Pictures and Syzygies: An exploration of pictures, cellular models and free resolutions

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Terminology

Let $G = \{\mathcal{X}, \mathcal{R}\}$ be a group presented by generators $x \in \mathcal{X}$ and relations $r \in \mathcal{R}$. The normal subgroup of the free group F generated by $\mathcal{R}$ will be denoted by R. So long as there is no confusion, the image of a generator $x \in \mathcal{X}$ in $G = F/R$ will still be denoted by x. Also, it will be useful to introduce a disjoint set $\mathcal{R}^{-1}$ corresponding to the inverse of relations. Throughout this talk, we will refer to the following presented group:

$$G = \mathbb{Z}^3 = \{x, y, z | r_x \equiv [y, z], r_y \equiv [z, x], r_z \equiv [x, y]\},$$

where $[-, -]$ denotes the commutator (e.g. $[x, y] = x^{-1}y^{-1}xy$).

An Introduction to Pictures

Let G be a group. Given a presentation

$$G = \{\mathcal{X}, \mathcal{R}\} = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$$

of the group, consider the set $\mathcal{P}$ of directed planar graphs satisfying three conditions:

1. Each edge of the graph is labeled with an element $x_i \in \mathcal{X}, i = 1, \ldots, n$.

2. Each vertex of the graph has a preferred sector denoted by $*$.

3. Each vertex v is associated with a word $r(v)$ by starting at the preferred sector for the vertex and reading the edge labels x_i^ϵ counterclockwise. $\epsilon = +1$ if the edge x_i is directed into the vertex and $\epsilon = -1$ otherwise. This word $r(v) \in \mathcal{R} \cup \mathcal{R}^{-1}$.

Definition 1. Any directed planar graph satisfying conditions (1)-(3) is called a *picture* for the presentation $\{\mathcal{X}, \mathcal{R}\}$ of G.

Defining an equivalence relation on pictures

Definition 2. If an embedded circle β in the plane intersects none of the vertices of a picture P, then β encloses a *subpicture* of P.

Now, to be able to consider a group of appropriate pictures, we define an equivalence relation $\backsim$ on $\mathcal{P}$ (see Figure 1):

1. A simply closed edge with empty interior is equivalent to the empty picture.

2. If v_i and v_j are vertices such that $r(v_i) = r(v_j)^{-1}$ and the preferred sectors for v_1 and v_j lie in the same region of the picture, the vertices v_i and v_j can be eliminated, with the associated edges connected appropriately.

3. If a region has two edges with identical label and identical orientation with respect to the region, the edges can be connected by a bridge move.

Picture Equivalences

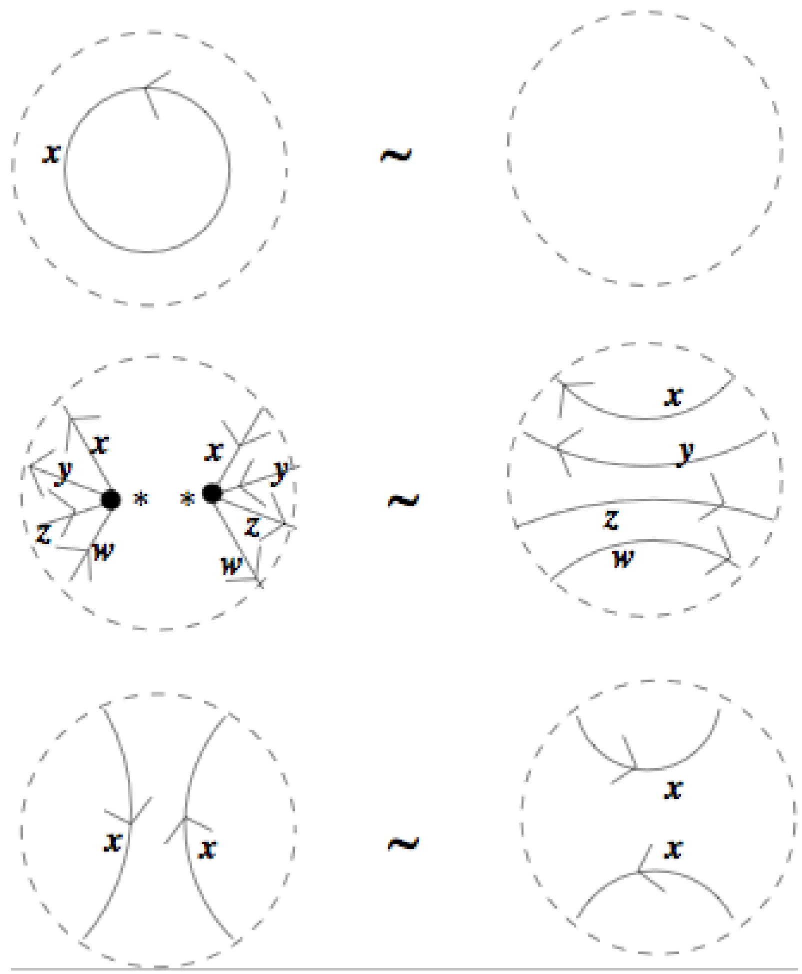

Figure 1: Picture Equivalences

The picture group

$\mathcal{P}/\backsim$ is a group under the operation of disjoint union. We will use $P(\mathcal{X},\mathcal{R})$ to denote the picture group $\mathcal{P}/\backsim$ of the group presentation $\{\mathcal{X},\mathcal{R}\} = G$. The empty picture is easily shown to be the identity element of $P(\mathcal{X},\mathcal{R})$. Furthermore, if P is a picture in $P(\mathcal{X},\mathcal{R})$, taking a planar reflection of P followed by a reversal of the edge orientations yields the inverse of P in $P(\mathcal{X},\mathcal{R})$. A more formal definition of pictures can be found in Bogley and Pride.

