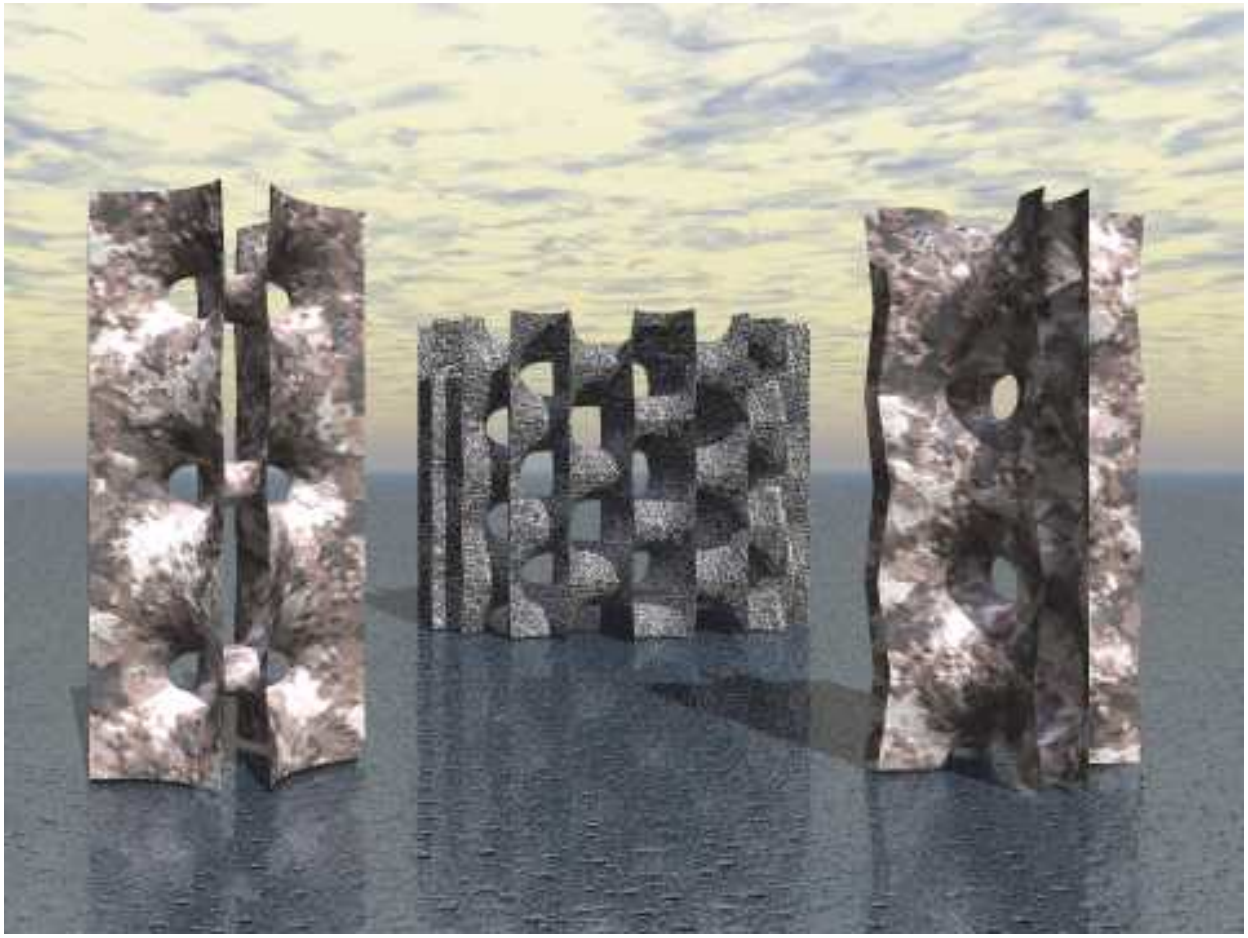
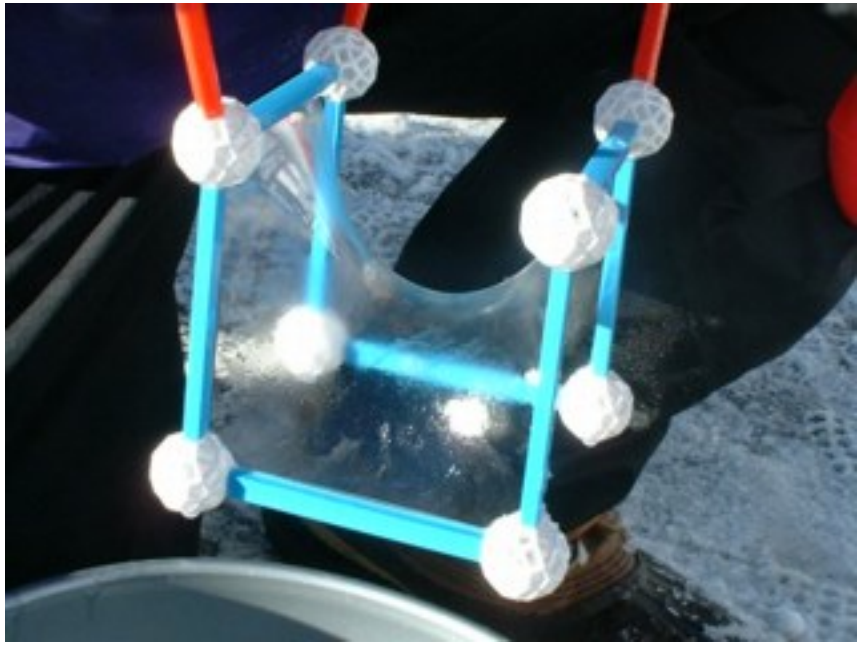
Minimal Surfaces



