g_L] \end{aligned}$$

from link-homotopy classes of 3-component links to homotopy classes of maps from the 3-torus to the 2-sphere.

$[S^1 \times S^1 \times S^1, S^2]$ was classified by Pontryagin by:

- The degrees p, q, r on the 2-dimensional subtori.
- A "Hopf invariant" ν, which is an integer modulo $2\gcd(p, q, r)$.

Interpreting link-homotopy invariants as homotopy invariants

Theorem A (with DeTurck et al.)

The map $g : \mathcal{L}(3) \to [S^1 \times S^1 \times S^1, S^2]$ *is injective and maps*

$$p \mapsto p$$
$$q \mapsto q$$
$$r \mapsto r$$
$$\mu \mapsto 2\mu$$

A more symmetric version of g along with a modification of Whitehead's integral for the Hopf invariant of a map $S^3 \to S^2$ yields:

Theorem B (with DeTurck et al.)

If the pairwise linking numbers of the three-component link $L \subset S^3$ are all zero, then

$$\mu(L) = \frac{1}{2} \int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L,$$

where $\omega_L = g_L^* \omega$ and φ is the fundamental solution of the Laplacian on T^3.

A more symmetric version of g along with a modification of Whitehead's integral for the Hopf invariant of a map $S^3 \to S^2$ yields:

Theorem B (with DeTurck et al.)

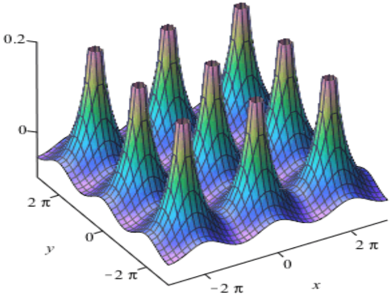
If the pairwise linking numbers of the three-component link $L \subset S^3$ are all zero, then

$$\mu(L) = \frac{1}{2} \int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L,$$

where $\omega_L = g_L^* \omega$ and φ is the fundamental solution of the Laplacian on T^3.

Note: The integrand is isometry-invariant.

The fundamental solution of the Laplacian



$$\varphi(\mathbf{x}) = \frac{1}{8\pi^3} \sum_{\mathbf{n}\in\mathbb{Z}^3\setminus\{\mathbf{0}\}} \frac{e^{i\mathbf{n}\cdot\mathbf{x}}}{|\mathbf{n}|^2}.$$

If

$$\omega_L = \sum_{\mathbf{n}\neq\mathbf{0}} (a_{\mathbf{n}} + ib_{\mathbf{n}}) e^{i\mathbf{n}\cdot\mathbf{x}} \cdot \star d\mathbf{x}$$

is the Fourier expansion of ω_L,

If

$$\omega_L = \sum_{\mathbf{n}\neq\mathbf{0}} (a_{\mathbf{n}} + i b_{\mathbf{n}}) e^{i\mathbf{n}\cdot\mathbf{x}} \cdot \star d\mathbf{x}$$

is the Fourier expansion of ω_L, then

$$\mu(L) = 8\pi^3 \sum_{\mathbf{n}\neq\mathbf{0}} \frac{a_{\mathbf{n}} \times b_{\mathbf{n}} \cdot \mathbf{n}}{|\mathbf{n}|^2}.$$

$$S^1 \times S^1 \times S^1 \hookrightarrow \mathrm{Conf}_3 S^3 \xrightarrow[\text{equiv.}]{\text{hom.}} S^3 \times S^2 \xrightarrow{\pi} S^2.$$

determines a map

$$\begin{aligned} g: \quad \mathcal{L}(3) &\longrightarrow [S^1 \times S^1 \times S^1, S^2] \\ L &\longmapsto [g_L] \end{aligned}$$

$$S^1 \times S^1 \times S^1 \hookrightarrow \mathrm{Conf}_3 S^3 \xrightarrow[\text{equiv.}]{\text{hom.}} S^3 \times S^2 \xrightarrow{\pi} S^2.$$

