

# Higher Helicities, Geometric Linking Integrals, and Koschorke's Conjecture

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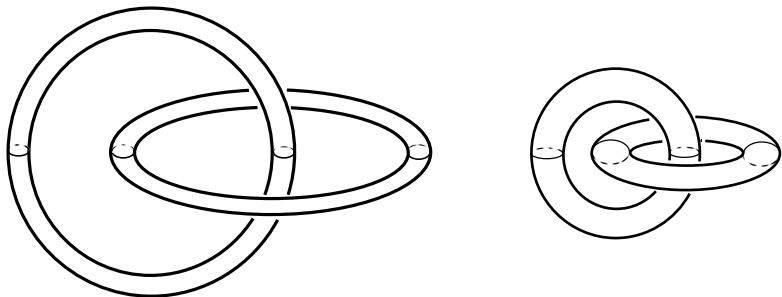
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Joint work with: Dennis DeTurck, Herman Gluck, Rafal Komendarczyk, Paul Melvin, and David Shea Vela-Vick

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Mathematically, are there obstructions to making the field energy arbitrarily small via volume-preserving diffeomorphisms homotopic to the identity?



Linked orbits prevent the energy from getting arbitrarily small.

The Gauss Linking Integral:

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Important: the integrand is isometry-invariant.

### Definition (Woltjer)

The *helicity* of a divergence-free vector field  $V$  on a compact domain  $\Omega \subset \mathbb{R}^3$  is

$$H(V) := \frac{1}{4\pi} \int_{\Omega \times \Omega} V(x) \times V(y) \cdot \frac{x - y}{|x - y|^3} d\text{vol}_x d\text{vol}_y.$$



Theorem (Woltjer, Moffatt, Arnol'd, ...)

*$H(V)$  is invariant under any volume-preserving diffeomorphism which is homotopic to the identity.*

Theorem (Arnol'd)

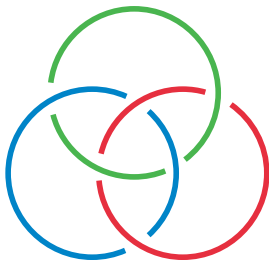
$$|H(V)| \leq R(\Omega)E(V)$$

*where  $R(\Omega)$  is a positive constant depending on the shape and size of  $\Omega$ .*

Arnol'd and Khesin:

*The dream is to define such a hierarchy of invariants for generic vector fields such that, whereas all the invariants of order  $\leq k$  have zero value for a given field and there exists a nonzero invariant of order  $k + 1$ , this nonzero invariant provides a lower bound for the field energy.*

# Idea: Higher-order linking invariants

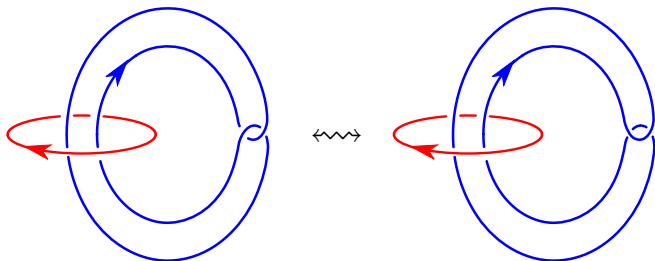


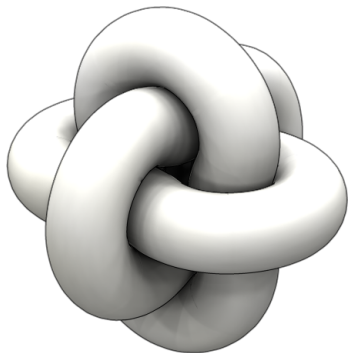
Three-component links were classified up to *link-homotopy* by Milnor:

- The pairwise linking numbers  $p, q, r$ .
- The *triple linking number*  $\mu = \bar{\mu}_{123}$ , which is an integer modulo  $\gcd(p, q, r)$ .

## Definition

A *link homotopy* of a link  $L$  is a deformation during which each component may cross itself, but distinct components must remain disjoint.





