Linking Integrals in Hyperspheres

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(Joint with David Shea Vela-Vick)

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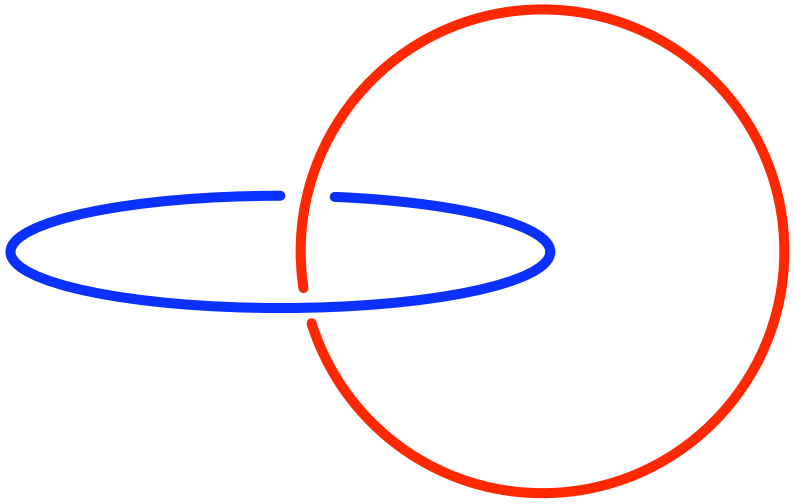
Sewanee
October 24, 2008

Links



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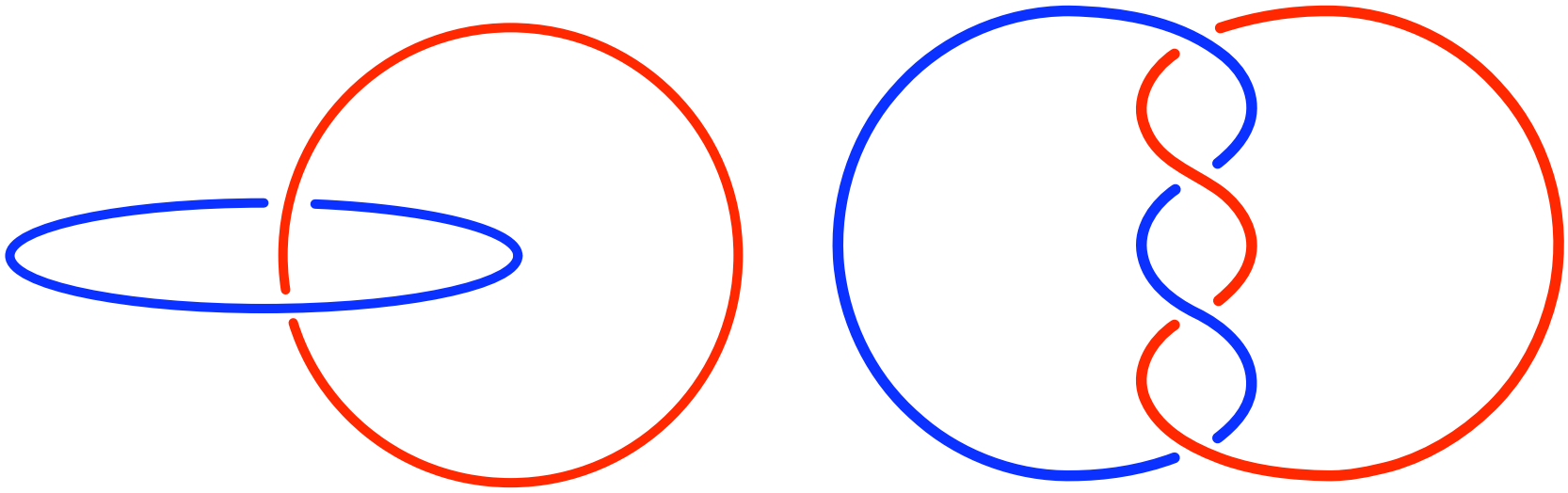

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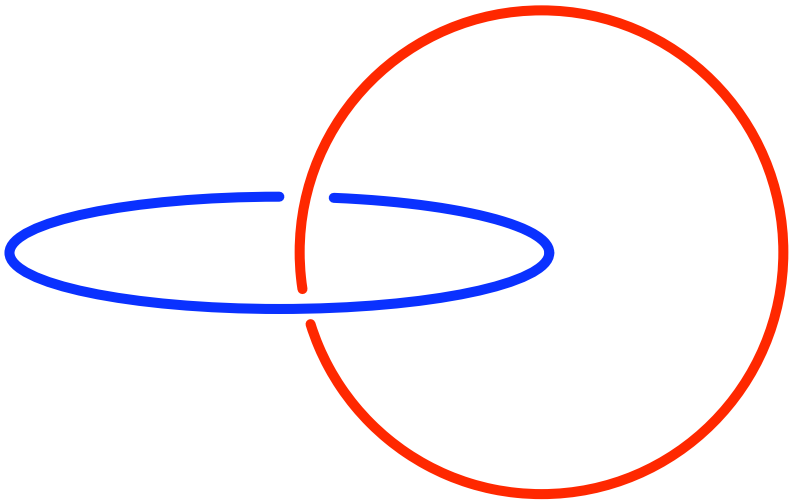