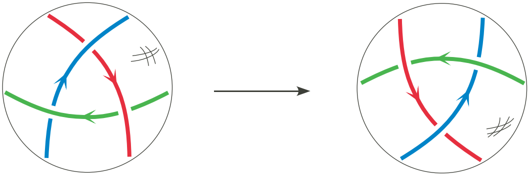
determines a map

$$\begin{array}{rccc} g: & \mathcal{L}(3) & \longrightarrow & [S^1 \times S^1 \times S^1, S^2] \\ & L & \longmapsto & [g_L] \end{array}$$

Theorem A

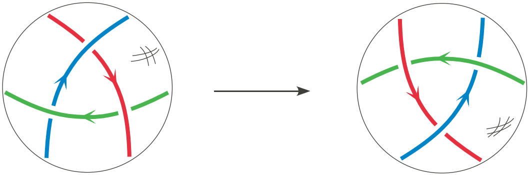
Let L be a three-component link in S^3. Then the pairwise linking numbers p, q, and r of L are equal to the degrees of its characteristic map $g_L : S^1 \times S^1 \times S^1 \to S^2$ on the two-dimensional coordinate subtori, while twice Milnor's μ-invariant for L is equal to Pontryagin's ν-invariant for g_L modulo $2\gcd(p, q, r)$.

The **delta move** does not change pairwise linking numbers.



Murakami and Nakanishi (1989) proved that any two links with the same number of components and the same pairwise linking numbers are related by a sequence of delta moves.

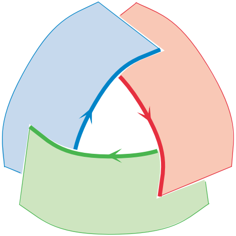
The **delta move** does not change pairwise linking numbers.

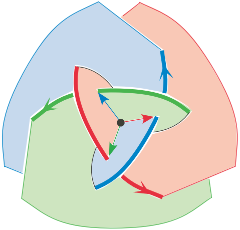


Murakami and Nakanishi (1989) proved that any two links with the same number of components and the same pairwise linking numbers are related by a sequence of delta moves.

The strategy for proving Theorem A is to analyze the effect of a delta move on the μ-invariant of a three component link and on the ν-invariant of its associated map.

We use Mellor and Melvin's (2003) geometric formulation of Milnor's μ-invariant to see that a delta move increases μ by 1.



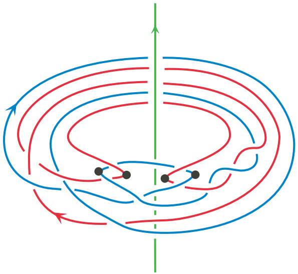


Delta moves and the ν-invariant

The heart of the proof of Theorem A consists of showing that a delta move increases Pontryagin's ν-invariant by 2.

The heart of the proof of Theorem A consists of showing that a delta move increases Pontryagin's ν-invariant by 2.

To see this, we put L in "generic position" with respect to the standard open book on S^3.



We develop an algorithm for computing the Pontryagin ν-invariant of a link in this position, and use it to show that a delta move increases ν by 2.

Theorem B

If the pairwise linking numbers of the three-component link $L \subset S^3$ are all zero, then:

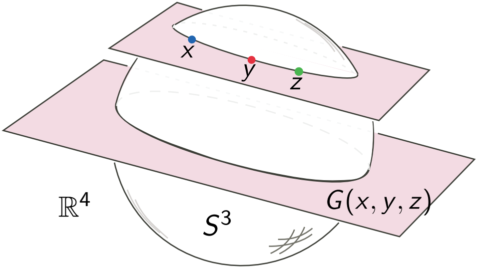
$$\mu(L) = \frac{1}{2} \int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L \tag{1}$$

$$= -\frac{1}{2} \int_{T^3 \times T^3} \vec{\mathbf{v}}_L(\mathbf{x}) \times \vec{\mathbf{v}}_L(\mathbf{y}) \cdot \nabla_{\mathbf{y}}\, \varphi(\mathbf{x} - \mathbf{y})\; d\mathbf{x}\, d\mathbf{y} \tag{2}$$

$$= 8\pi^3 \sum_{\mathbf{n} \neq \mathbf{0}} \mathbf{a_n} \times \mathbf{b_n} \cdot \mathbf{n} / \|\mathbf{n}\|^2\ . \tag{3}$$