If  $V$  is supported on flux tubes, then  $\mu$  provides a lower bound for the field energy (Monastyrsky–Retakh, Berger, etc.; also Freedman–He).

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$$(s, t) \longmapsto (x(s), y(t)) \longmapsto \frac{x(s) - y(t)}{|x(s) - y(t)|}$$

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If  $f$  is the composition of the above maps

$$Lk(K, L) = \int_{S^1 \times S^1} f^* \omega,$$

where  $\omega$  is the standard area form on  $S^2$ , scaled to have area 1 instead of  $4\pi$ .



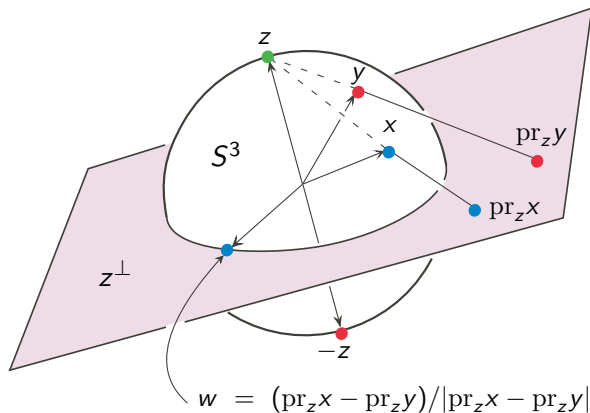
Consider a three-component link in  $S^3$ :

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# Three-component links

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This determines a map

$$g : \begin{array}{ccc} \mathcal{L}(3) & \longrightarrow & [S^1 \times S^1 \times S^1, S^2] \\ L & \longmapsto & [gL] \end{array}$$

from link-homotopy classes of 3-component links to homotopy classes of maps from the 3-torus to the 2-sphere.

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$[S^1 \times S^1 \times S^1, S^2]$  was classified by Pontryagin by:

- The degrees  $p, q, r$  on the 2-dimensional subtori.
- A “Hopf invariant”  $\nu$ , which is an integer modulo  $2 \gcd(p, q, r)$ .

# Interpreting link-homotopy invariants as homotopy invariants

Theorem A (with DeTurck et al.)

*The map  $g : \mathcal{L}(3) \rightarrow [S^1 \times S^1 \times S^1, S^2]$  is injective and maps*

$$p \mapsto p$$

$$q \mapsto q$$

$$r \mapsto r$$

$$\mu \mapsto 2\mu$$

A more symmetric version of  $g$  along with a modification of Whitehead's integral for the Hopf invariant of a map  $S^3 \rightarrow S^2$  yields:

**Theorem B (with DeTurck et al.)**

*If the pairwise linking numbers of the three-component link  $L \subset S^3$  are all zero, then*

$$\mu(L) = \frac{1}{2} \int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L,$$

*where  $\omega_L = g_L^* \omega$  and  $\varphi$  is the fundamental solution of the Laplacian on  $T^3$ .*

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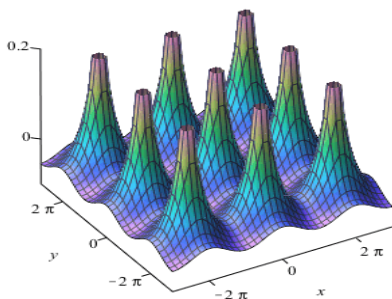
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Note: The integrand is isometry-invariant.

# The fundamental solution of the Laplacian



$$\varphi(\mathbf{x}) = \frac{1}{8\pi^3} \sum_{\mathbf{n} \in \mathbb{Z}^3 \setminus \{0\}} \frac{e^{i\mathbf{n} \cdot \mathbf{x}}}{|\mathbf{n}|^2}.$$



If

$$\omega_L = \sum_{n \neq 0} (a_n + ib_n) e^{in \cdot x} \cdot \star dx$$

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is the Fourier expansion of  $\omega_L$ , then

$$\mu(L) = 8\pi^3 \sum_{\mathbf{n} \neq \mathbf{0}} \frac{a_{\mathbf{n}} \times b_{\mathbf{n}} \cdot \mathbf{n}}{|\mathbf{n}|^2}.$$

$$S^1 \times S^1 \times S^1 \hookrightarrow \text{Conf}_3 S^3 \xrightarrow[\text{equiv.}]{\text{hom.}} S^3 \times S^2 \xrightarrow{\pi} S^2.$$