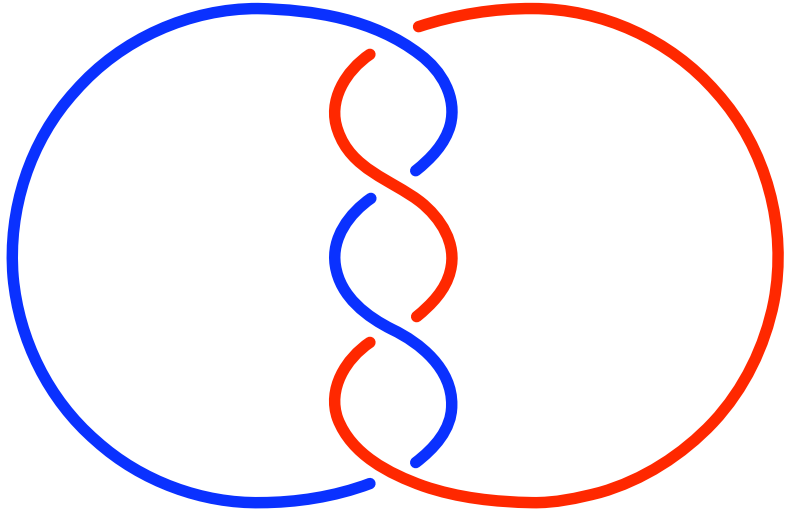

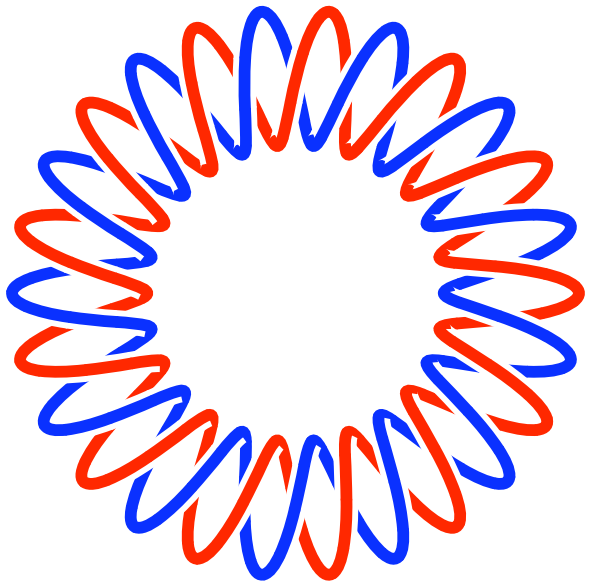

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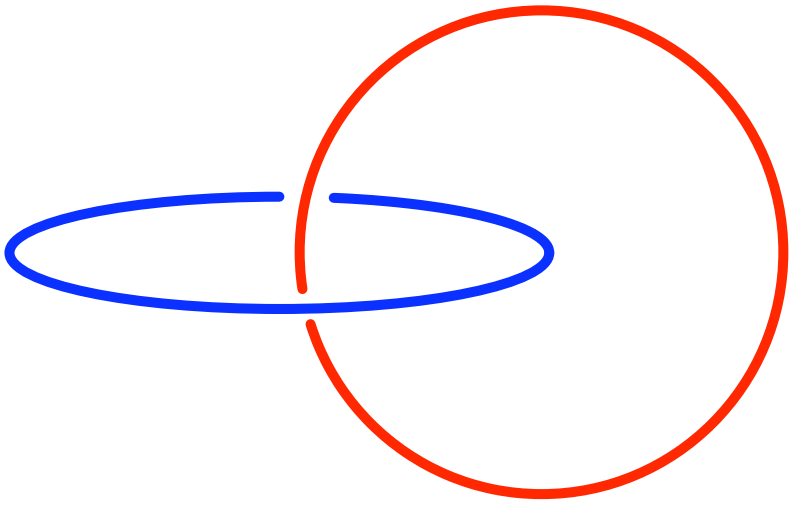