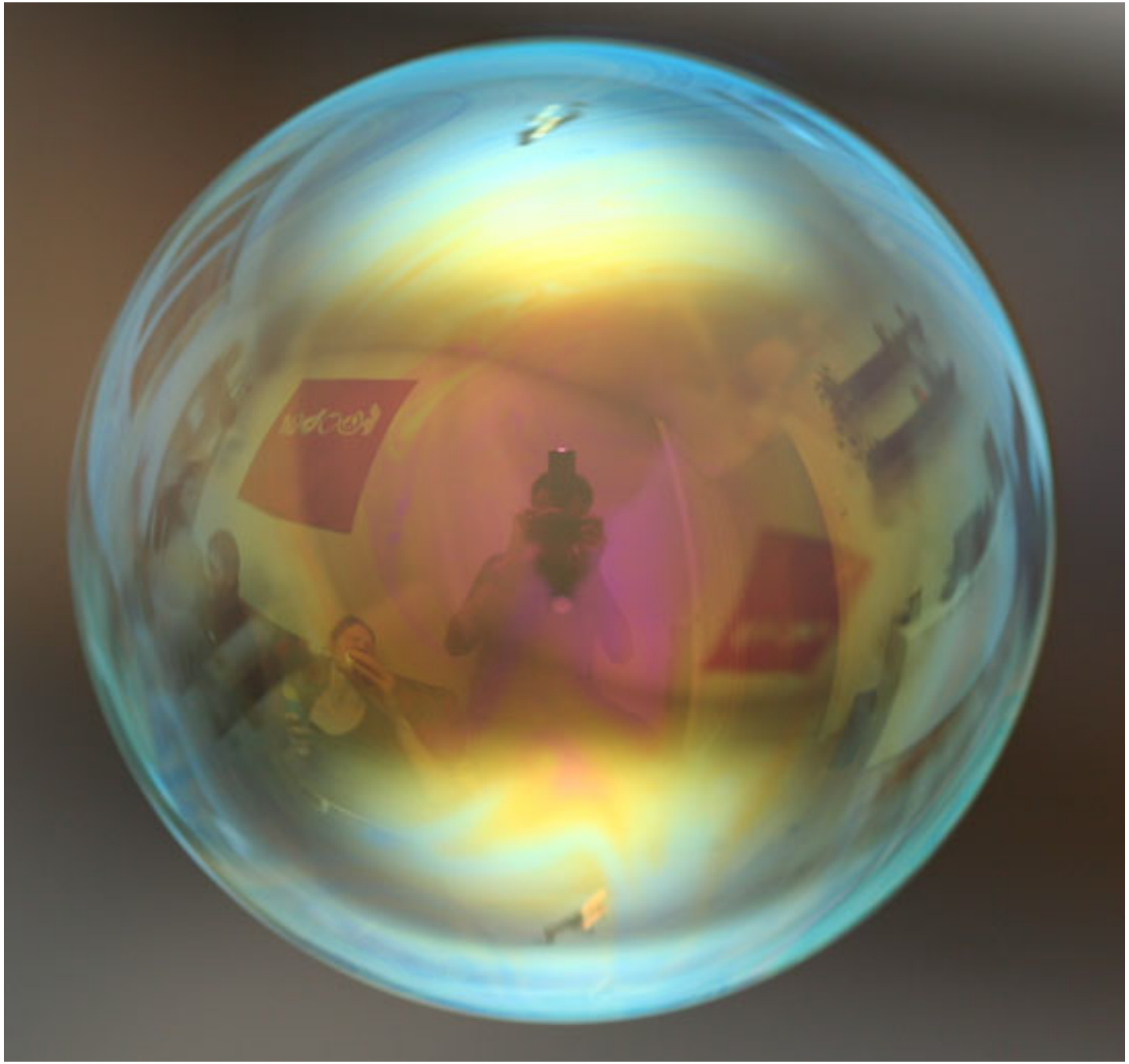
Clay Shonkwiler

October 18, 2006

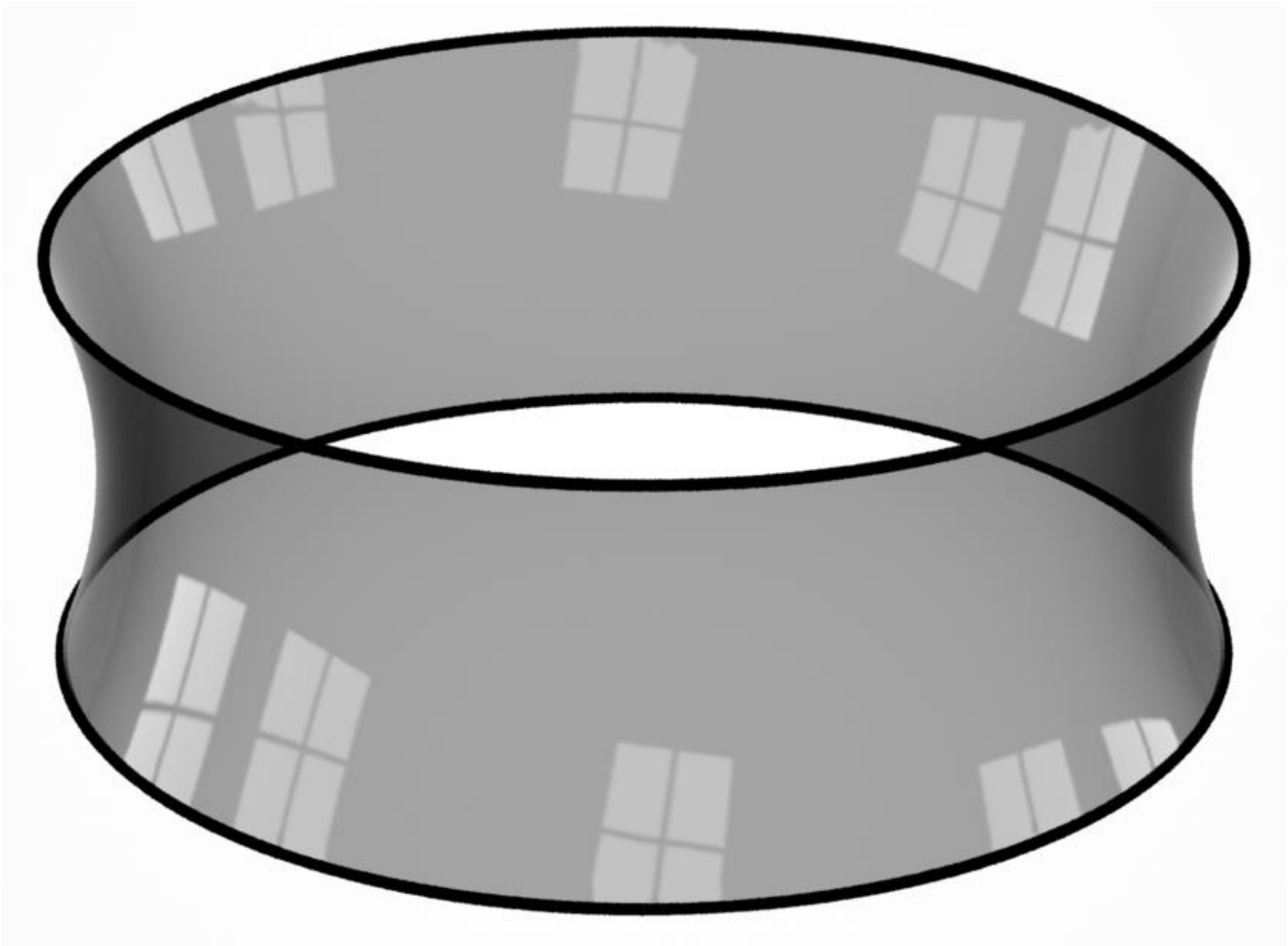
Soap Films



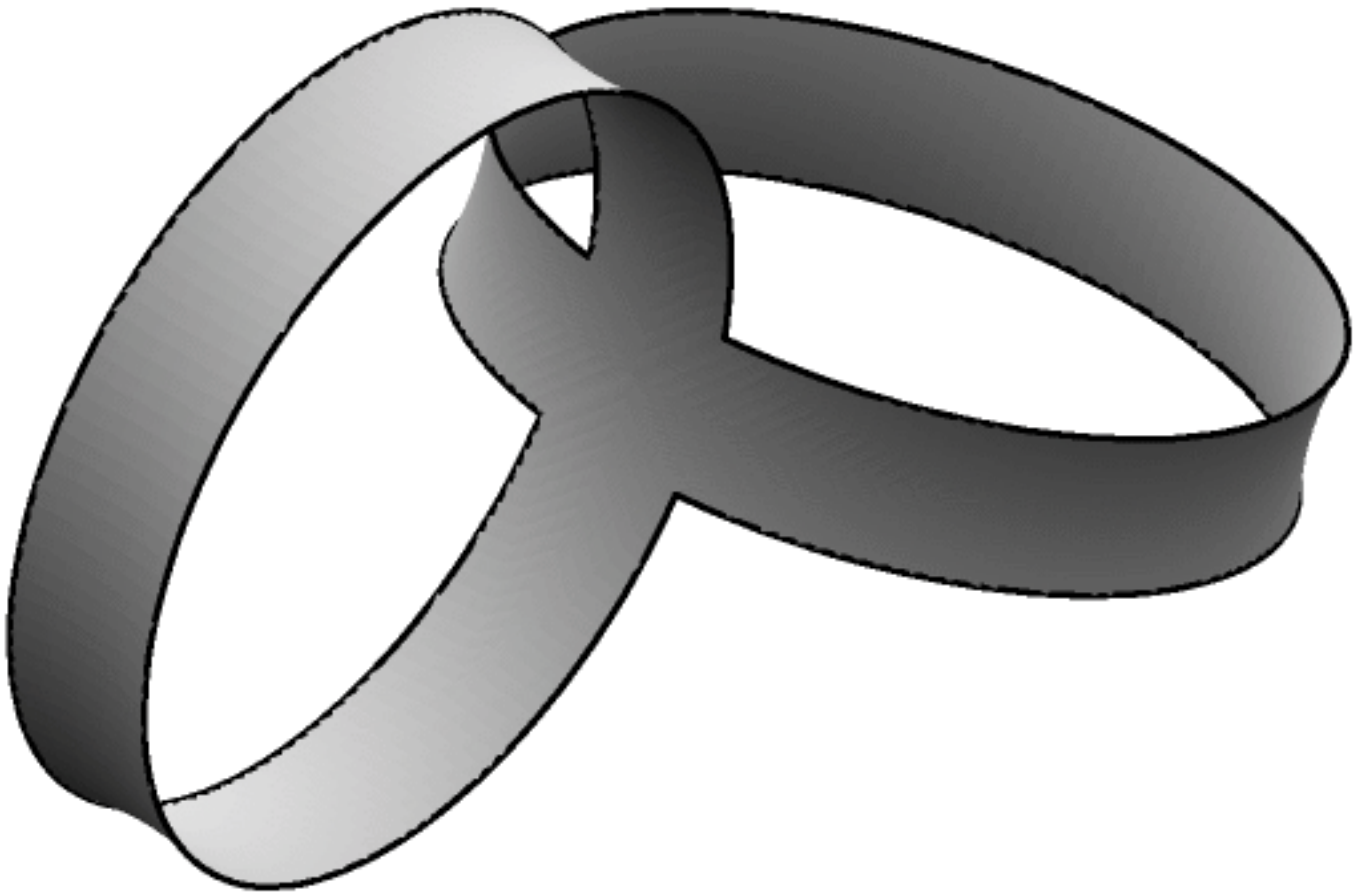
A soap film seeks to minimize its surface energy, which is proportional to area. Hence, a soap film achieves a minimum area among all surfaces with the same boundary.



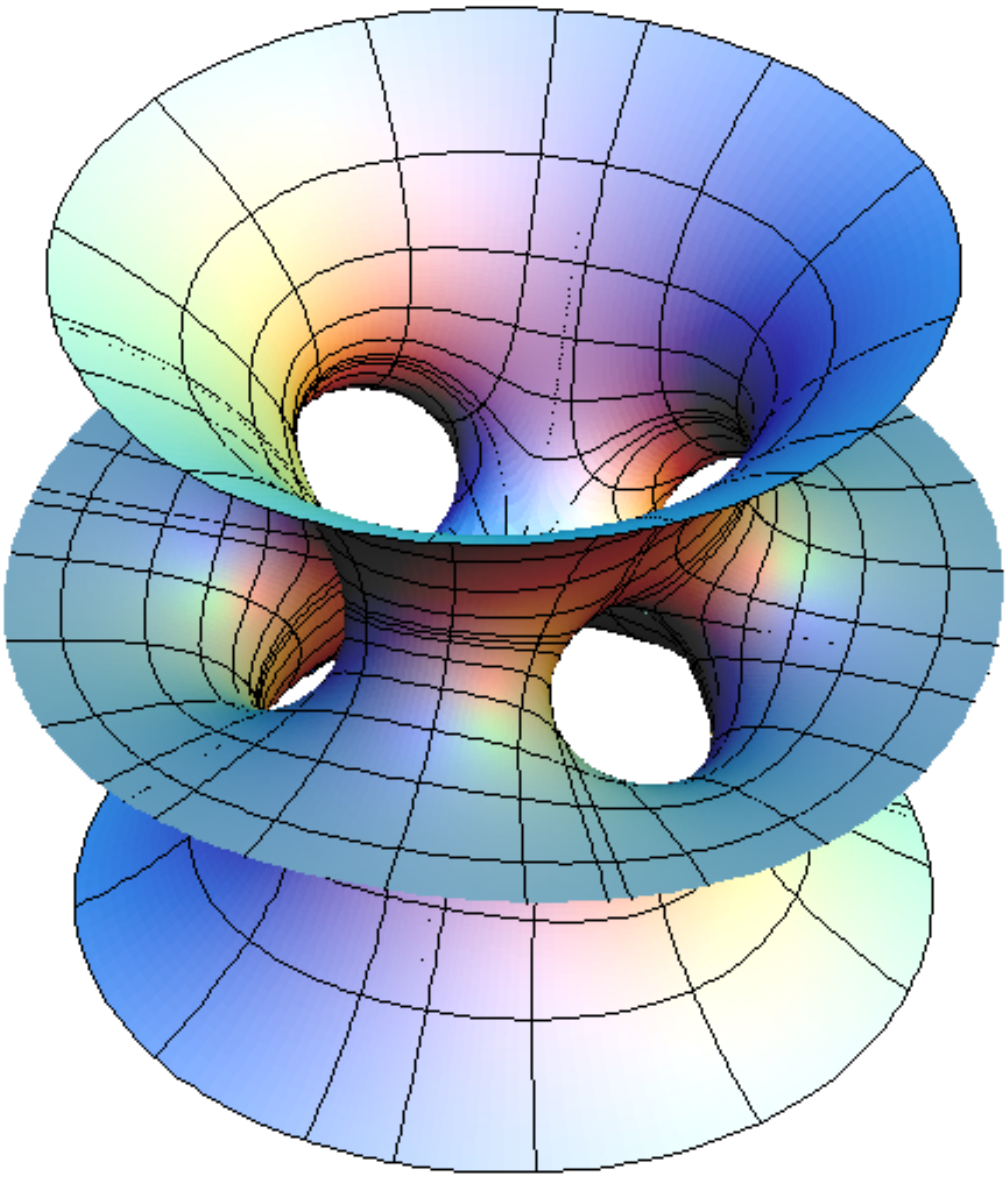
Computer-generated soap film



Double catenoid



Costa-Hoffman-Meeks Surface



Quick Background on Surfaces

Recall that, if $S \subset \mathbb{R}^3$ is a regular surface, the inner product $\langle\ ,\ \rangle_p$ on T_pS is a symmetric bilinear form. The associated quadratic form

$$I_p : T_pS \to \mathbb{R}$$

given by

$$I_p(W) = \langle W, W \rangle_p$$

is called the *first fundamental form* of S at p.

Quick Background on Surfaces

Recall that, if $S \subset \mathbb{R}^3$ is a regular surface, the inner product $\langle\ ,\ \rangle_p$ on T_pS is a symmetric bilinear form. The associated quadratic form

$$I_p : T_pS \to \mathbb{R}$$

given by

$$I_p(W) = \langle W, W \rangle_p$$

is called the *first fundamental form* of S at p.

If $X : U \subset \mathbb{R}^2 \to V \subset S$ is a local parametrization of S with coordinates (u, v) on U, then the vectors X_u and X_v form a basis for T_pS with $p \in V$. The first fundamental form I is completely determined by the three functions:

$$E(u, v) = \langle X_u, X_u \rangle$$
$$F(u, v) = \langle X_u, X_v \rangle$$
$$G(u, v) = \langle X_v, X_v \rangle$$

Area

Proposition 1. *Let $R \subset S$ be a bounded region of a regular surface contained in the coordinate neighborhood of the parametrization $X : U \to S$. Then*

$$\int_{X^{-1}(R)} \|X_u \times X_v\| du dv = \text{Area}(R).$$

Area

Proposition 1. *Let $R \subset S$ be a bounded region of a regular surface contained in the coordinate neighborhood of the parametrization $X : U \to S$. Then*

$$\int_{X^{-1}(R)} \|X_u \times X_v\| du dv = \text{Area}(R).$$

Note that

$$\|X_u \times X_v\|^2 + \langle X_u, X_v \rangle^2 = \|X_u\|^2 \|X_v\|^2,$$

so the above integrand can be written as

$$\|X_u \times X_v\| = \sqrt{EG - F^2}.$$

The Gauss Map and curvature

At a point $p \in S$, define the normal vector to p by the map $N : S \to S^2$ given by

$$N(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}(p).$$

This map is called the *Gauss map*.

The Gauss Map and curvature

At a point $p \in S$, define the normal vector to p by the map $N : S \to S^2$ given by

$$N(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}(p).$$

This map is called the $Gauss\ map$.

Remark: The differential $dN_p : T_pS \to T_pS^2 \simeq T_pS$ defines a self-adjoint linear map.

The Gauss Map and curvature

At a point $p \in S$, define the normal vector to p by the map $N : S \to S^2$ given by

$$N(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}(p).$$

This map is called the *Gauss map*.