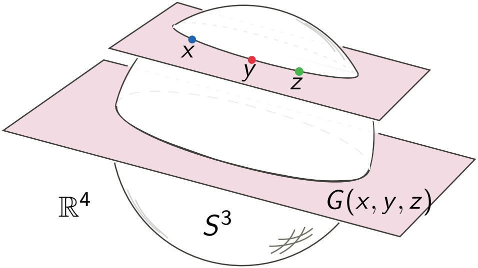
We need a more symmetric version of the characteristic map for Theorem B.

Suppose $(x, y, z) \in \mathrm{Conf}_3 S^3$. Then x, y, and z span a 2-plane in $\mathbb{R}^4$.



We need a more symmetric version of the characteristic map for Theorem B.

Suppose $(x, y, z) \in \mathrm{Conf}_3 S^3$. Then x, y, and z span a 2-plane in $\mathbb{R}^4$.



$G : \mathrm{Conf}_3 S^3 \to G_2\mathbb{R}^4 \simeq S^2 \times S^2$ is $SO(4)$-equivariant.

Why should there be an integral formula for μ

If $f : S^1 \times S^1 \times S^1 \to S^2$ is null-homotopic on the 2-skeleton (i.e., $p = q = r = 0$), then we can collapse the 2-skeleton, yielding a map $\overline{f} : S^3 \to S^2$; then $\nu(f)$ is equal to the Hopf invariant of $\overline{f}$.

Whitehead's integral formula for the Hopf invariant

- $h : S^3 \to S^2$ smooth.
- ω the normalized area form on S^2 ($\int_{S^2} \omega = 1$).

Why should there be an integral formula for μ

If $f : S^1 \times S^1 \times S^1 \to S^2$ is null-homotopic on the 2-skeleton (i.e., $p = q = r = 0$), then we can collapse the 2-skeleton, yielding a map $\overline{f} : S^3 \to S^2$; then $\nu(f)$ is equal to the Hopf invariant of $\overline{f}$.

Whitehead's integral formula for the Hopf invariant

- $h : S^3 \to S^2$ smooth.
- ω the normalized area form on S^2 ($\int_{S^2} \omega = 1$).
- Then $h^*\omega$ is closed and, therefore, exact: $h^*\omega = d\alpha$.

Why should there be an integral formula for μ

If $f : S^1 \times S^1 \times S^1 \to S^2$ is null-homotopic on the 2-skeleton (i.e., $p = q = r = 0$), then we can collapse the 2-skeleton, yielding a map $\overline{f} : S^3 \to S^2$; then $\nu(f)$ is equal to the Hopf invariant of $\overline{f}$.

Whitehead's integral formula for the Hopf invariant

- $h : S^3 \to S^2$ smooth.
- ω the normalized area form on S^2 ($\int_{S^2} \omega = 1$).
- Then $h^*\omega$ is closed and, therefore, exact: $h^*\omega = d\alpha$.
- Then

$$\mathrm{Hopf}(h) = \int_{S^3} \alpha \wedge h^*\omega = \int_{S^3} \alpha \wedge d\alpha.$$

If L is a three-component link in S^3 such that $p = q = r = 0$, then the characteristic map $g_L : T^3 \to S^2$ is null-homotopic on the coordinate two-tori.

Hence, $\omega_L = g_L^* \omega$ is an exact 2-form on T^3.