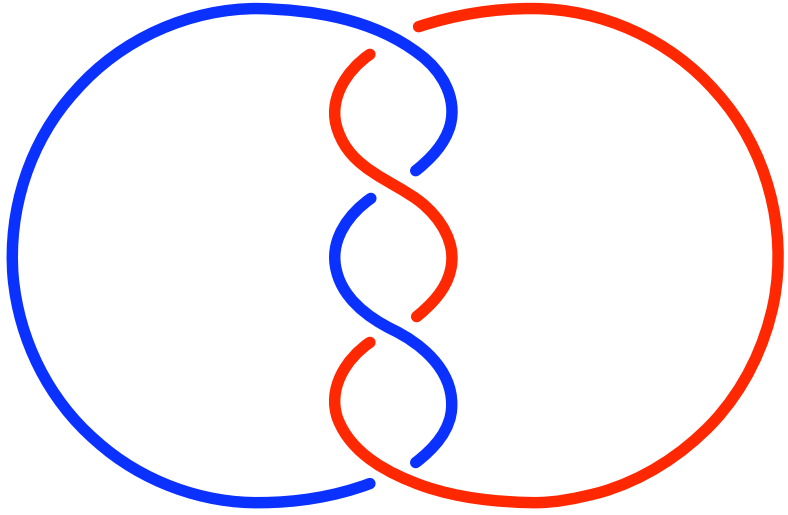

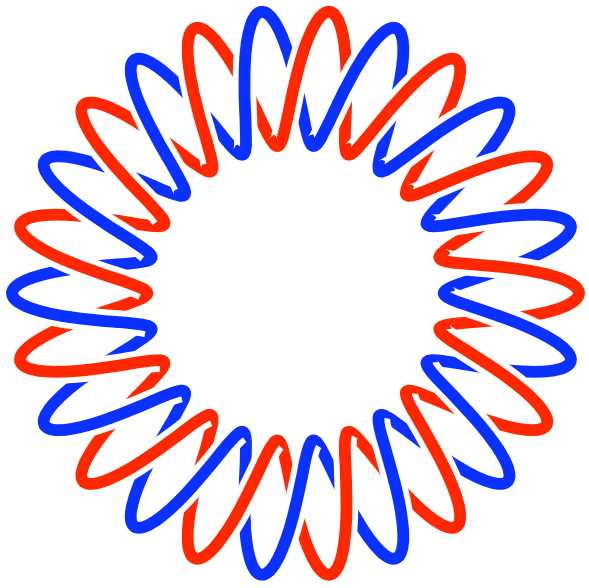

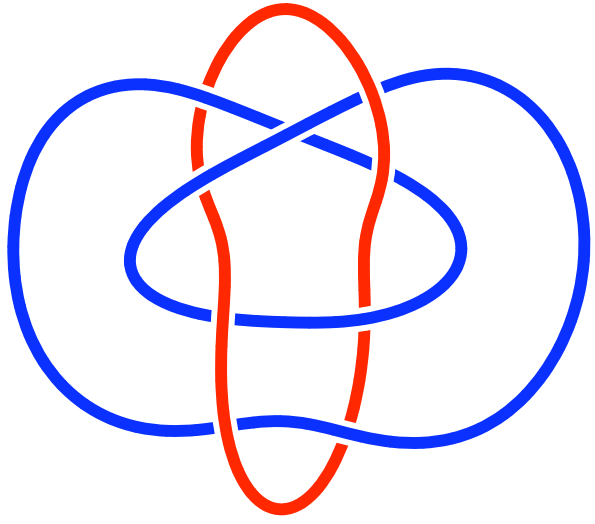

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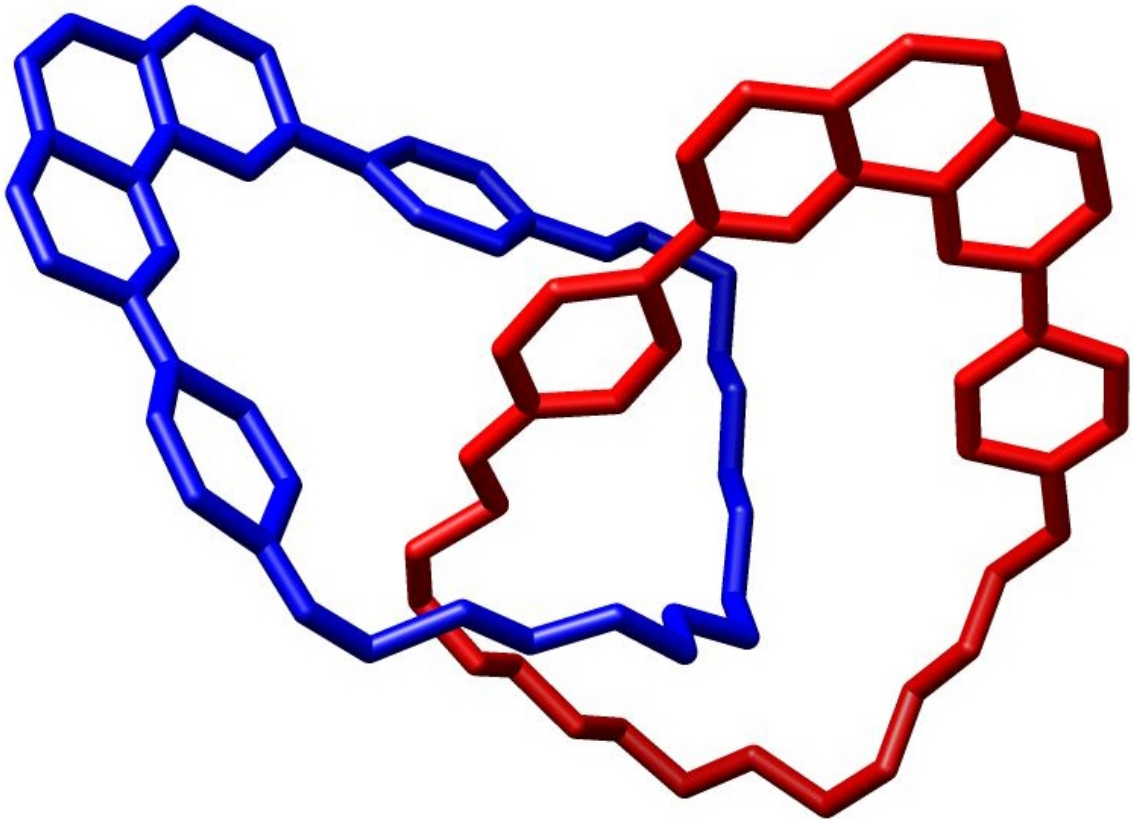

Definition

Let K and L be smooth, oriented closed curves embedded in $\mathbb{R}^3$. Then the *linking number* $Lk(K, L)$ between K and L is the (signed) number of intersections of K with a surface bounded by L.

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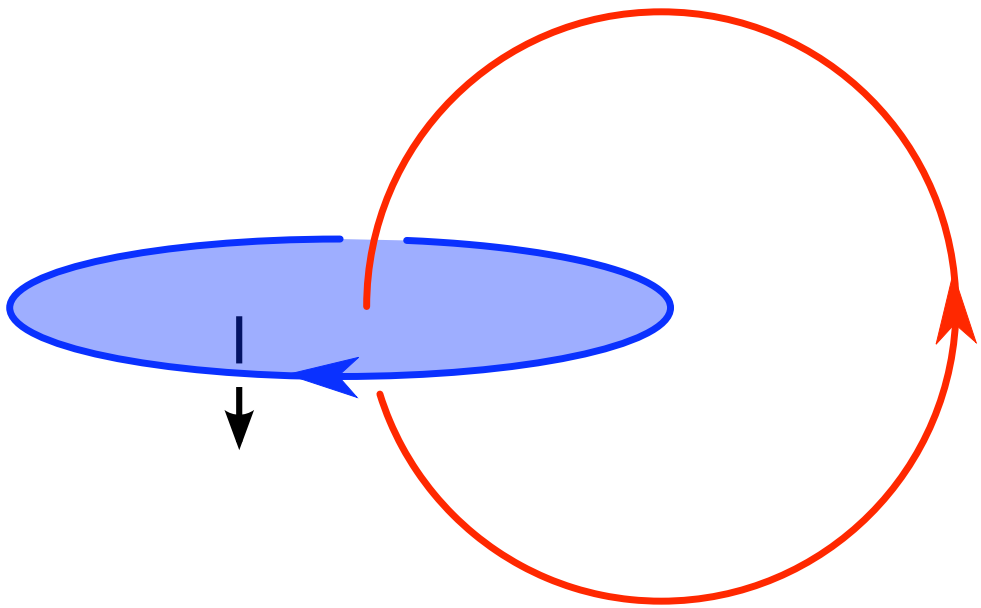