Remark: The differential $dN_p : T_pS \to T_pS^2 \simeq T_pS$ defines a self-adjoint linear map. Define

$$K = \det dN_p \text{ and } H = \frac{-1}{2}\text{trace } dN_p,$$

the *Gaussian curvature* and *mean curvature*, respectively, of S.

The Gauss Map and curvature

At a point $p \in S$, define the normal vector to p by the map $N : S \to S^2$ given by

$$N(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}(p).$$

This map is called the *Gauss map*.

Remark: The differential $dN_p : T_pS \to T_pS^2 \simeq T_pS$ defines a self-adjoint linear map. Define

$$K = \det dN_p \text{ and } H = \frac{-1}{2}\text{trace } dN_p,$$

the *Gaussian curvature* and *mean curvature*, respectively, of S.

Note: Since dN_p is self-adjoint, it can be diagonalized:

$$dN_p = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

k_1 and k_2 are called the *principal curvatures* of S at p and the associated eigenvectors are called the *principal directions*.

Minimal Surfaces

Definition 1. A regular surface $S \subset \mathbb{R}^3$ is called a *minimal surface* if its mean curvature is zero at each point.

Minimal Surfaces

Definition 1. A regular surface $S \subset \mathbb{R}^3$ is called a *minimal surface* if its mean curvature is zero at each point.

If S is minimal, then, when dN_p is diagonalized,

$$dN_p = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

Minimal Surfaces

Definition 1. A regular surface $S \subset \mathbb{R}^3$ is called a *minimal surface* if its mean curvature is zero at each point.

If S is minimal, then, when dN_p is diagonalized,

$$dN_p = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

If we give S^2 the opposite orientation (*i.e.* choose the inward normal instead of the outward normal), then

$$dN_p = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so N is a conformal map.

The Catenoid

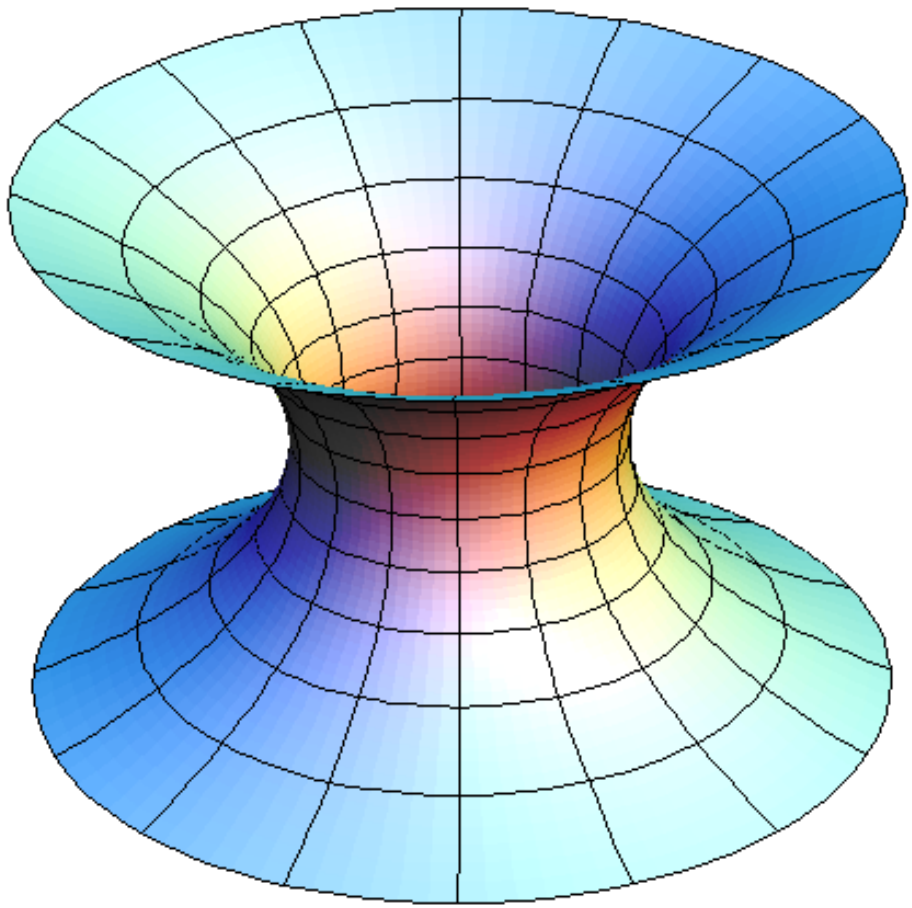


Figure 1: The catenoid is a minimal surface

The helicoid

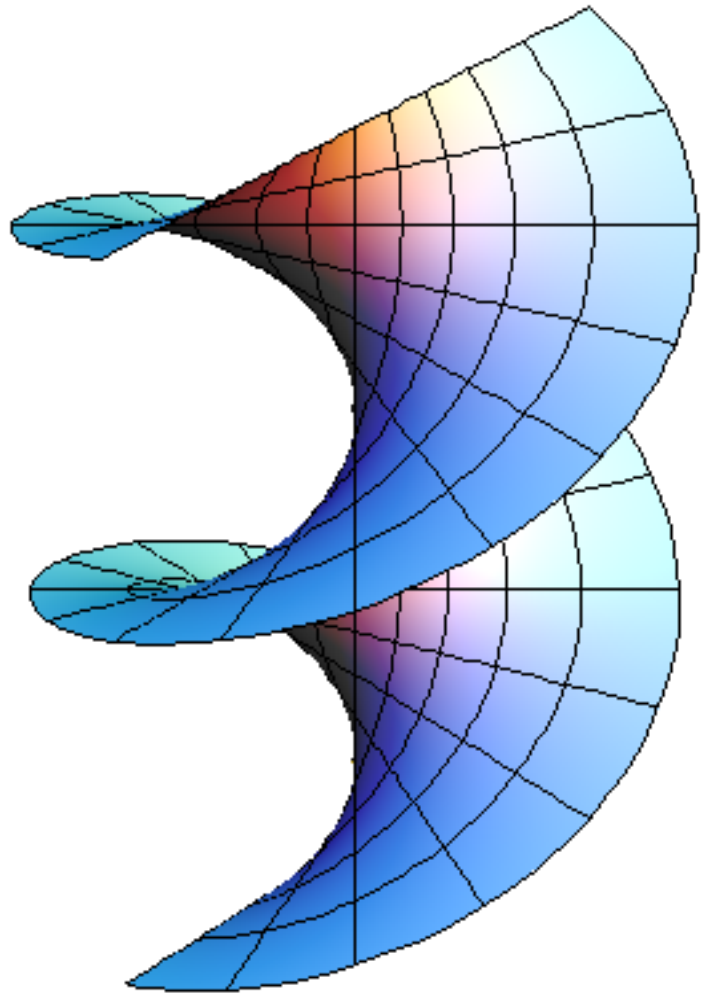
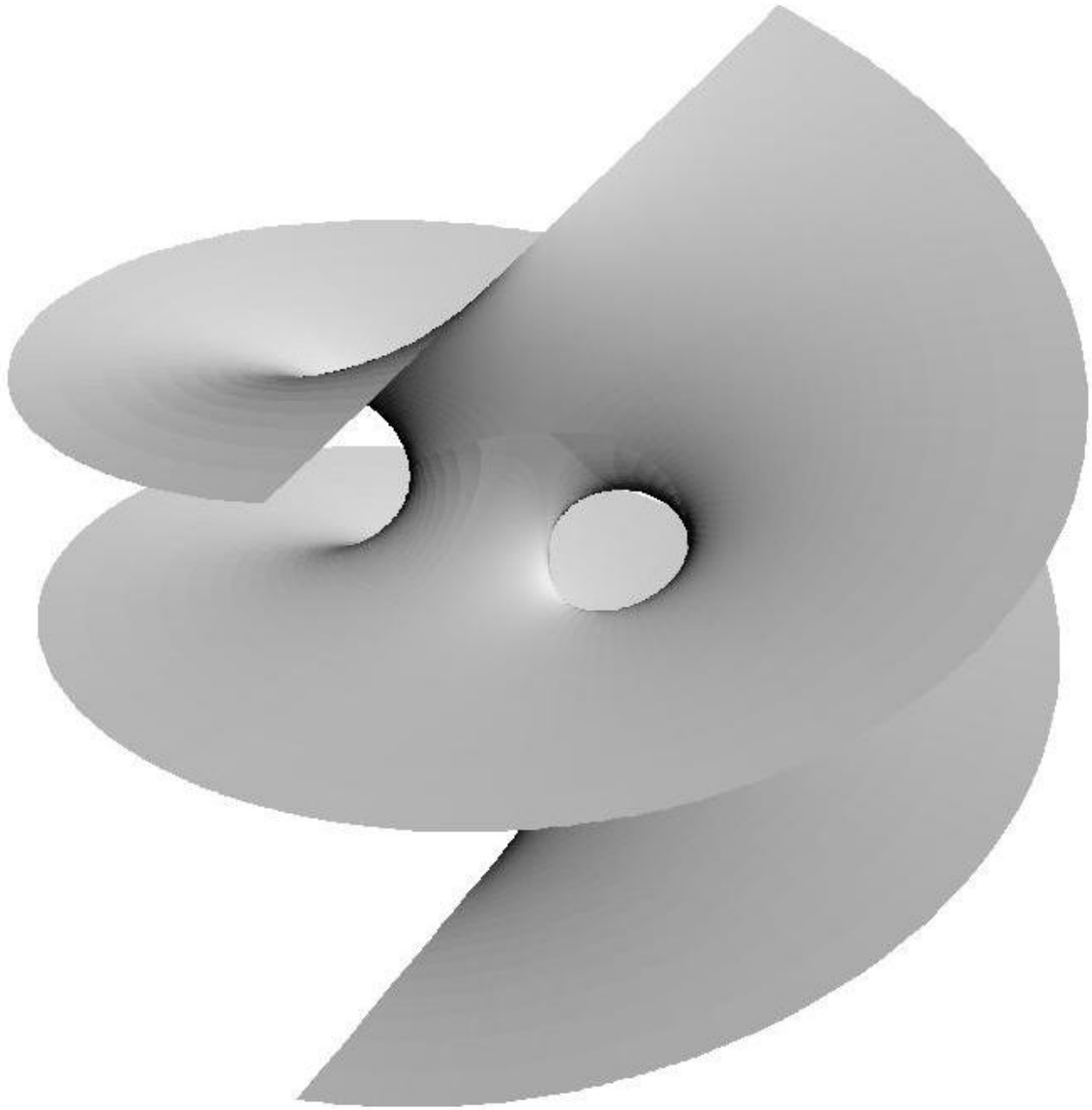