A G-action

We can define a G-action on $P(\mathcal{X}, \mathcal{R})$ as follows:

If $x_i \in \mathcal{X}$, the action $x_i P$ is determined by circling P with a simple closed edge labeled by x_i and oriented counterclockwise. Hence, if $g \in G$ and $g = x_{g_1}^{\epsilon_1} x_{g_2}^{\epsilon_2} \cdots x_{g_k}^{\epsilon_k}$, then the action gP will be to circle P with k concentric circles where each circle is appropriately labeled with one of the x_{g_i} and oriented counterclockwise if $\epsilon_i = +1$ and clockwise otherwise. Under this G-action, $P(\mathcal{X}, \mathcal{R})$ is a $\mathbb{Z}[G]$-module.

So what are pictures good for?

Given a presented group $G = \{\mathcal{X}, \mathcal{R}\}$, we would like to construct a cellular complex BG with homotopy type $K(G, 1)$, meaning that $\pi_1(BG) = G$ and all higher homotopy groups for BG are trivial.

Also, given a presented group, we would like to construct a free resolution of the trivial G-module $\mathbb{Z}$.

Homotopical 2-syzygies

In trying to perform this first procedure, we start with a 0-cell, to which we attach 1-cells, one for each generator, yielding the space $X(1)$. We then glue 2-cells onto $X(1)$, one for each relation, yielding the space $X(2)$. The Van Kampen theorem tells us that

$$\pi_1(X(2)) = G$$

but it is possible that $\pi_2(X(2)) \neq 0$. An element of this second homotopy group can be represented by a polytope decomposition of the 2-sphere along with a map $\phi : S^2 \to X(2)$. Such an element is called a *homotopical 2-syzygy* and, if we can determine a complete family of such syzygies, it becomes easy to "fill the holes" in $X(2)$ by attaching 3-cells.

Free Resolutions

On the other hand, the presentation of a group gives rise to the beginning of a free resolution:

$$C_2(G) = \oplus_{\mathcal{R}} \mathbb{Z}[G] \xrightarrow{d_2} C_1(G) = \oplus_{\mathcal{X}} \mathbb{Z}[G] \xrightarrow{d_1} C_0(G) = \mathbb{Z}[G] \xrightarrow{d_0} \mathbb{Z} \longrightarrow$$

where boundary maps are given by the Fox free differential calculus. This free resolution is important for several reasons, not least of which is that it makes calculation of the homology groups easy, as, for any complex C,

$$H_i(C) = \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}).$$

An element of Ker d_2 is called a *homological 2-syzygy* and is, in a sense, a sort of “meta-relation”. The dual use of the term “syzygy” is intentionally suggestive as, in fact, there exists an isomorphism between the homological and homotopical syzygy groups:

$$\pi_2(X(2)) \cong \mathrm{Ker}\ d_2.$$

The connection between homotopical and homological 2-syzygies

Ronnie Brown and Johannes Huebschmann have explored this isomorphism in terms of what are called *identities among relations*, which will be discussed in greater depth later. Kiyoshi Igusa, on the other hand, pioneered the use of what are now called "Igusa's pictures" to study this isomorphism.

In this talk, I follow, to a large degree, the development given by Jean-Louis Loday in constructing explicit maps from homotopical 2-syzygies to pictures, then to identities among relations and finally to homological 2-syzygies represented by the following schema:

$$\begin{array}{ccccccc}
\textit{topological} & \leftrightarrow & \textit{graphical} & \leftrightarrow & \textit{nonabelian group} & \leftrightarrow & \textit{abelian g} \\
\left\{\begin{array}{c}\text{homotopical}\\ \text{2-syzygies}\end{array}\right\} & \leftrightarrow & \left\{\begin{array}{c}\text{Igusa's}\\ \text{pictures}\end{array}\right\} & \leftrightarrow & \left\{\begin{array}{c}\text{identities}\\ \text{among}\\ \text{relations}\end{array}\right\} & \leftrightarrow & \left\{\begin{array}{c}\text{homolo}\\ \text{2-syzy}\end{array}\right. \\
 & \textit{dualization} & & \textit{linearization} & & \textit{abelianization} &
\end{array}$$

Homotopical syzygies: Constructing $X(1)$

Let $X(0) = *$ (a single point, or 0-cell). Then we let

$$X(1) = X(0) \vee \bigvee_{x \in \mathcal{X}} e_x^1$$

the wedge of 1-cells e^1, one per generator.