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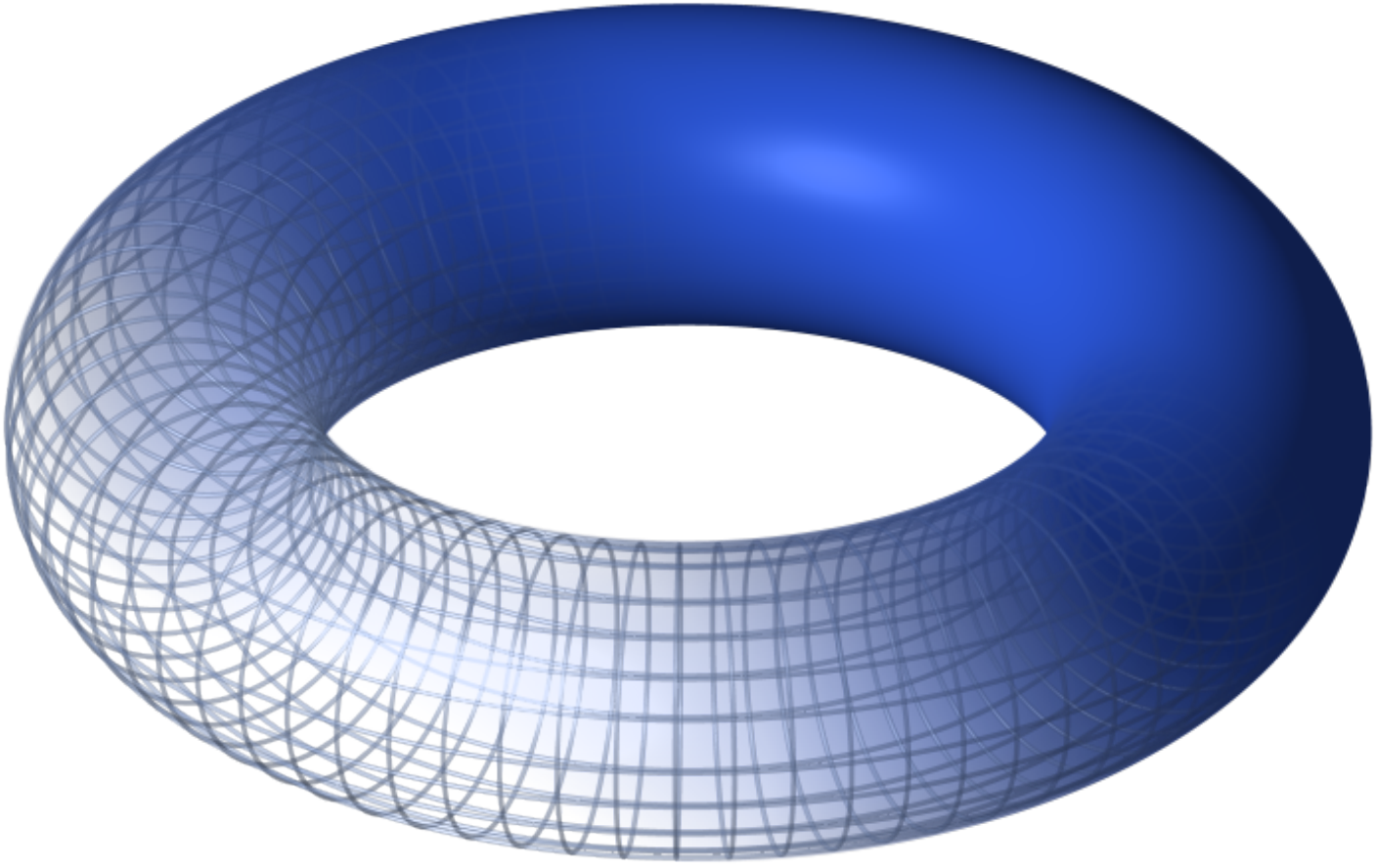

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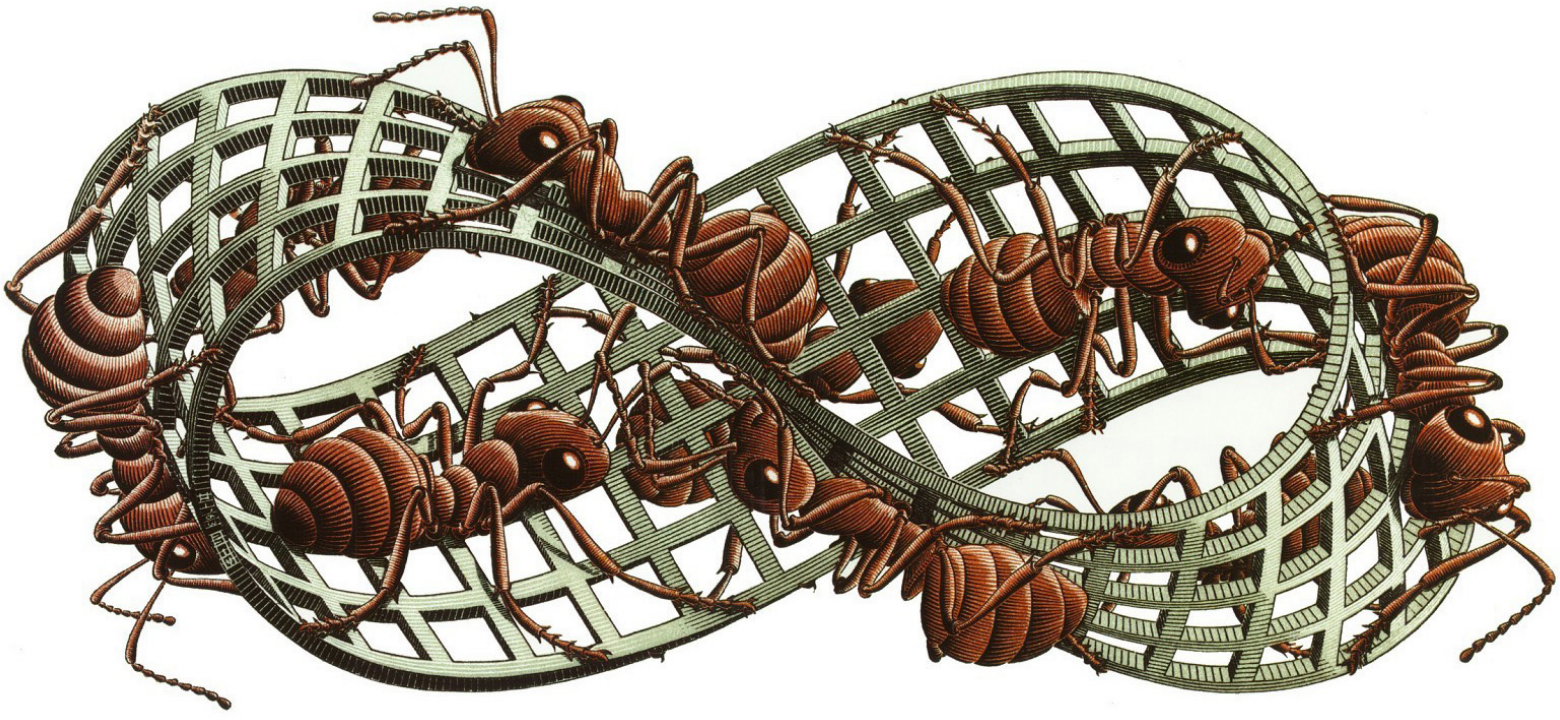