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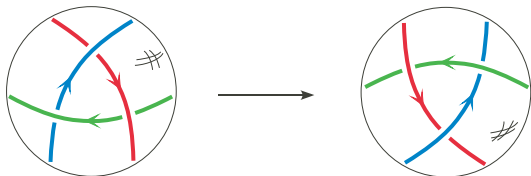
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### Theorem A

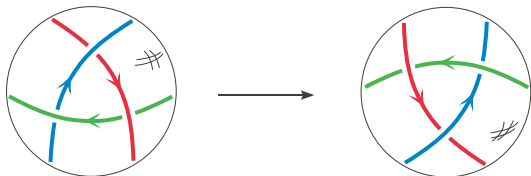
*Let  $L$  be a three-component link in  $S^3$ . Then the pairwise linking numbers  $p$ ,  $q$ , and  $r$  of  $L$  are equal to the degrees of its characteristic map  $g_L : S^1 \times S^1 \times S^1 \rightarrow S^2$  on the two-dimensional coordinate subtori, while twice Milnor's  $\mu$ -invariant for  $L$  is equal to Pontryagin's  $\nu$ -invariant for  $g_L$  modulo  $2 \gcd(p, q, r)$ .*

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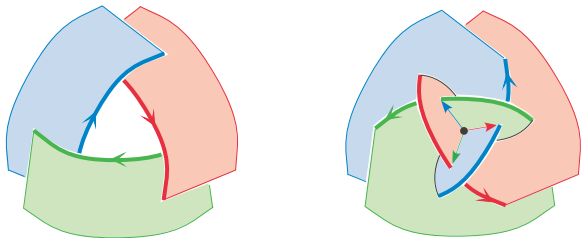


Murakami and Nakanishi (1989) proved that any two links with the same number of components and the same pairwise linking numbers are related by a sequence of delta moves.

The strategy for proving Theorem A is to analyze the effect of a delta move on the  $\mu$ -invariant of a three component link and on the  $\nu$ -invariant of its associated map.

# Delta moves and the $\mu$ -invariant

We use Mellor and Melvin's (2003) geometric formulation of Milnor's  $\mu$ -invariant to see that a delta move increases  $\mu$  by 1.



# Delta moves and the $\nu$ -invariant

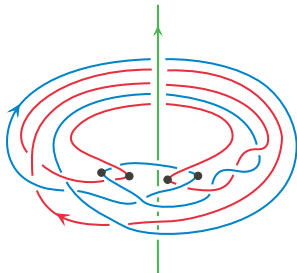
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To see this, we put  $L$  in “generic position” with respect to the standard open book on  $S^3$ .



We develop an algorithm for computing the Pontryagin  $\nu$ -invariant of a link in this position, and use it to show that a delta move increases  $\nu$  by 2.

## Theorem B

If the pairwise linking numbers of the three-component link  $L \subset S^3$  are all zero, then:

$$\mu(L) = \frac{1}{2} \int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L \quad (1)$$

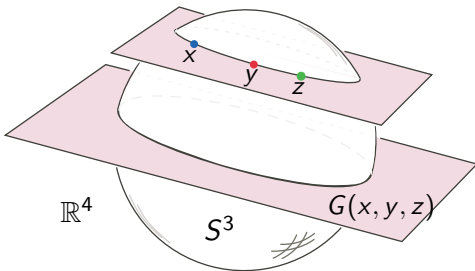
$$= -\frac{1}{2} \int_{T^3 \times T^3} \vec{\mathbf{v}}_L(\mathbf{x}) \times \vec{\mathbf{v}}_L(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \varphi(\mathbf{x} - \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} \quad (2)$$

$$= 8\pi^3 \sum_{\mathbf{n} \neq \mathbf{0}} \mathbf{a}_{\mathbf{n}} \times \mathbf{b}_{\mathbf{n}} \cdot \mathbf{n} / \|\mathbf{n}\|^2 . \quad (3)$$

# Symmetric characteristic map

We need a more symmetric version of the characteristic map for Theorem B.