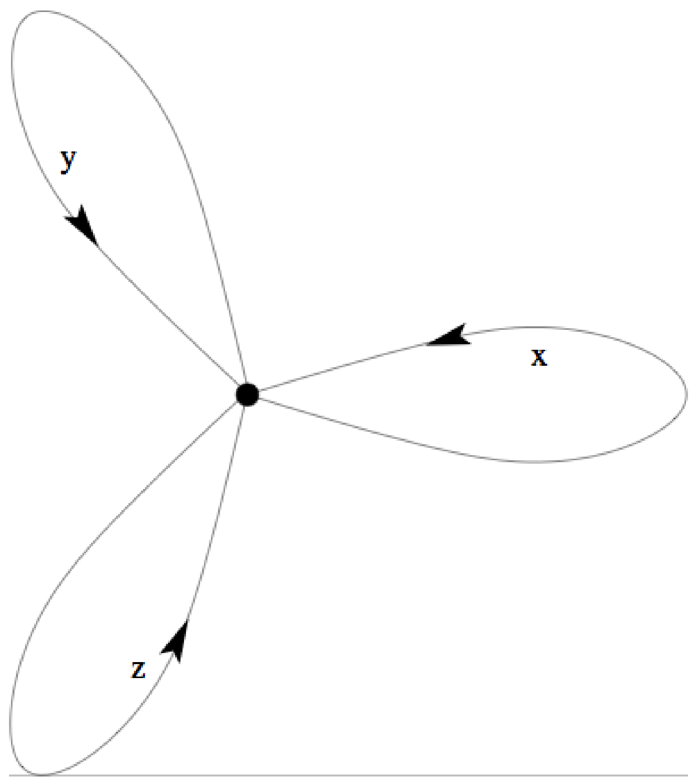

Figure 2: $X(1)$ of $\mathbb{Z}^3$

Homotopical syzygies: Constructing $X(2)$

Now, let

$$X(2) = X(1) \cup \bigcup_{r \in \mathcal{R}} e_r^2,$$

obtained by attaching 2-cells e^2, one per relation.

By the Van Kampen Theorem, the fundamental group of the space $X(2)$ is the group G.

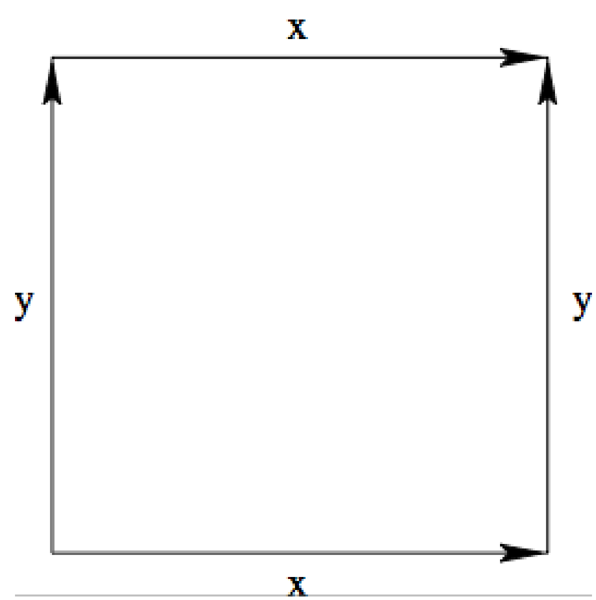

Figure 3: One of the 2-cells

Homotopical syzygies cont.

By definition, a *homotopical 2-syzygy* is a cellular decomposition of the 2-sphere $S^2 = \partial e^3$ together with a cellular map $\phi : S^2 \to X(2)$. Said otherwise, a homotopical 2-syzygy is completely determined by a polytope decomposition of S^2 together with a function from its set of oriented edges to $\mathcal{X}$, such that each face corresponds to a relation or the inverse of a relation.

From $X(2)$ to $X(3)$

A family $\{\phi_p\}_{p\in\mathcal{P}}$ of homotopical 2-syzygies is called *complete* if the homotopy classes $\{[\phi_p]\}_{p\in\mathcal{P}}$ generate $\pi_2(X(2))$. Given such a complete family of homotopical 2-syzygies, we can form

$$X(3) := X(2) \cup \bigcup_{p\in\mathcal{P}} e_p^3$$

by attaching a 3-cell e^3 to $X(2)$ by ϕ_p for each $p \in \mathcal{P}$. It turns out that

$$\pi_1(X(3)) = g \quad , \quad \pi_2(X(3)) = 0.$$

Our use of $\mathcal{P}$ as the index set of a complete family of 2-syzygies is intentionally suggestive, as will now be demonstrated.