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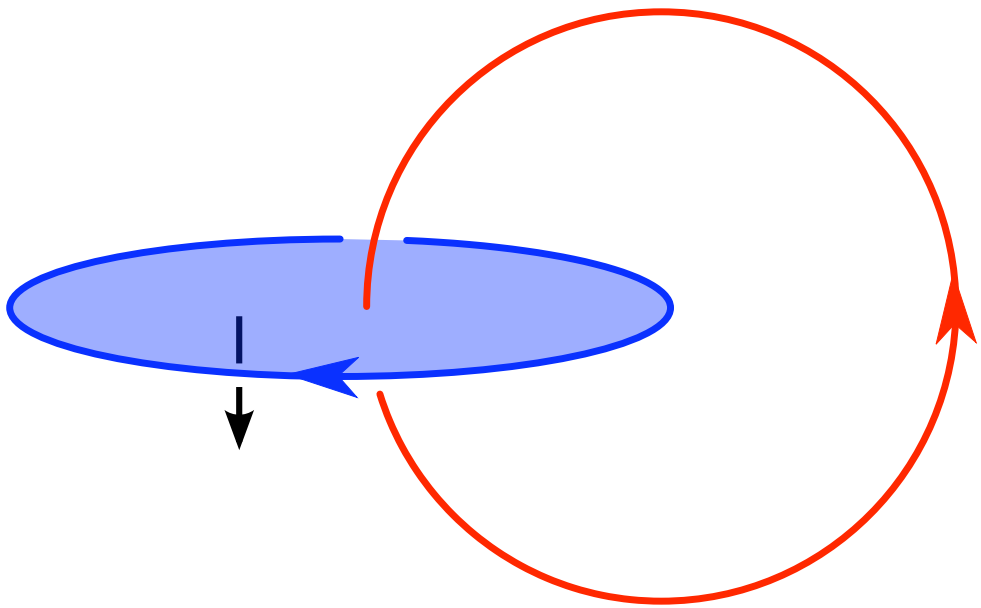

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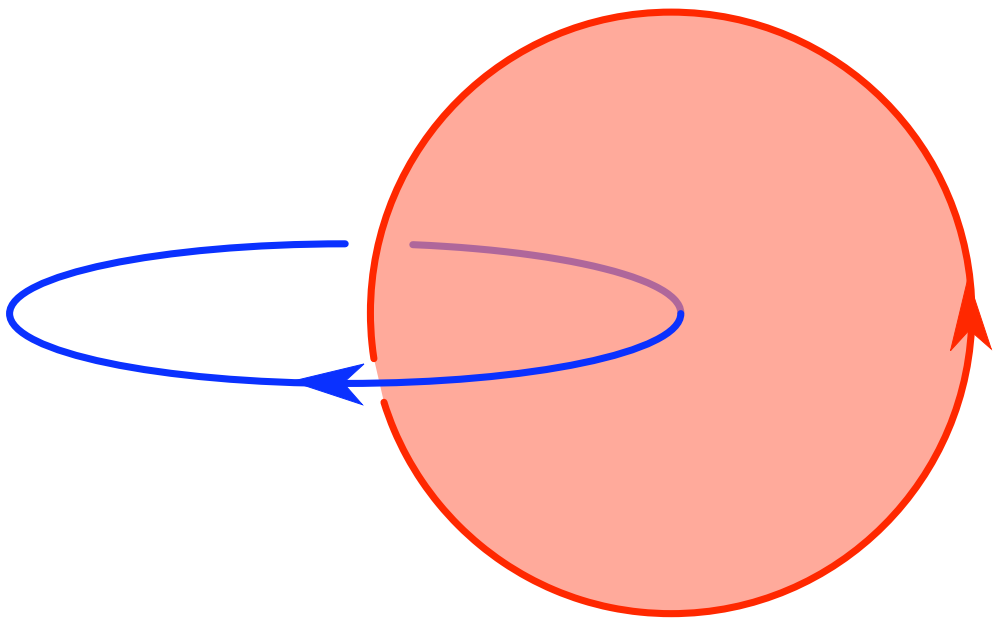