Figure 2: The helicoid is a minimal surface as well

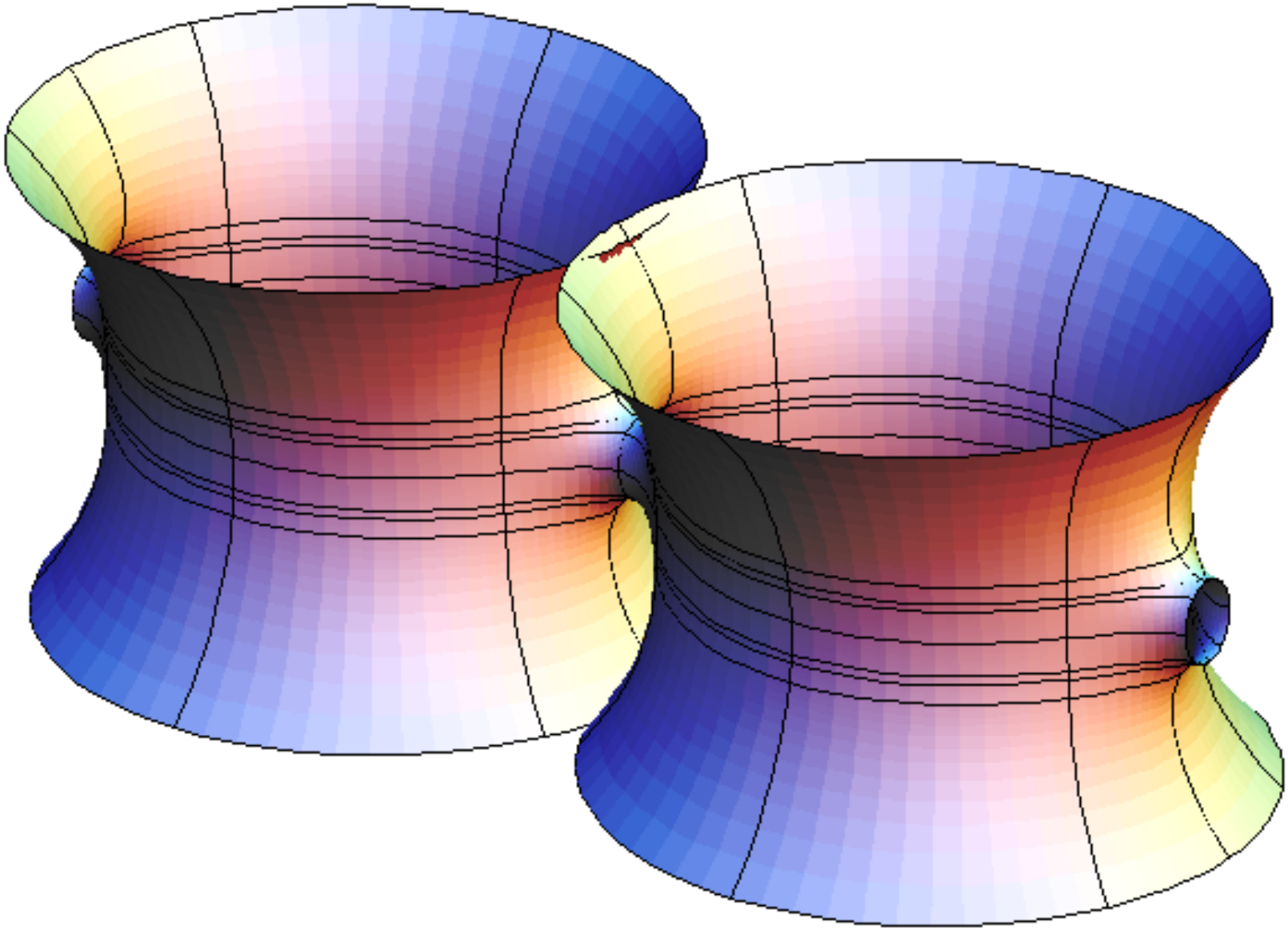
The helicoid and the catenoid are locally isometric