From homotopical 2-syzygies to pictures

We get from our homotopical syzygies to the picture group by a process of dualization. Let $\phi : S^2 \to X(2)$ be a syzygy as defined above. When we read the relation defined by a face, we will call the vertex from which we start to be the base-point of the face. We draw a path from the base-point of S^2 to the base-point of each face. Since S^2 is homeomorphic to $\mathbb{R}^2 \cup \{\infty\}$, we can draw this in the plane by choosing a point at infinity. In our example, this yields the following:

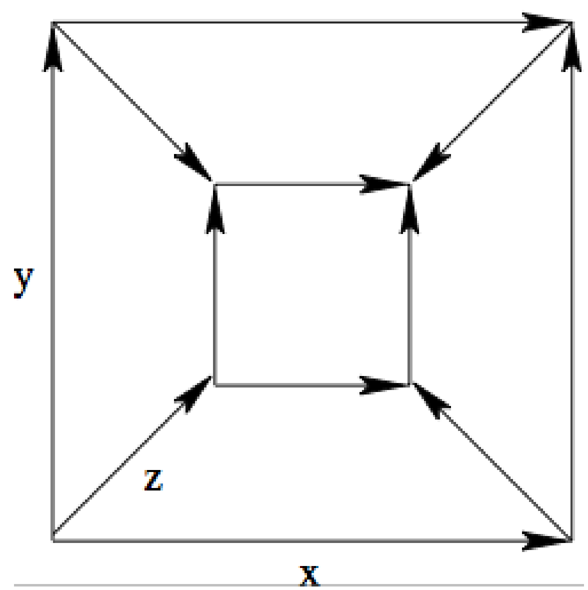

Figure 4:

Dualization

From this, we get an Igusa picture by taking the "Poincare dual".

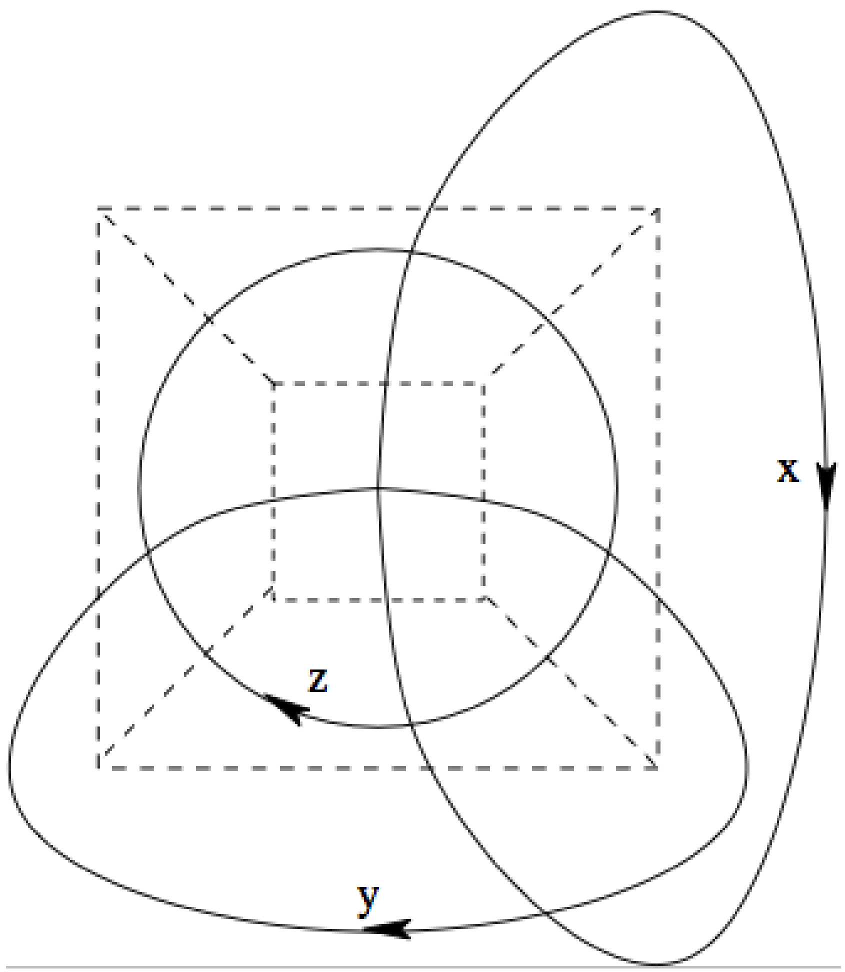

Figure 5: The Poincare dual

The Isomorphism

By passing to the quotient this construction gives an isomorphism of G-modules

$$d : \pi_2(X(2)) \to P.$$

Of course, we could go the other direction and construct a homotopical 2-syzygy out of an Igusa picture by the same kind of dualization.

Identities among relations

Definition 3. An *identity among relations* for $G = \{\mathcal{X}|\mathcal{R}\}$ is a sequence

$$(f_1, r_1; f_2, r_2; \ldots; f_n, r_n) \text{ where } f_i \in F(\mathcal{X}), r_i \in \mathcal{R} \cup \mathcal{R}^{-1}$$

such that

$$f_1 r_1 f_1^{-1} f_2 r_2 f_2^{-1} \cdots f_n r_n f_n^{-1} = 1 \text{ in } F(\mathcal{X}).$$

An Example

In our $\mathbb{Z}^3$ example the Jacobi-Witt-Hall identity

$$[x^y, r_x][y^z, r_y][z^x, r_z] = 1$$

is an identity among relations since it can be re-written:

$$(x^y)^{-1}r_x^{-1}x^y r_x (y^z)^{-1}r_y^{-1}y^z r_y (z^x)^{-1}r_z^{-1}z^x r_z = 1,$$

where $x^y = y^{-1}xy$ and $[-,-]$ is the commutator.