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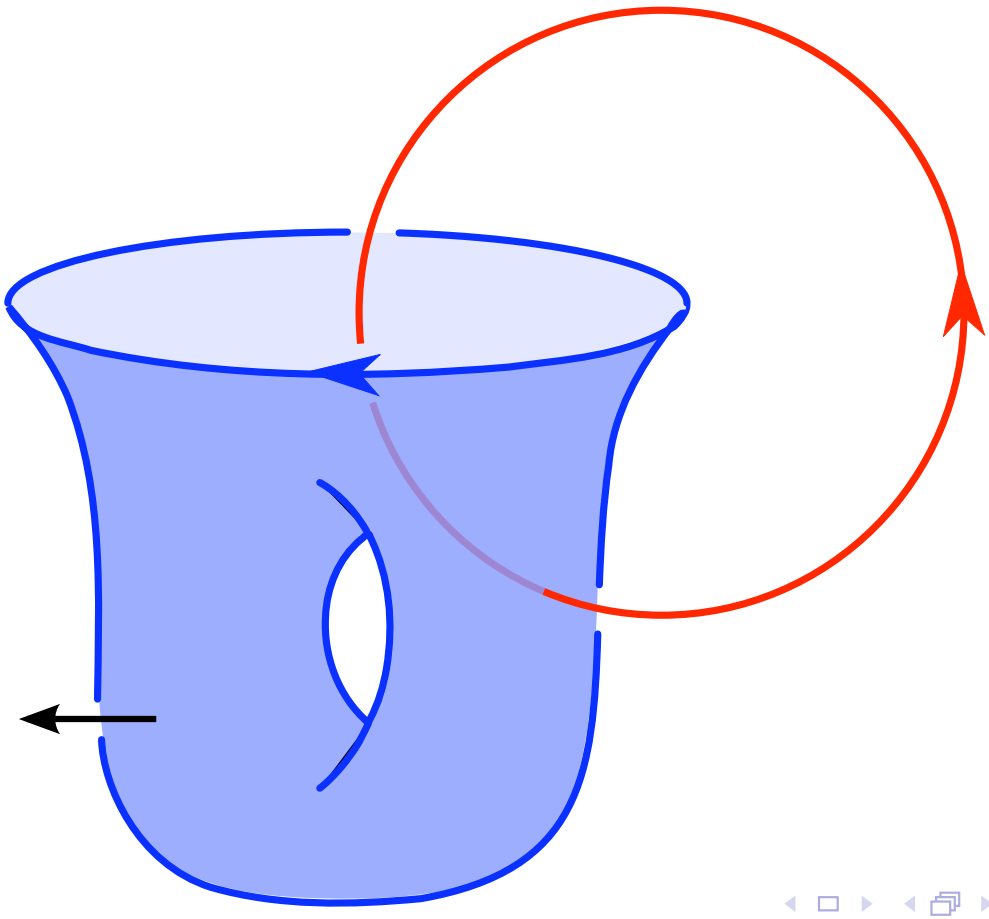

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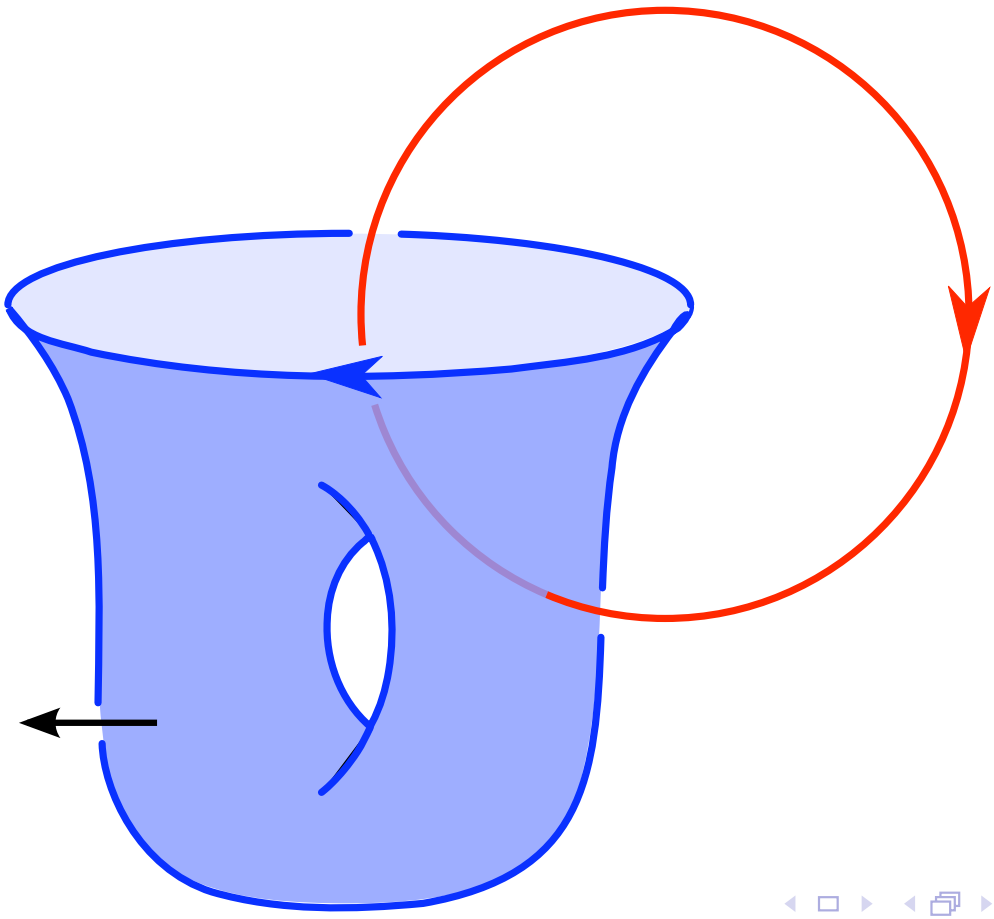