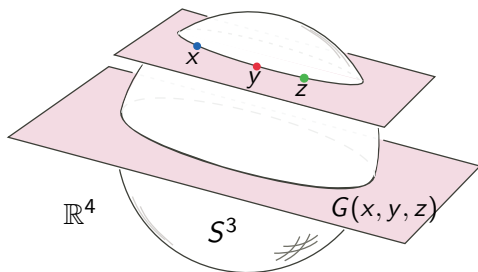
Suppose  $(x, y, z) \in \text{Conf}_3 S^3$ . Then  $x, y,$  and  $z$  span a 2-plane in  $\mathbb{R}^4$ .



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$G : \text{Conf}_3 S^3 \rightarrow G_2 \mathbb{R}^4 \simeq S^2 \times S^2$  is  $SO(4)$ -equivariant.

# Why should there be an integral formula for $\mu$

If  $f : S^1 \times S^1 \times S^1 \rightarrow S^2$  is null-homotopic on the 2-skeleton (i.e.,  $p = q = r = 0$ ), then we can collapse the 2-skeleton, yielding a map  $\bar{f} : S^3 \rightarrow S^2$ ; then  $\nu(f)$  is equal to the Hopf invariant of  $\bar{f}$ .

## Whitehead's integral formula for the Hopf invariant

- $h : S^3 \rightarrow S^2$  smooth.
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- Then

$$\text{Hopf}(h) = \int_{S^3} \alpha \wedge h^*\omega = \int_{S^3} \alpha \wedge d\alpha.$$

If  $L$  is a three-component link in  $S^3$  such that  $p = q = r = 0$ , then the characteristic map  $g_L : T^3 \rightarrow S^2$  is null-homotopic on the coordinate two-tori.

Hence,  $\omega_L = g_L^* \omega$  is an exact 2-form on  $T^3$ .



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Hence,  $\omega_L = g_L^* \omega$  is an exact 2-form on  $T^3$ .

A standard computation shows that

$$\alpha = \delta(\varphi * \omega_L)$$

satisfies  $d\alpha = \omega_L$ . Here  $\varphi$  is the fundamental solution of the scalar Laplacian on  $T^3$ ,  $\varphi * \omega_L$  is the convolution of  $\varphi$  and  $\omega_L$ , and  $\delta$  is the  $L^2$ -adjoint of the exterior derivative  $d$ .

Moreover,  $\alpha$  is the unique 1-form of smallest  $L^2$  norm satisfying  $d\alpha = \omega_L$ .

Then Whitehead's integral formula for the Hopf invariant, transplanted to  $T^3$ , implies

$$\nu(g_L) = \int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L.$$

By Theorem A,

$$\mu(L) = \frac{1}{2}\nu(g_L) = \frac{1}{2} \int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L.$$

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Then the vector field version is

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and the Fourier series version is

$$\mu(L) = 8\pi^3 \sum_{\mathbf{n} \neq \mathbf{0}} \frac{\mathbf{a}_{\mathbf{n}} \times \mathbf{b}_{\mathbf{n}} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}.$$

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- Is the representation  $\mathcal{L}(n) \rightarrow [T^n, \text{Conf}_n \mathbb{R}^3]$  faithful?

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- Is the representation  $\mathcal{L}(n) \rightarrow [T^n, \text{Conf}_n \mathbb{R}^3]$  faithful? Theorem A implies “yes” for  $n = 3$ , and Koschorke has shown that the answer is “yes” when one restricts to *almost trivial*  $n$ -component links.

Arnol'd and Khesin:

*The dream is to define such a hierarchy of invariants for generic vector fields such that, whereas all the invariants of order  $\leq k$  have zero value for a given field and there exists a nonzero invariant of order  $k + 1$ , this nonzero invariant provides a lower bound for the field energy.*



Thanks!