Crossed Modules

To elucidate the connection between identities among relations and pictures and homological 2-syzygies, we use the notion of a crossed module, first defined by Whitehead.

Definition 4. A *crossed module* is a group homomorphism

$$\delta : M \to N$$

with left action of N on M (which we denote by ${}^{n}m$ for $n \in N$ and $m \in M$) satisfying the following:

-δ is N-equivariant for the action of N on itself by conjugation: $\delta({}^{n}m) = nmn^{-1}$.

-the two action of M on itself agree: ${}^{\delta(M)}m' = mm'm^{-1}$.

These actions have the following consequences: Ker δ is abelian and is contained in the center of M. Also, the action of N on M induces a well-defined action of Coker δ on Ker δ which makes it a (Coker δ)-module.

A crossed module of identities among relations

Now, let $Q(G)$ be the free group generated by the (f, r) where $f \in F$ and $r \in \mathcal{R}$, modulo the relation

$$(f, r)(f', r') = (frf^{-1}f', r')(f, r)$$

for any $f, f' \in F$ and $r, r' \in \mathcal{R}$.

$\phi : Q(G) \to F$ is a crossed module

Here, F is acting on $Q(G)$ by ${}^{f}(f', r') := (ff', r')$. The map $\phi : Q(G) \to F$, induced by $\phi(f, r) = frf^{-1}$ is well-defined and F-equivariant. In addition,

$${}^{\phi(f,r)}(f', r') = {}^{frf^{-1}}(f', r') = (frf^{-1}f', r') = (f, r)(f', r')(f, r)^{-1}.$$

In other words, $\phi : Q(G) \to F$ is a crossed module. The image of ϕ is R and so Coker $\phi = F/R = G$. Any identity among relations defines an element $(f_1, r_1) \ldots (f_n, r_n)$ in $Q(G)$ which belongs to Ker ϕ. We define this kernel $I :=$ Ker ϕ to be the *module of identities* of the presented group G. It is a module over G (cf. Brown and Huebschmann).

From pictures to identities among relations

The relationship between pictures and identities among relations is illustrated by a process of linearization. Remember that each picture has a preferred sector (marked by $*$) at each vertex. Choose a point (∞) in the outside face and draw a line (*tail*) from ∞ to $*$ for each vertex such that tails do not intersect. Next, draw a line ($horizon$) from ∞ to ∞ by paralleling the left side of the left-most tail, turning around the vertex, coming back to ∞ along the right side of the tail, then repeating for each successive tail until the last.

Making the horizon horizontal

This picture can then be redrawn up to isotopy such that the horizon is horizontal, putting all the vertices above the horizon. Reading off the generators as they cross the horizon we get a word of the form $f_1 r_1 f_1^{-1} \dots f_n r_n f_n^{-1}$, where $f_i \in F$ and $r_i \in \mathcal{R}$ or $\mathcal{R}^{-1}$. In other words, we get an identity among relations of the form $(f_1, r_1) \dots (f_n, r_n)$.

An Example

In our $\mathbb{Z}^3$ example we see that, by reading on the horizon, we get the Jacobi-Witt-Hall identity:

$$(y^{-1}x^{-1}y, r_x^{-1})(1, r_x)(z^{-1}y^{-1}z, r_y^{-1})(1, r_y)(x^{-1}z^{-1}x, r_z^{-1})(1, r_z).$$

The isomorphism between pictures and identities among relations

Despite the choice made in obtaining an identity among relations from a picture, this construction yields a well-defined map when passing to the quotients:

Proposition 1. *The linearization construction described above induces a well-defined G-module isomorphism $l : P \cong I$.*

Linearization is well-defined

First, we show this linearization is well-defined. To show independence of choice of tails, it is sufficient to examine what happens when we interchange two adjacent tails. Suppose the i-th and the $(i+1)$-st tails are so interchanged. Then $(f_i, r_i)(f_{i+1}, r_{i+1})$ is changed to $(f_i r_i f_i^{-1} f_{i+1}, r_{i+1})(f_i, r_i)$. By the relation of definition for $Q(G)$, these are equal.

If we change the point at infinity, then the identity is changed by conjugation by an element of $Q(G)$. However, since Ker ϕ is in the center of $Q(G)$, this does not take the identity out of its original equivalence class in $Q(G)$.

Compatibility with the equivalence relation on pictures

Let us show the choice of tails is compatible with relations (1), (2) and (3) on the pictures (see slide 5). The change in identity induced by (1) will modify only some f_i by changing xx^{-1} to 1, so this is acceptable. For (2) there is an obvious way to choose the tails such that the identity is unchanged. For (3), suppose a tail passes between the two vertices. Before performing (3), we can simply interchange this tail with the tail of one of the vertices, leaving the identity unchanged.