Helicoid with genus

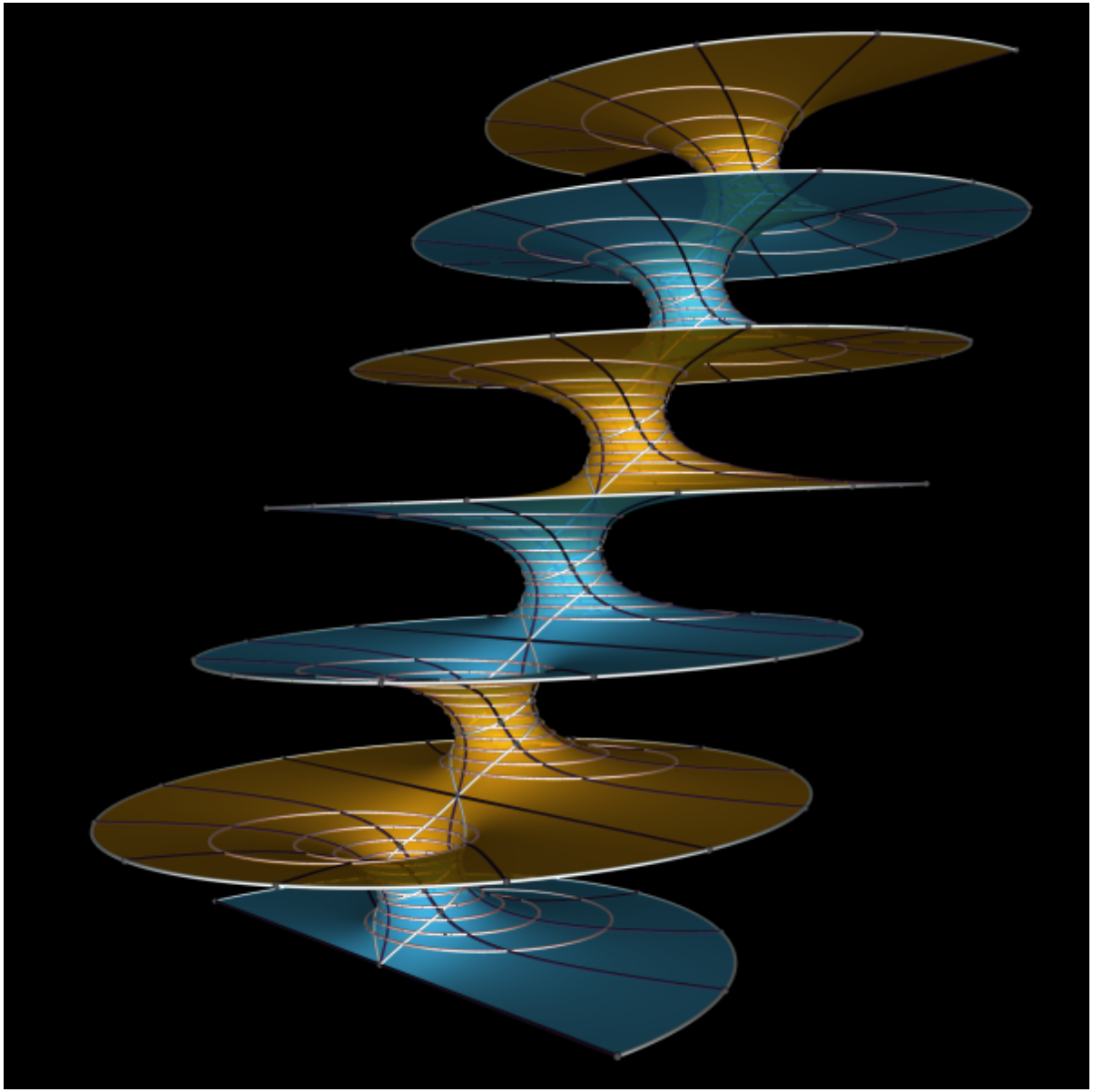


Helicoid with genus

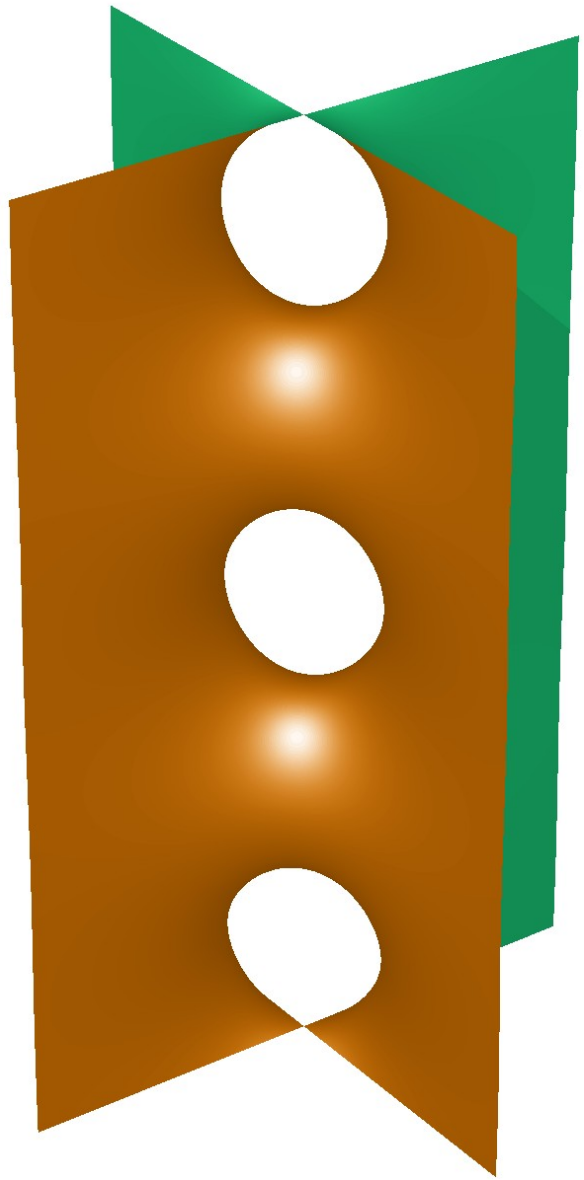
Catenoid Fence



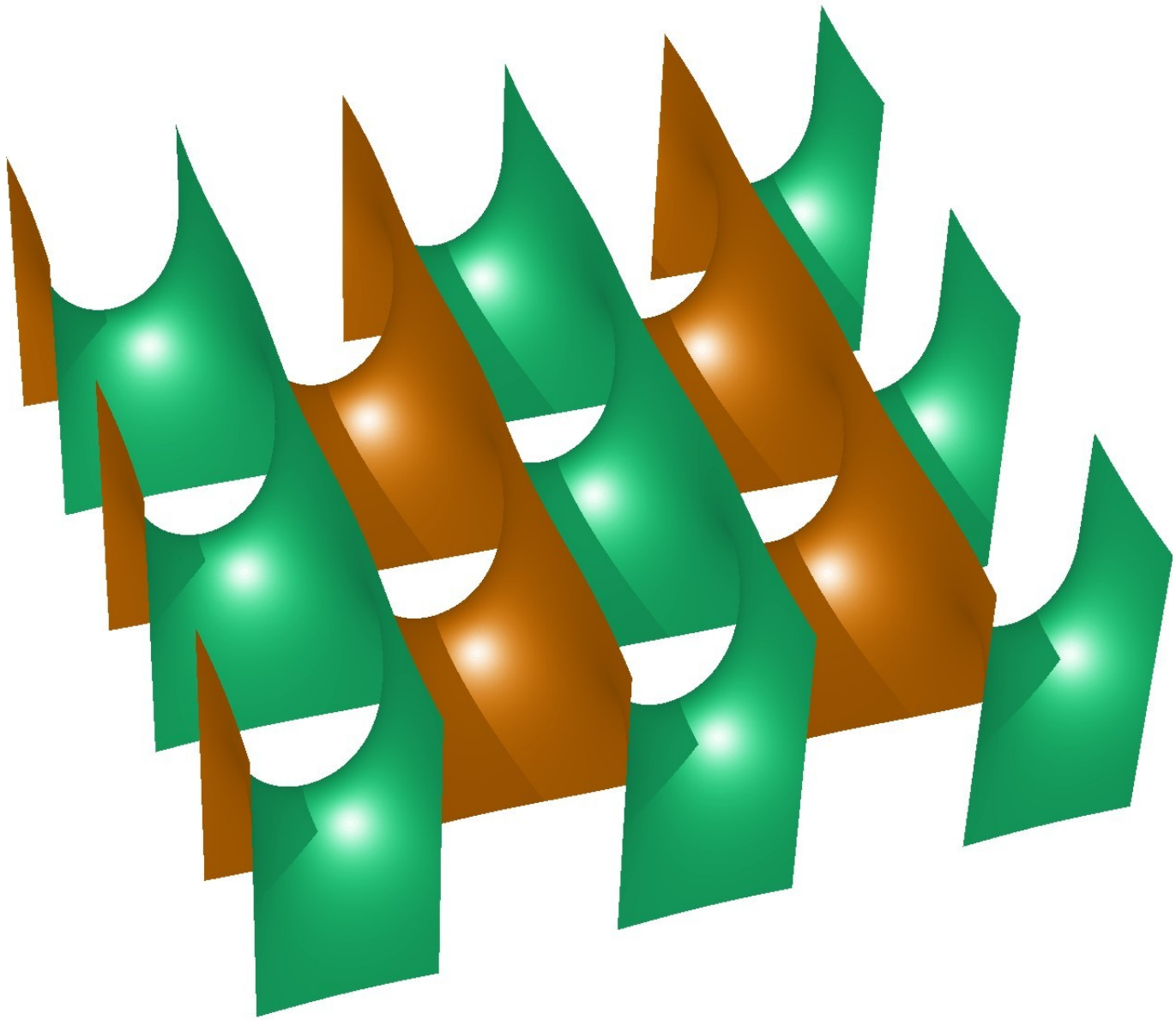
Riemann's minimal surface



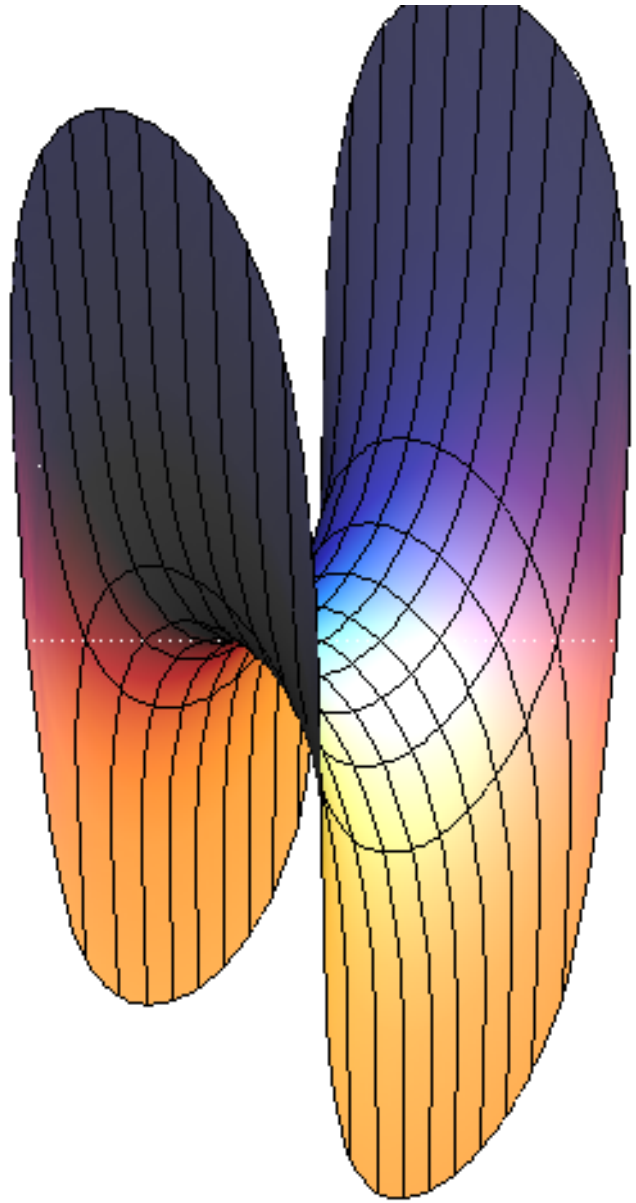
Singly-periodic Scherk surface



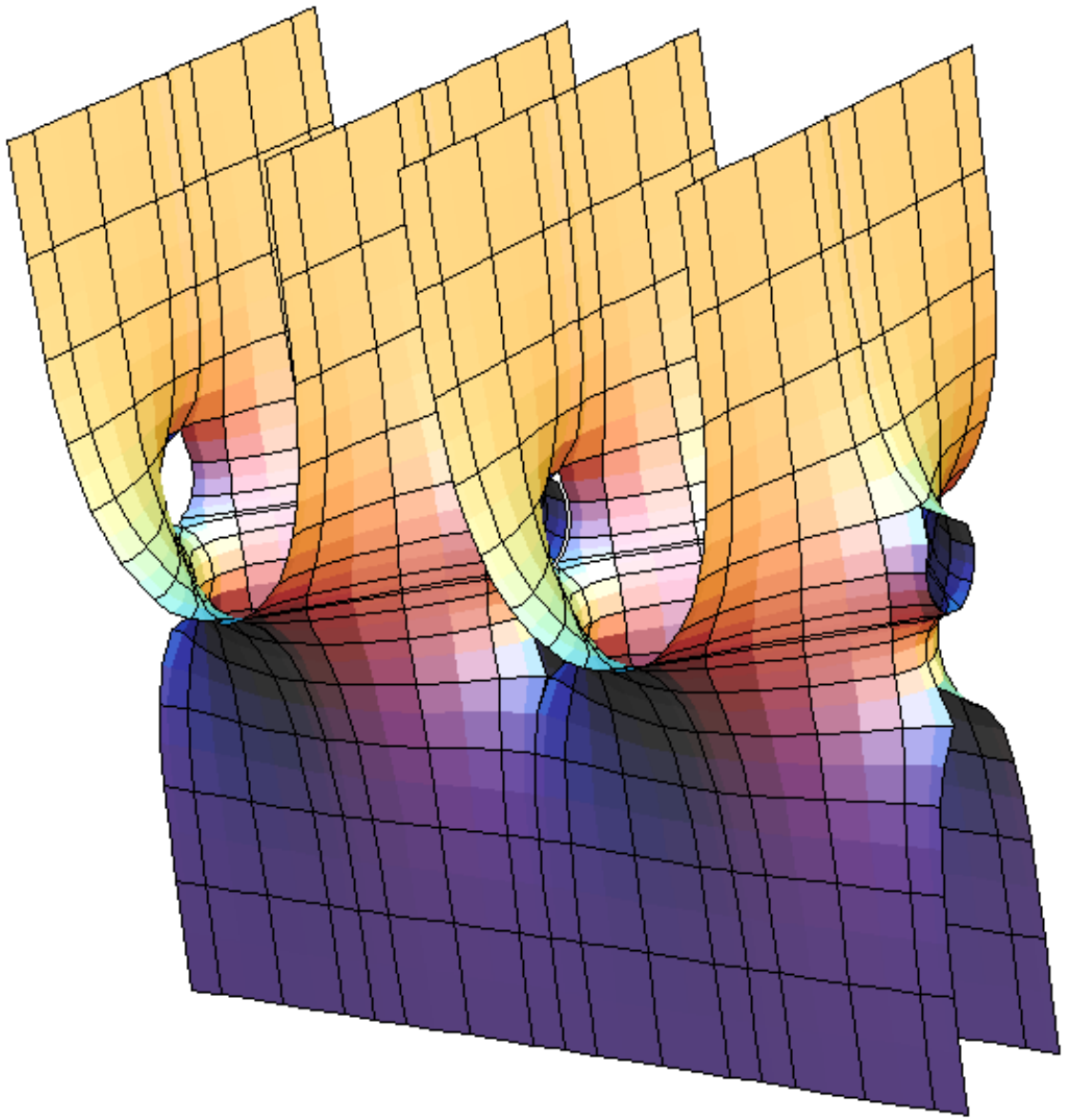
Doubly-periodic Scherk surface



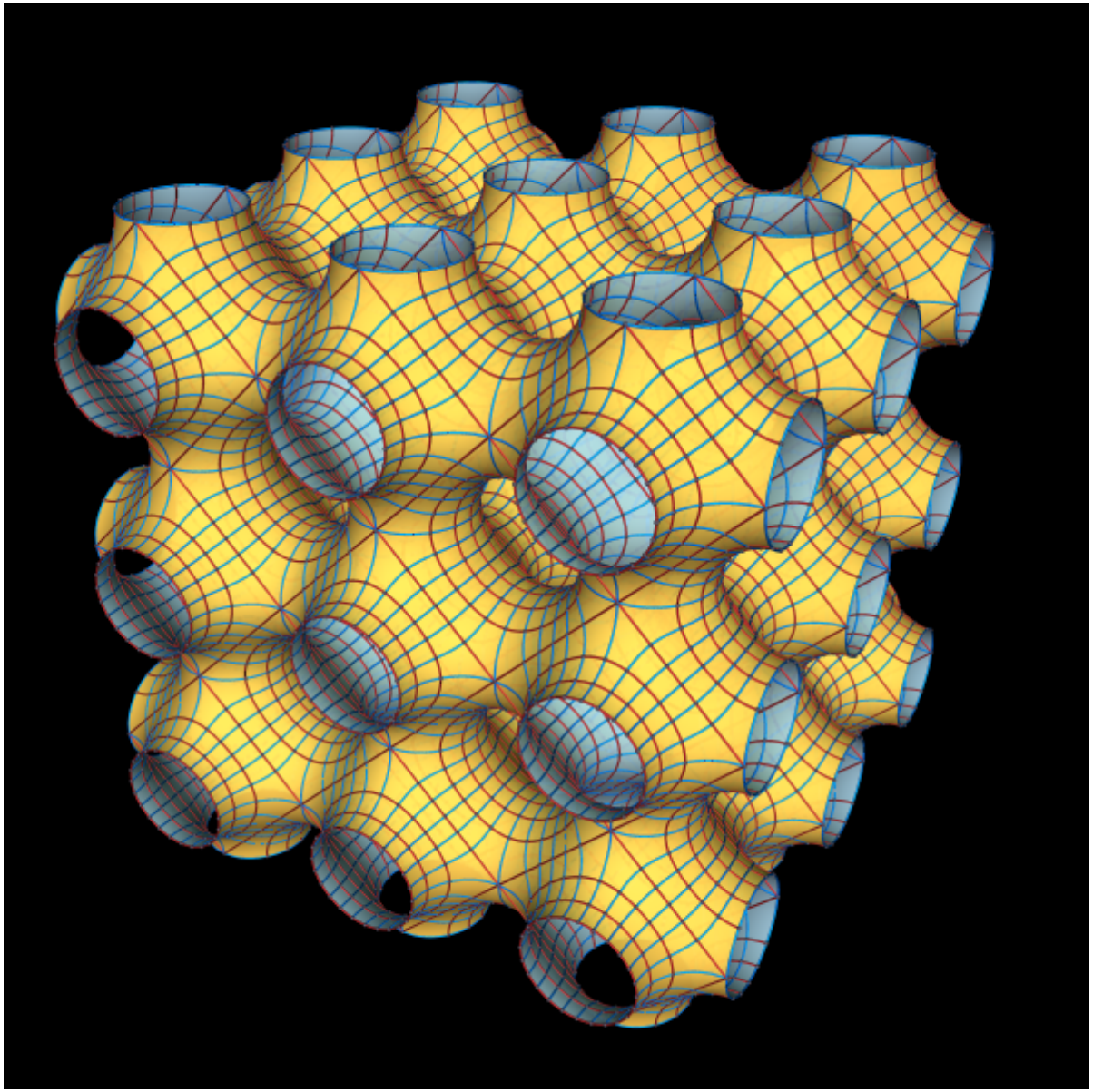
Fundamental domain for Scherk's surface



Sherk's surface with handles



Triply-periodic surface (Schwarz)



Why are these surfaces "minimal"?

From this definition, it's not at all clear what is "minimal" about these surfaces.

The idea is that minimal surfaces should locally minimize surface area.

Why are these surfaces "minimal"?

From this definition, it's not at all clear what is "minimal" about these surfaces.

The idea is that minimal surfaces should locally minimize surface area.