If L is a three-component link in S^3 such that $p = q = r = 0$, then the characteristic map $g_L : T^3 \to S^2$ is null-homotopic on the coordinate two-tori.

Hence, $\omega_L = g_L^* \omega$ is an exact 2-form on T^3.

A standard computation shows that

$$\alpha = \delta(\varphi * \omega_L)$$

satisfies $d\alpha = \omega_L$. Here φ is the fundamental solution of the scalar Laplacian on T^3, $\varphi * \omega_L$ is the convolution of φ and ω_L, and δ is the L^2-adjoint of the exterior derivative d.

Moreover, α is the unique 1-form of smallest L^2 norm satisfying $d\alpha = \omega_L$.

Then Whitehead's integral formula for the Hopf invariant, transplanted to T^3, implies

$$\nu(g_L) = \int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L.$$

By Theorem A,

$$\mu(L) = \frac{1}{2}\nu(g_L) = \frac{1}{2}\int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L.$$

Then Whitehead's integral formula for the Hopf invariant, transplanted to T^3, implies

$$\nu(g_L) = \int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L .$$

By Theorem A,

$$\mu(L) = \frac{1}{2}\nu(g_L) = \frac{1}{2}\int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L .$$

Then the vector field version is

$$\mu(L) = -\frac{1}{2}\int_{T^3 \times T^3} \vec{\mathbf{v}}_L(\mathbf{x}) \times \vec{\mathbf{v}}_L(\mathbf{y}) \cdot \nabla_{\mathbf{y}}\, \varphi(\mathbf{x} - \mathbf{y})\; d\mathbf{x}\, d\mathbf{y}$$

and the Fourier series version is

$$\mu(L) = 8\pi^3 \sum_{\mathbf{n} \neq \mathbf{0}} \frac{\mathbf{a_n} \times \mathbf{b_n} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \; .$$

- Find the analogues of Theorems A and B for three-component links in $\mathbb{R}^3$.

What next?

- Find the analogues of Theorems A and B for three-component links in $\mathbb{R}^3$.
  In other words, the representation of link-homotopy classes of links into the set of homotopy classes of maps should be **equivariant** with respect to the action of $\mathrm{Isom}^+(\mathbb{R}^3)$, and the integral formula for μ should be **invariant** under the action of $\mathrm{Isom}^+(\mathbb{R}^3)$.

What next?

- Find the analogues of Theorems A and B for three-component links in $\mathbb{R}^3$.
  In other words, the representation of link-homotopy classes of links into the set of homotopy classes of maps should be **equivariant** with respect to the action of $\mathrm{Isom}^+(\mathbb{R}^3)$, and the integral formula for μ should be **invariant** under the action of $\mathrm{Isom}^+(\mathbb{R}^3)$.

- Is the representation $\mathcal{L}(n) \to [T^n, \mathrm{Conf}_n\mathbb{R}^3]$ faithful?

What next?

- Find the analogues of Theorems A and B for three-component links in $\mathbb{R}^3$.
  In other words, the representation of link-homotopy classes of links into the set of homotopy classes of maps should be **equivariant** with respect to the action of $\mathrm{Isom}^+(\mathbb{R}^3)$, and the integral formula for μ should be **invariant** under the action of $\mathrm{Isom}^+(\mathbb{R}^3)$.

- Is the representation $\mathcal{L}(n) \to [T^n, \mathrm{Conf}_n\mathbb{R}^3]$ faithful?
  Theorem A implies "yes" for $n = 3$, and Koschorke has shown that the answer is "yes" when one restricts to *almost trivial* n-component links.

Arnol′d and Khesin:

The dream is to define such a hierarchy of invariants for generic vector fields such that, whereas all the invariants of order $\leq k$ *have zero value for a given field and there exists a nonzero invariant of order* $k + 1$*, this nonzero invariant provides a lower bound for the field energy.*

Thanks!