Equivalent pictures yield equivalent identities

Now, let us show that two equivalent pictures give equivalent identities. With regard to axiom (1) it is clear it does not affect the element $\Pi_i(f_i, r_i)$. For axiom (2), we consider a tail crossing the two edges in the left picture. It does not cross anything in the right picture. The contribution in the left case is merely xx^{-1}, so axiom (2) is fulfilled. For (3) we need only examine the case where two consecutive vertices are labeled r and r^{-1} respectively. This contributes rr^{-1} to the identity for the left picture and 1 for the right.

Going the other direction

A similar manipulation allows us to construct a picture from an identity among relations:

-write the word $f_1 r_1 f_1^{-1} \ldots f_n r_n f_n^{-1}$ on the horizon,

-since the value of the word is 1 in F, cancellation allows us to draw edges below the horizon connecting all the generators of the word,

-put one vertex per relation above the horizon and draw the appropriate edges (with preferred sector)

Independence of choices

This gives a picture. Now, we need to show that the class of picture in P does not depend on our choices and that the defining relation of $Q(G)$ holds. The relation $(f, r)(f, r)^{-1} = 1$ in $Q(G)$ holds in the picture thanks to (a):

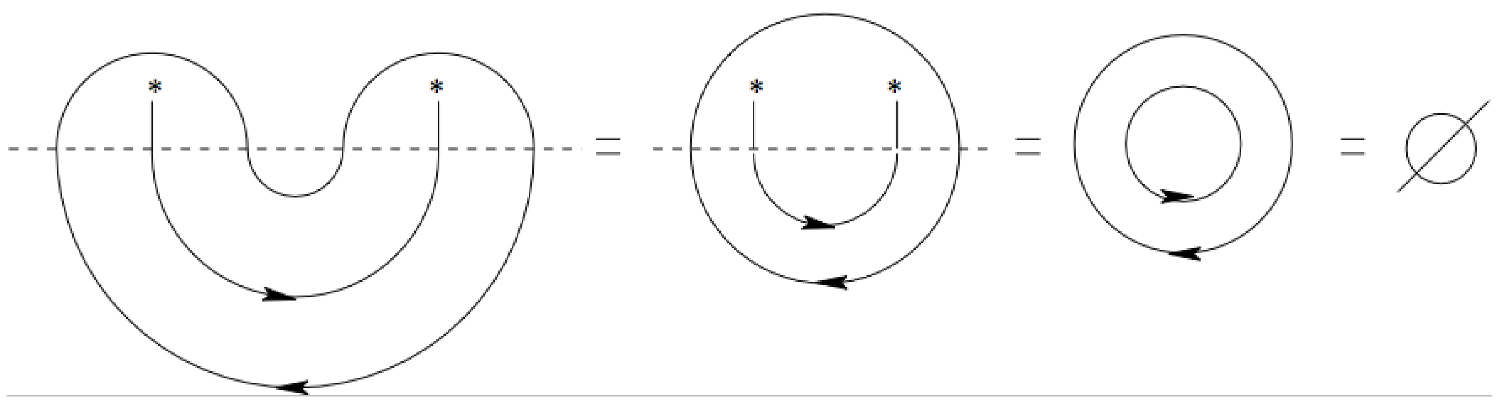

Figure 6: Picture Equivalences

Independence of choices (cont.)

The choices made in the cancellation procedure are irrelevant due to (b). The defining relation of $Q(G)$ stands thanks to (b) and (c):

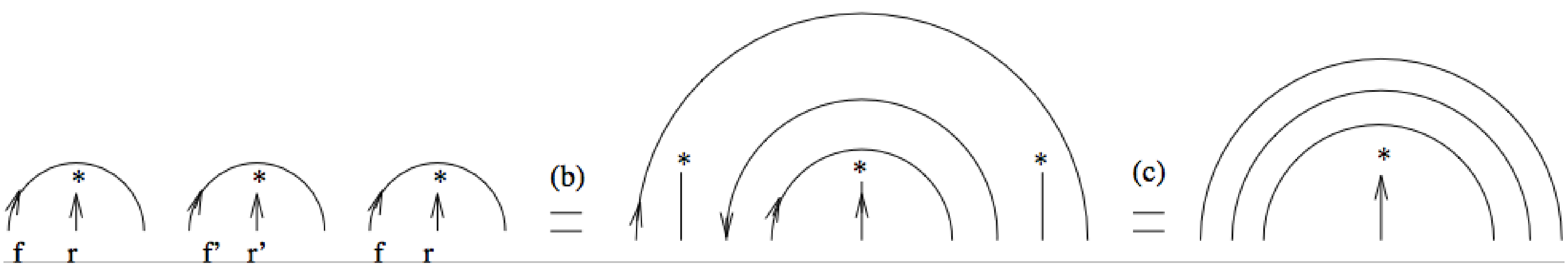

Figure 7: Picture Equivalences

Finally, the map is a G-module as a result of (a) and (b).