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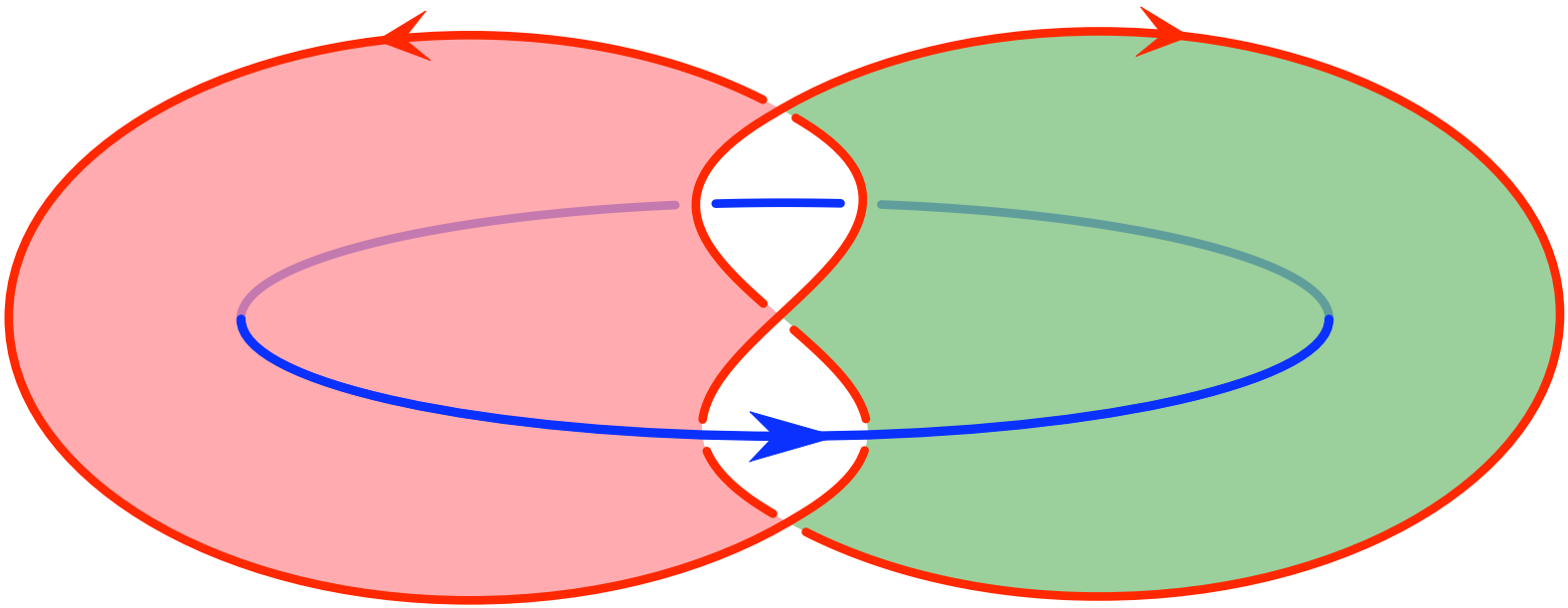

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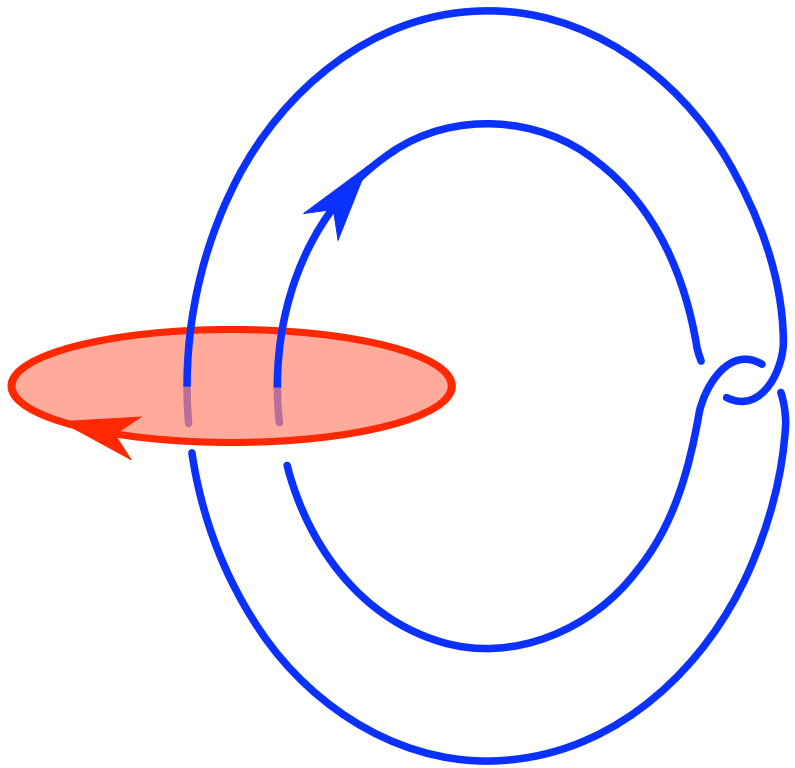

Theorem (Gauß, 1833)

Let $K = \{\mathbf{x}(s)\}$ and $L = \{\mathbf{y}(t)\}$ be disjoint, smooth, oriented closed curves embedded in $\mathbb{R}^3$. Then the linking number between K and L is given by

$$Lk(K, L) = \frac{1}{4\pi} \int_{K \times L} \frac{\mathbf{x}(s) - \mathbf{y}(t)}{|\mathbf{x}(s) - \mathbf{y}(t)|^3} \cdot \left(\frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \right) ds\, dt$$

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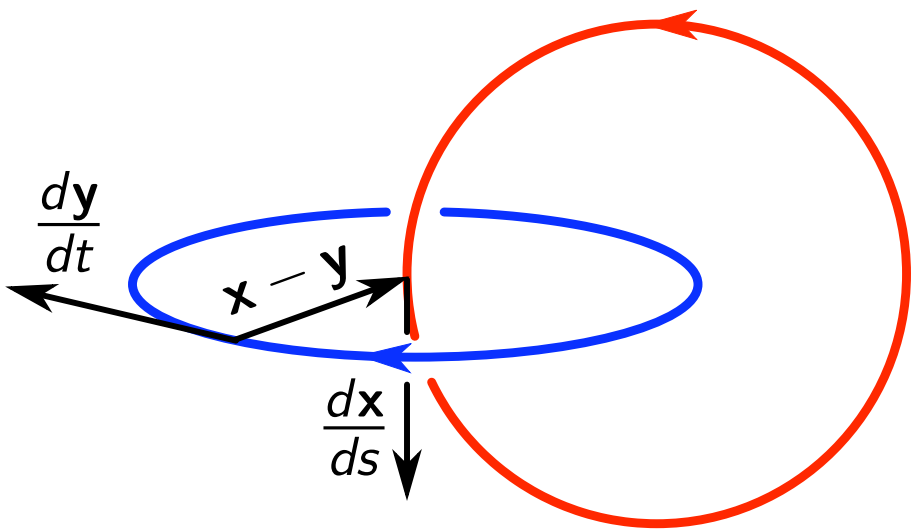