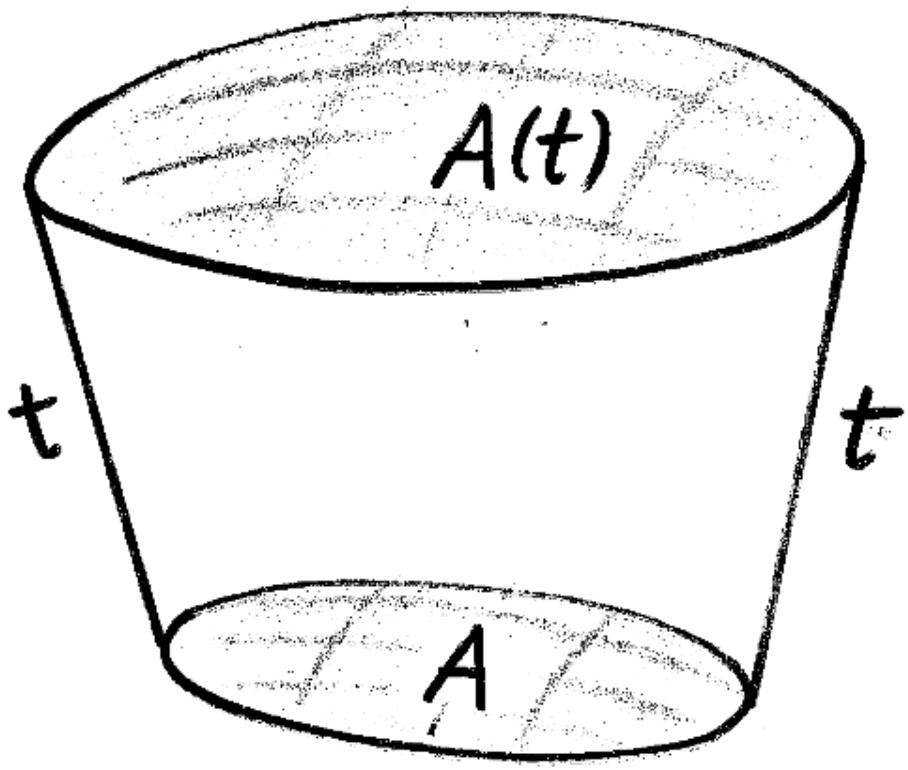


Figure 3: The surface and a normal variation

Normal Variations

Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface. Let $D \subset U$ be a bounded domain and let $h : \bar{D} \to \mathbb{R}$ be differentiable. Then the *normal variation* of $X(\bar{D})$ determined by h is given by

$$\varphi : \bar{D} \times (-\epsilon, \epsilon) \to \mathbb{R}^3$$

where

$$\varphi(u, v, t) = X(u, v) + th(u, v)N(u, v).$$

Normal Variations

Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface. Let $D \subset U$ be a bounded domain and let $h : \bar{D} \to \mathbb{R}$ be differentiable. Then the *normal variation* of $X(\bar{D})$ determined by h is given by

$$\varphi : \bar{D} \times (-\epsilon, \epsilon) \to \mathbb{R}^3$$

where

$$\varphi(u, v, t) = X(u, v) + th(u, v)N(u, v).$$

For small ϵ, $X^t(u, v) = \varphi(u, v, t)$ is a regular parametrized surface with

$$X_u^t = X_u + thN_u + th_u N$$
$$X_v^t = X_v + thN_v + th_v N.$$

Normal Variations

Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface. Let $D \subset U$ be a bounded domain and let $h : \bar{D} \to \mathbb{R}$ be differentiable. Then the *normal variation* of $X(\bar{D})$ determined by h is given by

$$\varphi : \bar{D} \times (-\epsilon, \epsilon) \to \mathbb{R}^3$$

where

$$\varphi(u, v, t) = X(u, v) + th(u, v)N(u, v).$$

For small ϵ, $X^t(u, v) = \varphi(u, v, t)$ is a regular parametrized surface with

$$X_u^t = X_u + thN_u + th_u N$$
$$X_v^t = X_v + thN_v + th_v N.$$

Then it's straightforward to see that

$$E^t G^t - (F^t)^2 = EG - F^2 - 2th(Eg - 2Ff + Ge) + R$$
$$= (EG - F^2)(1 - 4thH) + R$$

where $lim_{t \to 0} \frac{R}{t} = 0$.

Equivalence of curvature and variational definitions of minimal

The area $A(t)$ of $X^t(\bar{D})$ is given by

$$\begin{aligned} A(t) &= \int_{\bar{D}} \sqrt{E^t G^t - (F^t)^2} du dv \\ &= \int_{\bar{D}} \sqrt{1 - 4thH + \bar{R}} \sqrt{EG - F^2} du dv, \end{aligned}$$

where $\bar{R} = \frac{R}{EG - F^2}$.

Equivalence of curvature and variational definitions of minimal

The area $A(t)$ of $X^t(\bar{D})$ is given by

$$\begin{aligned} A(t) &= \int_{\bar{D}} \sqrt{E^t G^t - (F^t)^2} du dv \\ &= \int_{\bar{D}} \sqrt{1 - 4thH + \bar{R}} \sqrt{EG - F^2} du dv, \end{aligned}$$

where $\bar{R} = \frac{R}{EG - F^2}$.

Hence, if ϵ is small, A is differentiable and

$$A'(0) = -\int_{\bar{D}} 2hH \sqrt{EG - F^2} du dv.$$

Equivalence of curvature and variational definitions of minimal

The area $A(t)$ of $X^t(\bar{D})$ is given by

$$\begin{aligned} A(t) &= \int_{\bar{D}} \sqrt{E^t G^t - (F^t)^2} du dv \\ &= \int_{\bar{D}} \sqrt{1 - 4thH + \bar{R}} \sqrt{EG - F^2} du dv, \end{aligned}$$

where $\bar{R} = \frac{R}{EG - F^2}$.

Hence, if ϵ is small, A is differentiable and

$$A'(0) = -\int_{\bar{D}} 2hH \sqrt{EG - F^2} du dv.$$

Therefore:

Proposition 2. *Let $X : U \to \mathbb{R}^3$ be a regular parametrized surface and let $D \subset U$ be a bounded domain. Then X is minimal (i.e. $H \equiv 0$) if and only if $A'(0) = 0$ for all such D and all normal variations of $X(\bar{D})$.*

Isothermal coordinates

Definition 2. Given a regular surface $S \subset \mathbb{R}^3$, a system of local coordinates $X : U \to V \subset S$ is said to be $isothermal$ if

$$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle \text{ and } \langle X_u, X_v \rangle = 0.$$

$i.e.$ $G = E$ and $F = 0$.

Isothermal coordinates

Definition 2. Given a regular surface $S \subset \mathbb{R}^3$, a system of local coordinates $X : U \to V \subset S$ is said to be $isothermal$ if

$$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle \text{ and } \langle X_u, X_v \rangle = 0.$$

$i.e.$ $G = E$ and $F = 0$.

Note: A parametrization $X : U \to V \subset S$ is isothermal if and only if it is conformal.

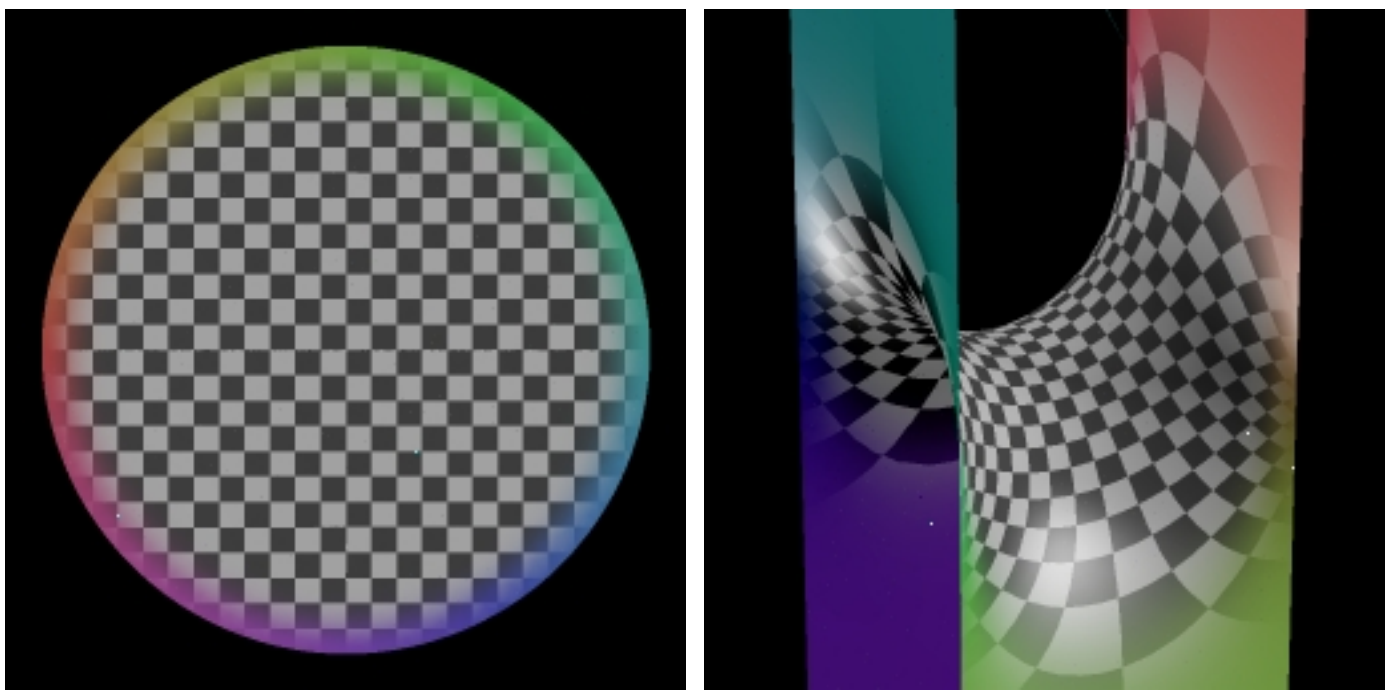
Isothermal coordinates

Definition 2. Given a regular surface $S \subset \mathbb{R}^3$, a system of local coordinates $X : U \to V \subset S$ is said to be *isothermal* if

$$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle \text{ and } \langle X_u, X_v \rangle = 0.$$

i.e. $G = E$ and $F = 0$.

Note: A parametrization $X : U \to V \subset S$ is isothermal if and only if it is conformal.



Facts about isothermal parametrizations

Proposition 3. *Let $X : U \to V \subset S$ be an isothermal local parametrization of a regular surface S. Then*

$$\Delta X = X_{uu} + X_{vv} = 2\lambda^2 \mathbf{H},$$

where $\lambda^2(u,v) = \langle X_u, X_u \rangle = \langle X_v, X_v \rangle$ and $\mathbf{H} = HN$ is the mean curvature vector field.

Facts about isothermal parametrizations

Proposition 3. *Let $X : U \to V \subset S$ be an isothermal local parametrization of a regular surface S. Then*

$$\Delta X = X_{uu} + X_{vv} = 2\lambda^2 \mathbf{H},$$

where $\lambda^2(u,v) = \langle X_u, X_u \rangle = \langle X_v, X_v \rangle$ and $\mathbf{H} = HN$ is the mean curvature vector field.

Corollary 4. *Let $X(u,v) = (x^1(u,v), x^2(u,v), x^3(u,v))$ be an isothermal parametrization of a regular surface S in $\mathbb{R}^3$. Then S is minimal (within the range of this parametrization) if and only if the three coordinate functions $x^1(u,v)$, $x^2(u,v)$ and $x^3(u,v)$ are harmonic.*

Facts about isothermal parametrizations

Proposition 3. *Let $X : U \to V \subset S$ be an isothermal local parametrization of a regular surface S. Then*

$$\Delta X = X_{uu} + X_{vv} = 2\lambda^2 \mathbf{H},$$

where $\lambda^2(u,v) = \langle X_u, X_u \rangle = \langle X_v, X_v \rangle$ and $\mathbf{H} = HN$ is the mean curvature vector field.

Corollary 4. *Let $X(u,v) = (x^1(u,v), x^2(u,v), x^3(u,v))$ be an isothermal parametrization of a regular surface S in $\mathbb{R}^3$. Then S is minimal (within the range of this parametrization) if and only if the three coordinate functions $x^1(u,v)$, $x^2(u,v)$ and $x^3(u,v)$ are harmonic.*

Corollary 5. *The three component functions x^1, x^2 and x^3 are locally the real parts of holomorphic functions.*

Facts about isothermal parametrizations

Proposition 3. *Let $X : U \to V \subset S$ be an isothermal local parametrization of a regular surface S. Then*

$$\Delta X = X_{uu} + X_{vv} = 2\lambda^2 \mathbf{H},$$

where $\lambda^2(u,v) = \langle X_u, X_u \rangle = \langle X_v, X_v \rangle$ and $\mathbf{H} = HN$ is the mean curvature vector field.