Conclusion

This construction is clearly inverse to the linearization construction; hence, we have a G-module isomorphism:

$$l : P \cong I.$$

Homological 2-syzygies

The presentation $\{\mathcal{X}|\mathcal{R}\}$ of G gives rise to the beginning of a free resolution of the trivial G-module $\mathbb{Z}$ as follows. Let

$$C_0(G) := \mathbb{Z}[G].[1], \;\; C_1(G) := \oplus_{x\in\mathcal{X}} \mathbb{Z}[G].[x], \;\; C_2(G) = \oplus_{r\in\mathcal{R}} \mathbb{Z}[G].[r]$$

be free modules over G. We define G-maps $d_0 : C_0(G) \to \mathbb{Z}, d_1 : C_1(G) \to C_0(G)$ and $d_2 : C_2(G) \to C_1(G)$ such that:

$$\begin{aligned} d_0([1]) &= 1, \\ d_1([x]) &= (1-x)[1], \\ d_2(x_1x_2\ldots x_k) &= [x_1] + x_1[x_2] + \ldots + x_1x_2\cdots x_{k-1}[x_k], \end{aligned}$$

with the convention that if $x_i = x^{-1}$, where x is a generator, then $[x^{-1}] = -x^{-1}[x]$, and $[1] = 0$ in $C_1(G)$.

Free resolutions and homological 2-syzygies

Then it is well-known that

$$C_2(G) = \bigoplus_{\mathcal{R}} \mathbb{Z}[G] \xrightarrow{d_2} C_1(G) = \bigoplus_{\mathcal{X}} \mathbb{Z}[G] \xrightarrow{d_1} C_0(G) = \mathbb{Z}[G] \xrightarrow{d_0} \mathbb{Z} \longrightarrow$$

is the beginning of a free resolution of $\mathbb{Z}$.

Definition 5. An element of Ker d_2 is called a *homological 2-syzygy*.

An Example

In our $\mathbb{Z}^3$ example we get, for instance, $d_2 r_x = (1-x).[y] + (y-1).[z]$, and

$$(1-x)[r_x] + (1-y)[r_y] + (1-z)[r_z]$$

is a homological 2-syzygy which generates the free G-module Ker $d_2 \cong \mathbb{Z}$.

From identities among relations to homological 2-syzygies

The relationship between identies and homological syzygies is established by means of abelianization. The abelianization of $Q(G)$ is the free abelian group on $G \times \mathcal{R}$, $\mathbb{Z}[G][\mathcal{R}]$. If we denote by $\bar{f}$ the image in G of the element $f \in F$, then the map $Q(G) \twoheadrightarrow Q(G)_{ab} = \mathbb{Z}[G][\mathcal{R}]$ is induced by $(f, r) \mapsto \bar{f}.[r]$.

Defining a map

Define a set-map ∂ from F to $\mathbb{Z}[G][\mathcal{X}]$ by the following:

-$\partial x = [x]$ if x is a generator,

-$\partial(uv) = \partial u + \bar{u}.\partial v$, for $u, v \in F$.

Then we see immediately that $\partial 1 = 0$ and $\partial x^{-1} = -\bar{x}^{-1}[x]$, when x is a generator (Fox derivative calculus).

Whitehead's Lemma

Lemma [Whitehead] 2. *The following is a commutative diagram:*

$$\begin{array}{ccc} Q(G) & \xrightarrow{\phi} & F \\ {\scriptstyle (-)_{ab}}\downarrow & & \downarrow{\scriptstyle \partial} \\ Q(G)_{ab} = \mathbb{Z}[G][\mathcal{R}] & \xrightarrow{d_2} & \mathbb{Z}[G][\mathcal{X}] \end{array}$$

Proof of Whitehead's Lemma

Proof. On one hand, we see:

$$\begin{aligned}\partial\phi(f,r) = \partial(frf^{-1}) &= \partial f + \bar{f}\partial(r) + \overline{fr}\partial f^{-1} \\ &= \partial f + \bar{f}\partial(r) - \overline{frf^{-1}}\partial f \\ &= \partial f + \bar{f}\partial(r) - \partial f \\ &= \bar{f}\partial(r).\end{aligned}$$

On the other hand, we see:

$$d_2((f,r)_{ab}) = d_2(\bar{f}.[r]) = \bar{f}(d_2[r]).$$

Hence, we need only check that $d_2[r] = \partial r$ for $r \in \mathcal{R}$. For $r = x_1 \cdots x_n$, where x_i is a generator or the inverse of a generator, we have:

$$\partial(x_1 \cdots x_n) = [x_1] + x_1[x_2] + \ldots + x_1 \cdots x_{n-1}[x_n] = d_2[r].$$

□

Brown and Huebschmann's Proposition

From the above lemma, we perceive a map

$$a : \text{Ker } \phi \to \text{Ker } d_2.$$

Proposition [Brown and Huebschmann] 3. *Abelianization of $Q(G)$ induces a map*

$$a : I = \textit{Ker } \phi \to \textit{Ker } d_2$$

which is an isomorphism of G-modules.

