Poincaré Duality Angles on Riemannian Manifolds with Boundary

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Let $M^n$ be a compact Riemannian manifold with non-empty boundary $\partial M$. 
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\[ H^p(M; \mathbb{R}) \]
\[ H^p(M, \partial M; \mathbb{R}) \]
de Rham’s Theorem

Suppose $M^n$ is a compact, oriented, smooth manifold. Then

$$H^p(M; \mathbb{R}) \cong C^p(M)/\mathcal{E}^p(M),$$

where $C^p(M)$ is the space of closed $p$-forms on $M$ and $\mathcal{E}^p(M)$ is the space of exact $p$-forms.
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$$\langle \omega, \eta \rangle := \int_M \omega \wedge \ast \eta.$$
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Kodaira called this the space of harmonic $p$-fields on $M$. 
Hodge’s Theorem

If $M^n$ is a closed, oriented, smooth Riemannian manifold,

\[ H^p(M; \mathbb{R}) \cong \mathcal{H}^p(M). \]
Define $i : \partial M \hookrightarrow M$ to be the natural inclusion.
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The $L^2$-orthogonal complement of the exact forms inside the space of closed forms is now:

$$\mathcal{H}_N^p(M) := \{ \omega \in \Omega^p(M) : d\omega = 0, \delta \omega = 0, i^* \star \omega = 0 \}.$$
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If $i^*\star \omega = 0$, then

$$\star \omega = \left( \sum f_I dx_I \right) \land dx_n,$$

meaning that $\omega$ has no $dx_n$ in it.
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\]

Then

\[
H^p(M; \mathbb{R}) \cong \mathcal{H}^p_N(M).
\]
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\[ \mathcal{H}^p_D(M) := \{ \omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^*\omega = 0 \}. \]
Non-orthogonality

The concrete realizations of $H^p(M; \mathbb{R})$ and $H^p(M, \partial M; \mathbb{R})$ meet only at the origin:

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...but they are not orthogonal!
Interior and boundary subspaces

Interior subspace of $\mathcal{H}_N^p(M)$:

$$\ker i^* \text{ where } i^*: H^p(M; \mathbb{R}) \to H^p(\partial M; \mathbb{R})$$
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$$\mathcal{E}_{\partial} \mathcal{H}_N^p(M) := \{ \omega \in \mathcal{H}_N^p(M) : i^* \omega = d\varphi, \varphi \in \Omega^{p-1}(\partial M) \}.$$
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Interior subspace of $\mathcal{H}_D^p(M)$:

$$\star \mathcal{E}_\partial \mathcal{H}_N^{n-p}(M) = c \mathcal{E}_\partial \mathcal{H}_D^p(M)$$

$$= \{\eta \in \mathcal{H}_D^p(M) : i^* \star \eta = d\psi, \psi \in \Omega^{n-p-1}(\partial M)\}.$$
To prove that there is an element of $\mathcal{E}H^p_N(M)$ having arbitrary preassigned periods on $c p_1 \hookrightarrow \ldots c p_g$, it suffices to show that $(F_1 \hookrightarrow \ldots \hookrightarrow F_g) \mapsto (C_1 \hookrightarrow \ldots \hookrightarrow C_g)$ is an isomorphism.

Suppose some set of $F$-values gives all zero $C$-values, meaning that $i^* \eta$ is zero in the cohomology of $\partial M$. In other words, the form $i^* \eta$ is exact, meaning that $\eta \in \mathcal{E}H^p_N(M)$, the interior subspace of $H^p_N(M)$. Since $\mathcal{E}H^p_N(M)$ is orthogonal to $\mathcal{E}H^p_N(M)$, this implies that $\eta = 0$, so $\tilde{\eta} = \pm \star \eta = 0$ and hence the periods $F_i$ of $\tilde{\eta}$ must have been zero.

Therefore, the map $(F_1 \hookrightarrow \ldots \hookrightarrow F_g) \mapsto (C_1 \hookrightarrow \ldots \hookrightarrow C_g)$ is an isomorphism, completing Step 1.

Step 2: Let $\omega \in H^p_N(M)$ and let $C_1 \hookrightarrow \ldots \hookrightarrow C_g$ be the periods of $\omega$ on the above $p$-cycles $c p_1 \hookrightarrow \ldots \hookrightarrow c p_g$. Let $\alpha \in \mathcal{E}H^p_N(M)$ be the unique form guaranteed by Step 1 having the same periods on this homology basis.