Corollary 4. *Let $X(u,v) = (x^1(u,v), x^2(u,v), x^3(u,v))$ be an isothermal parametrization of a regular surface S in $\mathbb{R}^3$. Then S is minimal (within the range of this parametrization) if and only if the three coordinate functions $x^1(u,v)$, $x^2(u,v)$ and $x^3(u,v)$ are harmonic.*

Corollary 5. *The three component functions x^1, x^2 and x^3 are locally the real parts of holomorphic functions.*

Theorem 6. *Isothermal coordinates exist on any minimal surface $S \subset \mathbb{R}^3$.*

Remark: In fact, isothermal coordinates exist for any C^2 surface.

Connections with complex analysis

If $\sigma : S^2 \to \mathbb{C} \cup \{\infty\}$ is stereographic projection and $N : S \to S^2$ is the Gauss map, then

$$g := \sigma \circ N : S \to \mathbb{C} \cup \{\infty\}$$

is orientation-preserving and conformal whenever $dN \neq 0$. Therefore g is a meromorphic function on S (now thought of as a Riemann surface).

Connections with complex analysis

If $\sigma : S^2 \to \mathbb{C} \cup \{\infty\}$ is stereographic projection and $N : S \to S^2$ is the Gauss map, then

$$g := \sigma \circ N : S \to \mathbb{C} \cup \{\infty\}$$

is orientation-preserving and conformal whenever $dN \neq 0$. Therefore g is a meromorphic function on S (now thought of as a Riemann surface).

In fact, we have:

Theorem 7 (Osserman). *If S is a complete, immersed minimal surface of finite total curvature, then S can be conformally compactified to a Riemann surface Σ_k by closing finitely many punctures. Moreover, the Gauss map $N : S \to S^2$, which is meromorphic, extends to a meromorphic function on Σ_k.*

Enneper's Surface

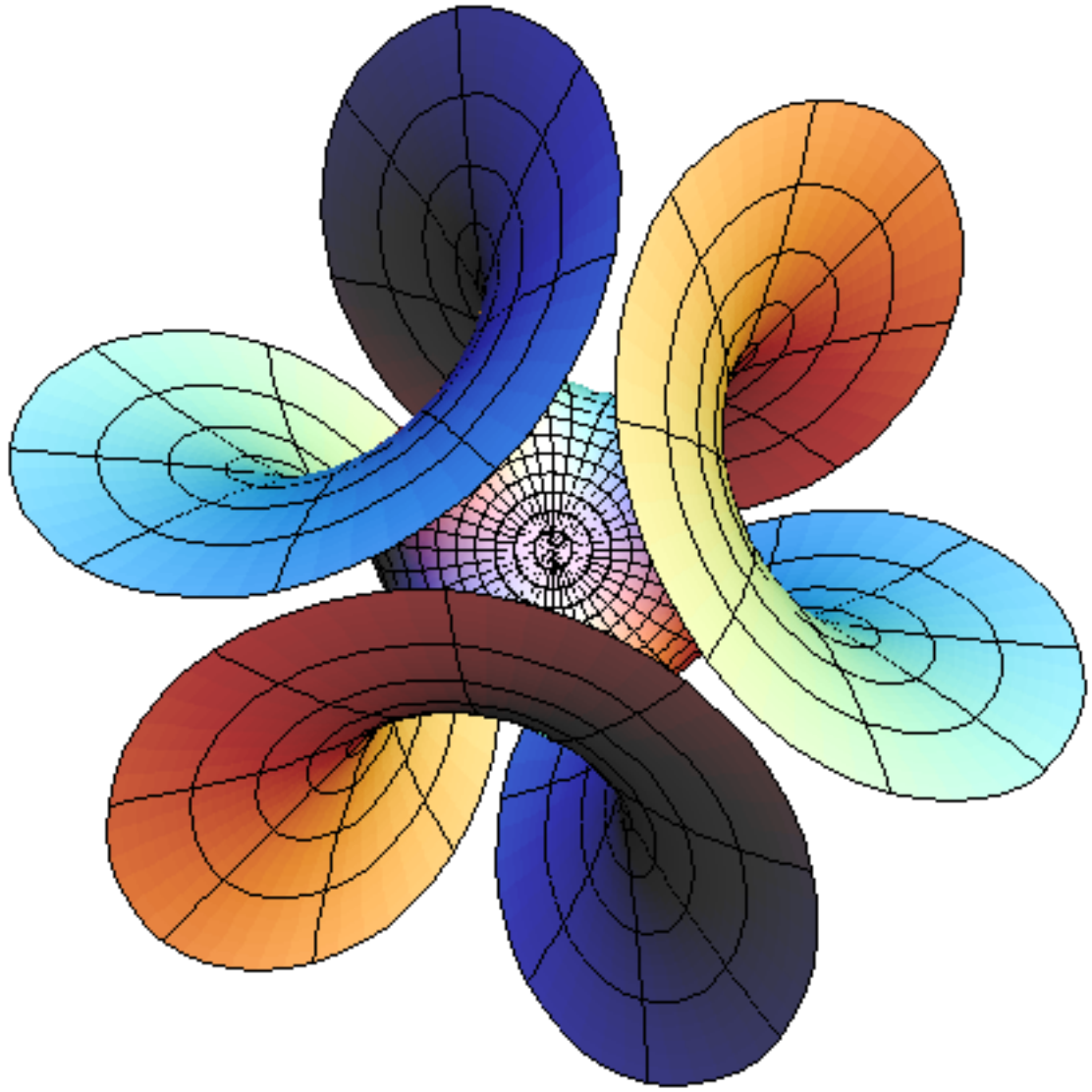
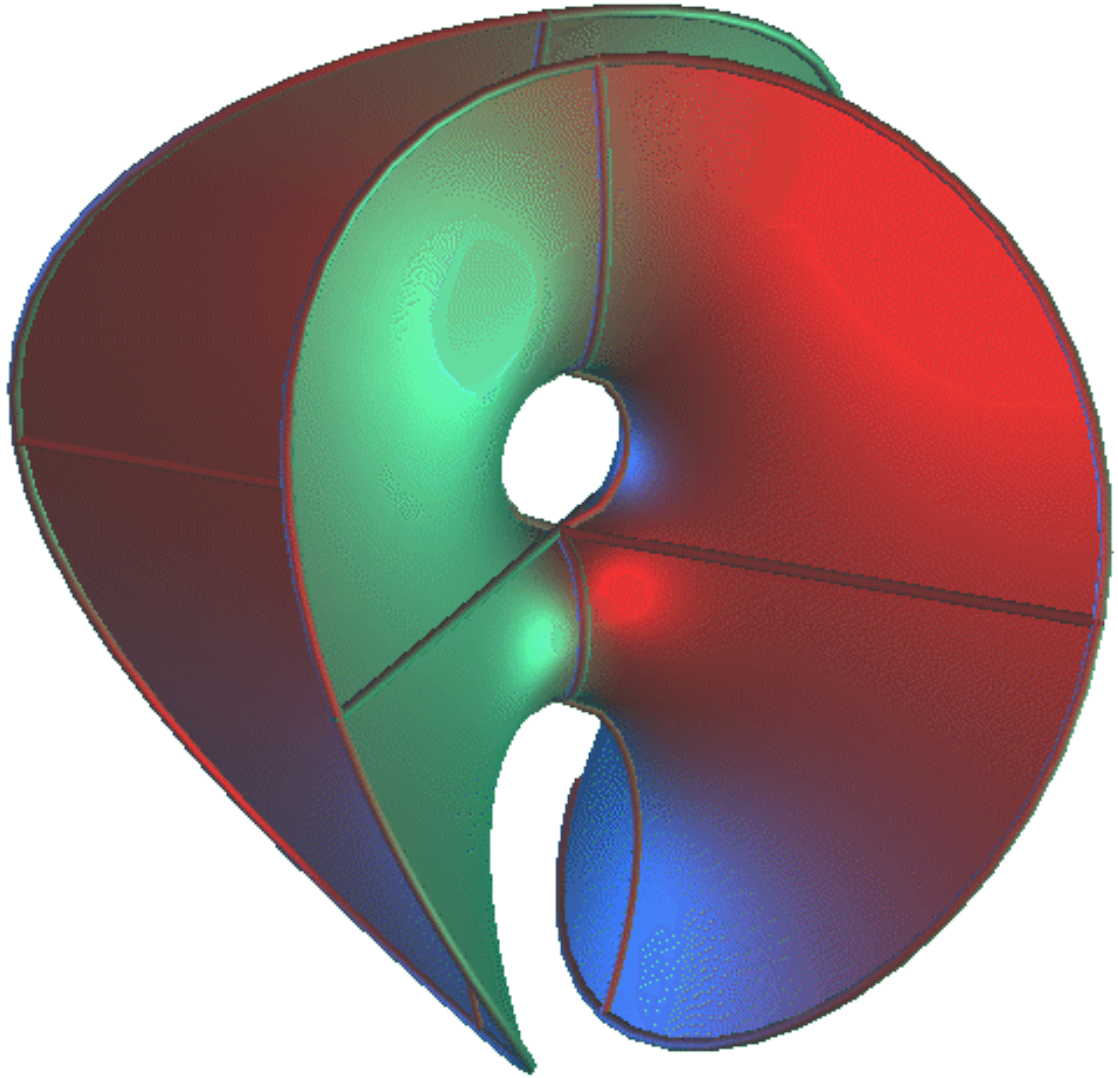
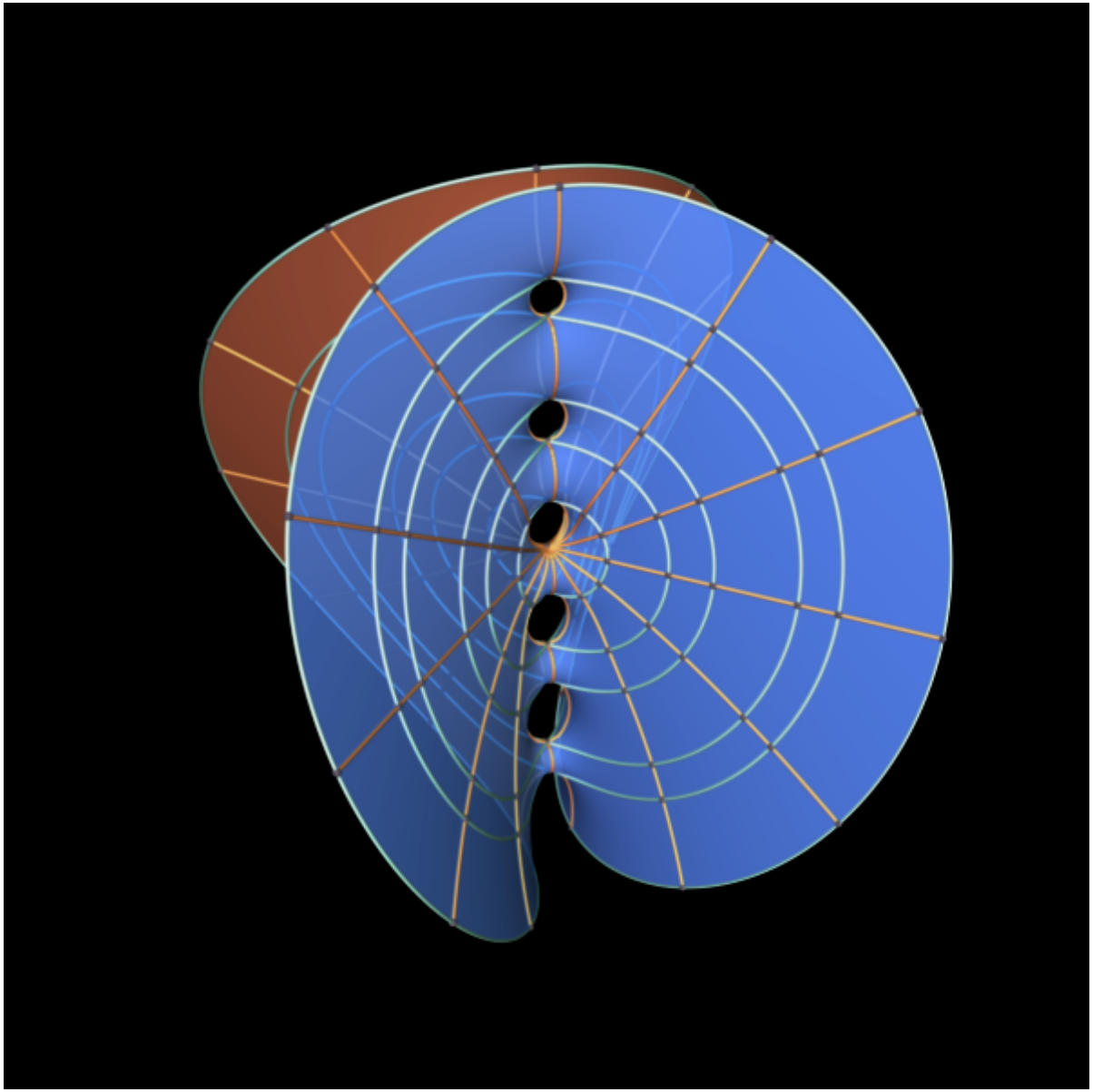