From homotopical to homological syzygies

The composition

$$\pi_2(X(2)) \xrightarrow{d} P \xrightarrow{l} I \xrightarrow{a} \text{Ker } d_2$$

from homotopical to homological syzygies is simply the Hurewicz map applied to the universal cover of $X(2)$:

$$\pi_2(X(2)) \cong \pi_2(\widetilde{X(2)}) \to H_2(\widetilde{X(2)}) \cong \text{Ker } d_2$$

since the chain complex of $\widetilde{X(2)}$ is precisely the chain complex $C_2 \to C_1 \to C_0$ described above.

The crossed module $\phi : Q(G) \to F$ is isomorphic to the crossed module in Brown and Huebschmann. The composite $d \circ l \circ a$ is precisely the algebraic map constructed in Brown and Huebschmann's paper, where it is shown to coincide with the Hurewicz map.

Other uses for Pictures: Knots

Let K be a link and let D be an oriented diagram for K with n arcs labeled $x_1, x_2, \ldots, x_n$ and m crossings labeled $c_1, c_2, \ldots, c_m$. The *Wirtinger presentation* of $\pi_1(S^3 \backslash K)$ is $\langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ where r_i is the relation obtained from c_i by choosing the preferred sector of c_i such that the first two letters of r_i are elements of $\{x_1, \ldots, x_n\}$. Clearly, $D \in P(\mathcal{X}, \mathcal{R})$, but, in fact, D generates $P(\mathcal{X}, \mathcal{R})$:

Theorem 4. *Let D be an oriented labeled knot diagram for a knot K and let $G = \{\mathcal{X}, \mathcal{R}\}$ be the Wirtinger Presentation of $\pi_1(S^3 \backslash K)$ given by D. Then each picture $P \in P(\mathcal{X}, \mathcal{R})$ is of the form $\sum_{g \in G} n_g g D$ where $n_g \in \mathbb{Z}$. In other words, the picture D of the link K generates the picture group of $P(X, R)$.*

This proposition is equivalent to Papakyriakopoulos' Asphericity Theorem.

Making a picture from the Trefoil knot

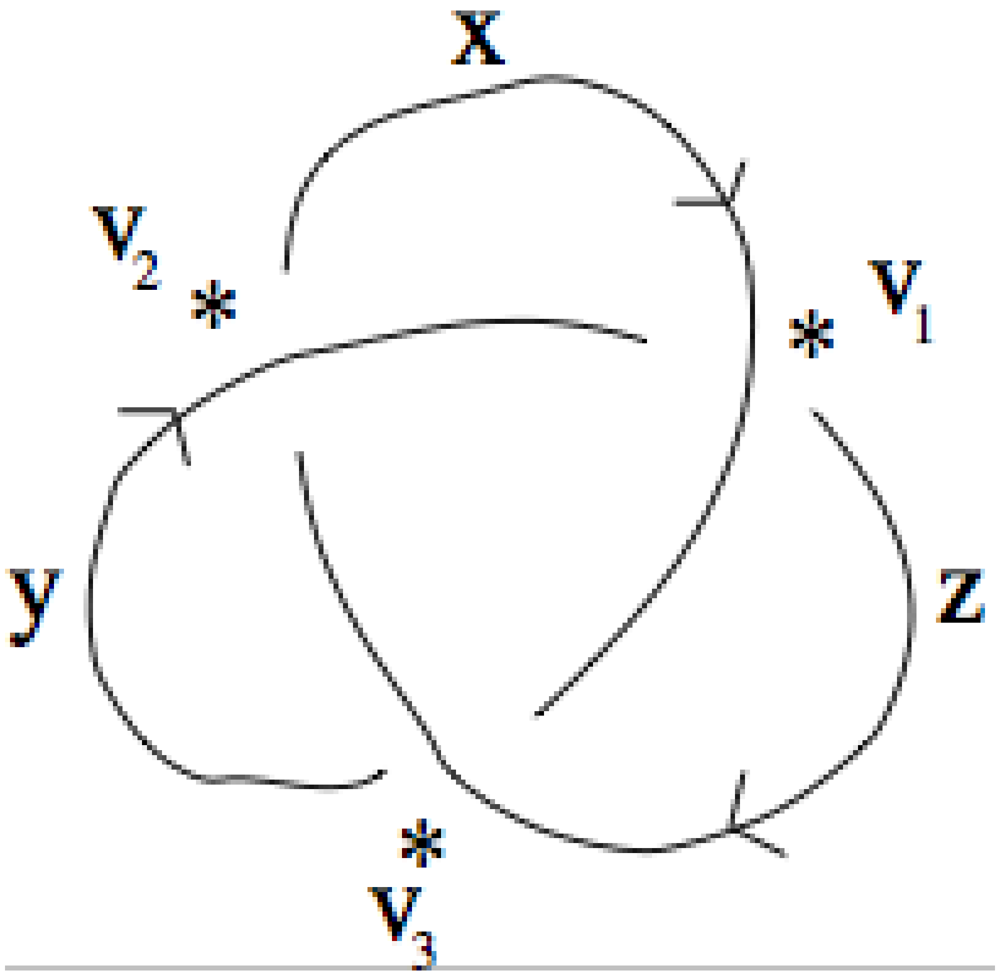

Figure 8: The Trefoil Knot