Then $\beta = \omega - \alpha$ has zero periods on the $p$-cycles $c p_1 \hookrightarrow \ldots \hookrightarrow c p_g$; since $\beta$ is a closed form on $M$, it certainly has zero period on each $p$-cycle of $\partial M$ which bounds in $M$. Hence, $\beta$ has zero periods on all $p$-cycles of $\partial M$, meaning that $i^* \beta$ is exact, so $\beta \in \mathcal{E}H^p_N(M)$.

Therefore, $\omega = \alpha + \beta \in \mathcal{E}H^p_N(M) + \mathcal{E}H^p_D(M)$, so $H^p_N(M)$ is indeed the sum of these two subspaces, as claimed in (2.1.8). This completes the proof of the theorem.

Theorem 2.1.2 allows the details of Figure 1.1 to be filled in, as shown in Figure 2.1.

**Definition (DeTurck–Gluck)**

The *Poincaré duality angles* of the Riemannian manifold $M$ are the principal angles between the interior subspaces.
Guess

If $M$ is “almost” closed, the Poincaré duality angles of $M$ should be small.
Consider $\mathbb{CP}^2$ with its usual Fubini-Study metric. Let $p \in \mathbb{CP}^2$. Then define

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Therefore, \( M_r \) has a single Poincaré duality angle \( \theta_r \) between \( \mathcal{H}^2_N(M_r) \) and \( \mathcal{H}^2_D(M_r) \).
So the goal is to find closed and co-closed 2-forms on $M_r$ which satisfy Neumann and Dirichlet boundary conditions.
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Find closed and co-closed $SU(2)$-invariant forms on $M_r$ satisfying Neumann and Dirichlet boundary conditions.
The hypersurfaces at constant distance $t$ from $\mathbb{CP}^1$ are Berger 3-spheres:

$$S^3(\cos t) \sin t$$
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The Poincaré duality angle for $M_r$

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Generalizes to $\mathbb{CP}^n - B_r(p)$.
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**Theorem**

- As \( r \to 0 \), all the Poincaré duality angles of \( N_r \) go to zero.
- As \( r \) approaches its maximum value of \( \pi/2 \), all the Poincaré duality angles of \( N_r \) go to \( \pi/2 \).
Conjecture

If $M^n$ is a closed Riemannian manifold and $N^k$ is a closed submanifold of codimension $\geq 2$, the Poincaré duality angles of

$$M - \nu_r(N)$$

go to zero as $r \to 0$. 
What can you learn about the topology of $M$ from knowledge of $\partial M$?
Electrical Impedance Tomography

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Then $f$ extends to a potential $u$ on $M$, where

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If $\gamma$ is the conductivity on $M$, the current flux through $\partial M$ is given by

$$(\gamma \nabla u) \cdot \nu = -\gamma \frac{\partial u}{\partial \nu}$$
The Dirichlet-to-Neumann map

The map $\Lambda_{cl} : C^\infty(\partial M) \rightarrow C^\infty(\partial M)$ defined by

$$f \mapsto \frac{\partial u}{\partial \nu}$$

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**Theorem (Lee-Uhlmann)**

*If \( M^n \) is a compact, analytic Riemannian manifold with boundary, then \( M \) is determined up to isometry by \( \Lambda_{\text{cl}} \).*
Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

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If \( f \in \Omega^0(\partial M) \),

\[ \Lambda f = i^* \ast du = \frac{\partial u}{\partial \nu} \text{dvol}_{\partial M} = (\Lambda_{\text{cl}} f) \text{dvol}_{\partial M} \]
Theorem (Belishev–Sharafutdinov)

The data \((\partial M, \Lambda)\) completely determines the cohomology groups of \(M\).
Define the *Hilbert transform* $T := d\Lambda^{-1}$. 
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**Theorem**

If $\theta_1, \ldots, \theta_k$ are the Poincaré duality angles of $M$ in dimension $p$, then the quantities

$$(-1)^{np+n+p} \cos^2 \theta_i$$

are the non-zero eigenvalues of an appropriate restriction of $T^2$. 
Idea of the Proof

\[ \cos \theta \]

\[ \cos^2 \theta \]

The Hilbert transform \( T \) recaptures the orthogonal projection \( H \mathbb{P} \mathbb{N}(M) \rightarrow H \mathbb{P} \mathbb{D}(M) \).
The Hilbert transform $T$ recaptures the orthogonal projection $\mathcal{H}_N^p(M) \rightarrow \mathcal{H}_D^p(M)$. 
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**Theorem**

*The mixed cup product*

\[ \cup : H^p(M; \mathbb{R}) \times H^q(M, \partial M; \mathbb{R}) \rightarrow H^{p+q}(M, \partial M; \mathbb{R}) \]

is completely determined by the data $(\partial M, \Lambda)$ when the relative class is restricted to come from the boundary subspace.
Thanks!