Figure 4: $z \in \mathbb{C}$, $g(z) := z$

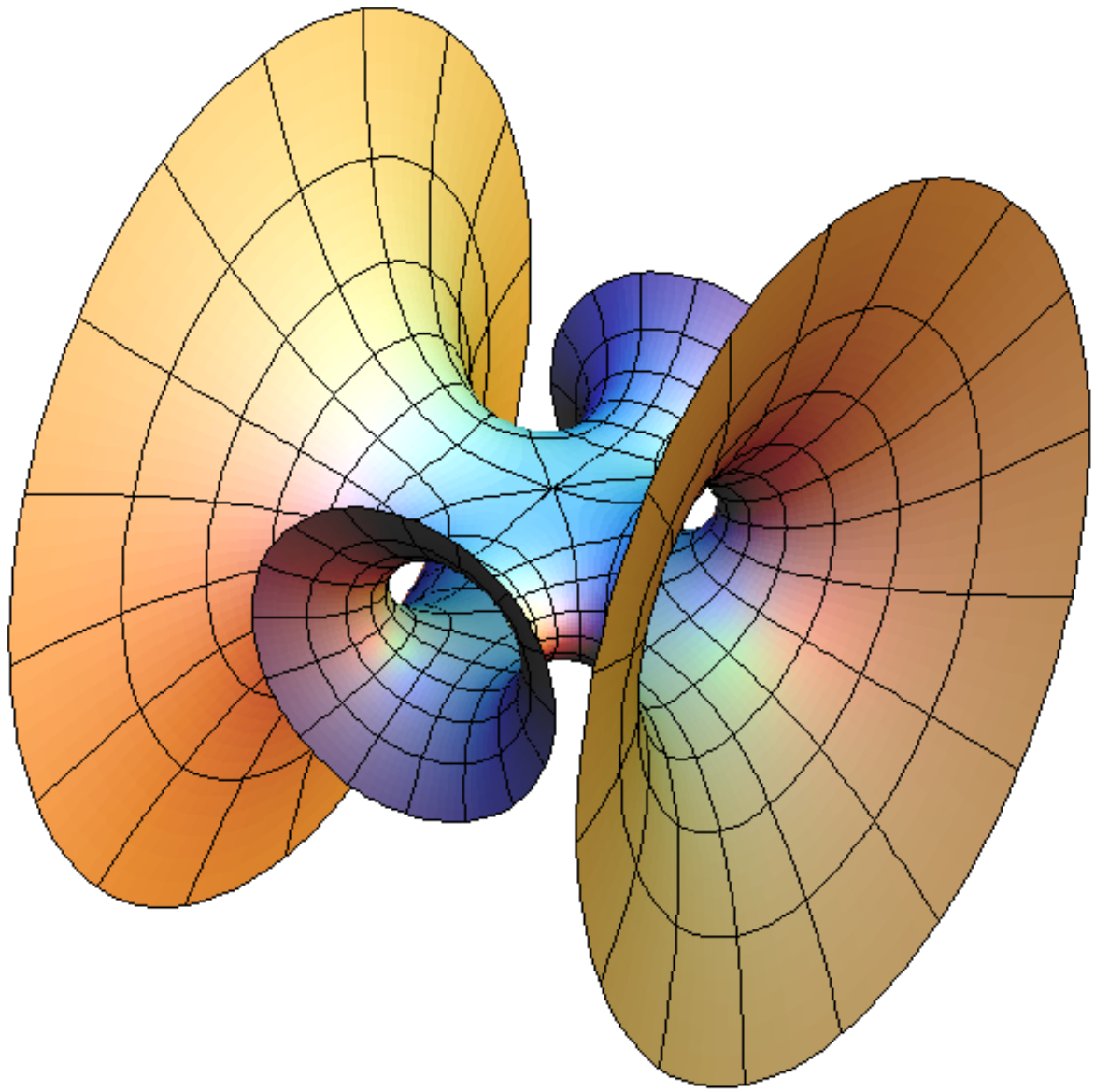
Chen-Gackstatter Surface



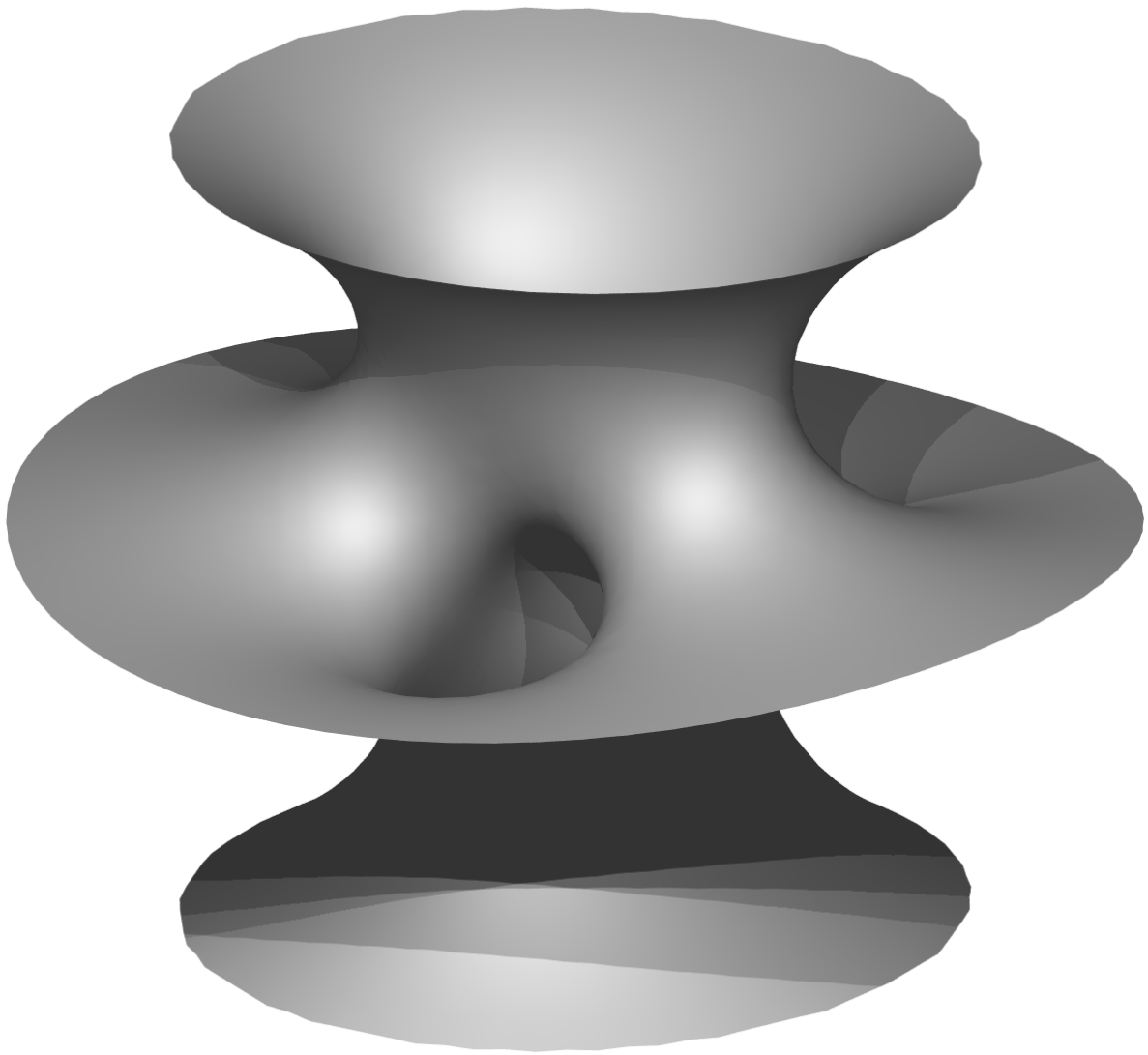
Chen-Gackstatter Surface with higher genus



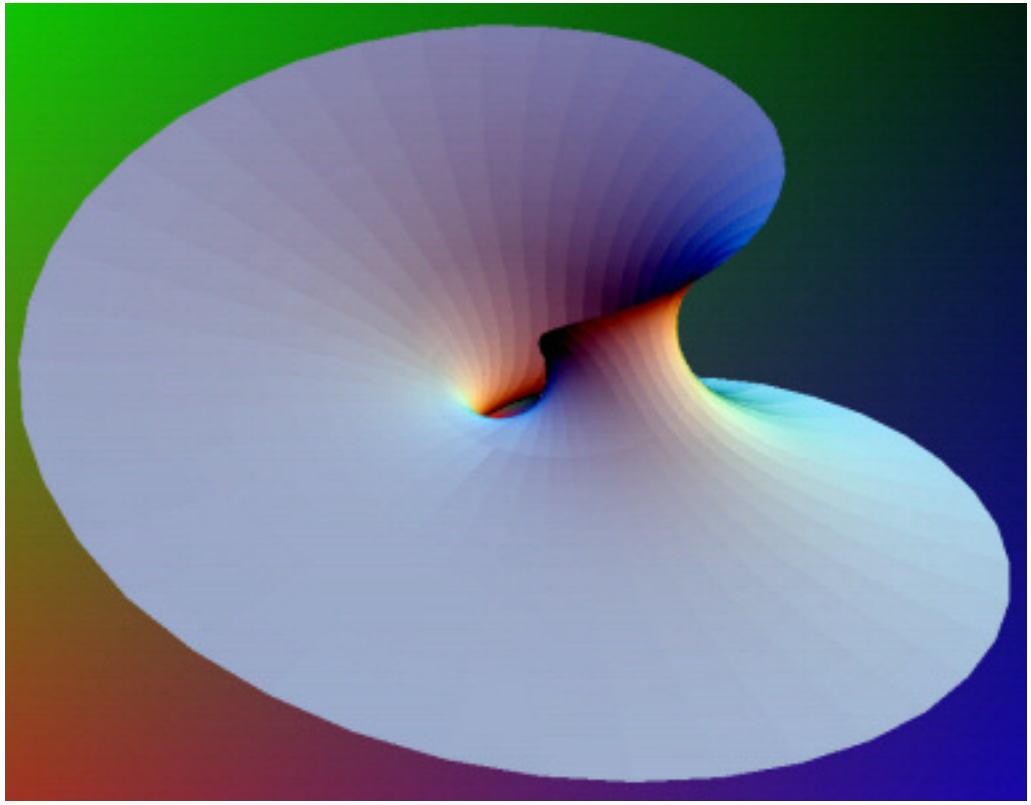
Symmetric 4-Noid



Costa surface



Meeks' minimal Möbius strip



Thanks!