The "Visual" Proof of the Gauß Linking Integral

Define $f : K \times L \longrightarrow S^2$ by

$$f(s,t) = \frac{\mathbf{x}(s) - \mathbf{y}(t)}{|\mathbf{x}(s) - \mathbf{y}(t)|}.$$

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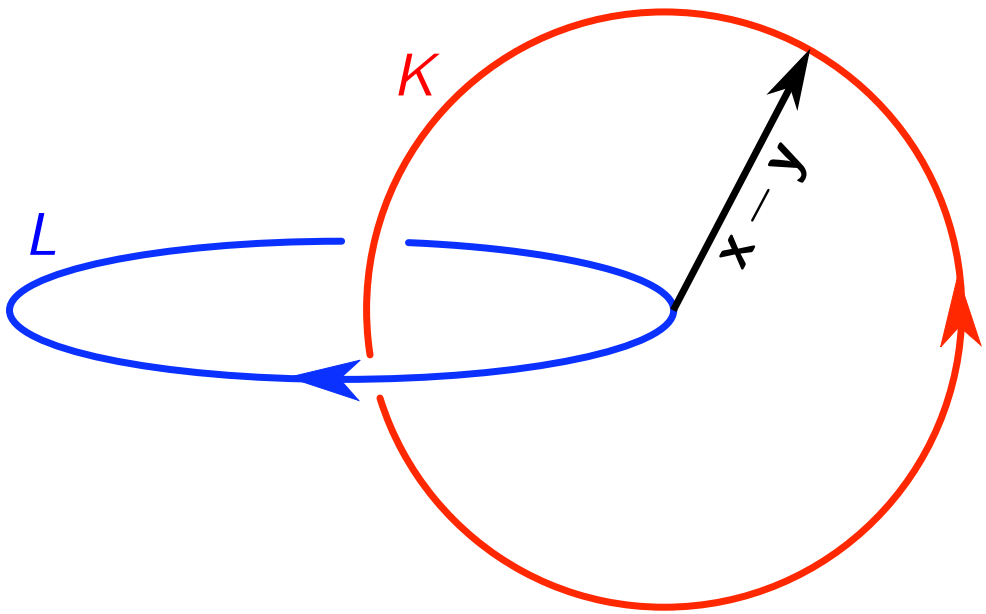

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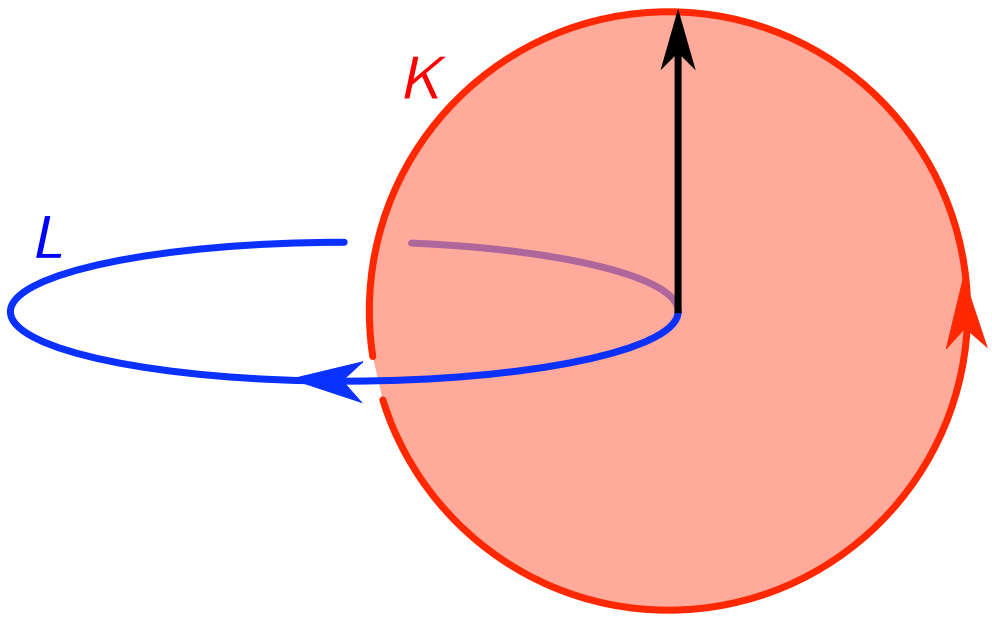

$Lk(K,L)$ = number of times $f(s,t)$ is the north pole.

Covering the Sphere

Linking Number as Area

$$Lk(K,L) = \frac{1}{4\pi} \int_{K\times L} \frac{\mathbf{x}(s) - \mathbf{y}(t)}{|\mathbf{x}(s) - \mathbf{y}(t)|^3} \cdot \left(\frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \right) ds\, dt$$

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Area of S^2

$$\frac{1}{|\mathbf{x}(s)-\mathbf{y}(t)|^3} \det \begin{pmatrix} x_1 - y_1 & x_2 - y_2 & x_3 - y_3 \\ \frac{dx_1}{ds} & \frac{dx_2}{ds} & \frac{dx_3}{ds} \\ \frac{dy_1}{dt} & \frac{dy_2}{dt} & \frac{dy_3}{dt} \end{pmatrix}$$

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Note

The integrand is invariant under rigid motions of $\mathbb{R}^3$.

Goal

Find analogous linking integrals in higher dimensions and in non-Euclidean spaces.

Higher-Dimensional Euclidean Spaces

In higher-dimensional Euclidean spaces, the same idea yields an equivalent linking integral.

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Theorem

If $K = \{\mathbf{x}(s_1, s_2)\}$ is a closed surface and $L = \{\mathbf{y}(t)\}$ is a closed curve, both living in $\mathbb{R}^4$, then

$$Lk(K, L) = \frac{1}{2\pi^2} \int_{K \times L} \frac{-1}{|\mathbf{x} - \mathbf{y}|^4} [\mathbf{x} - \mathbf{y}, d\mathbf{x}, d\mathbf{y}]$$

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What about spheres?

The 3-sphere S^3 is defined as:

$$S^3 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

Intersection with a (hyper)plane:

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Stereographic projection:

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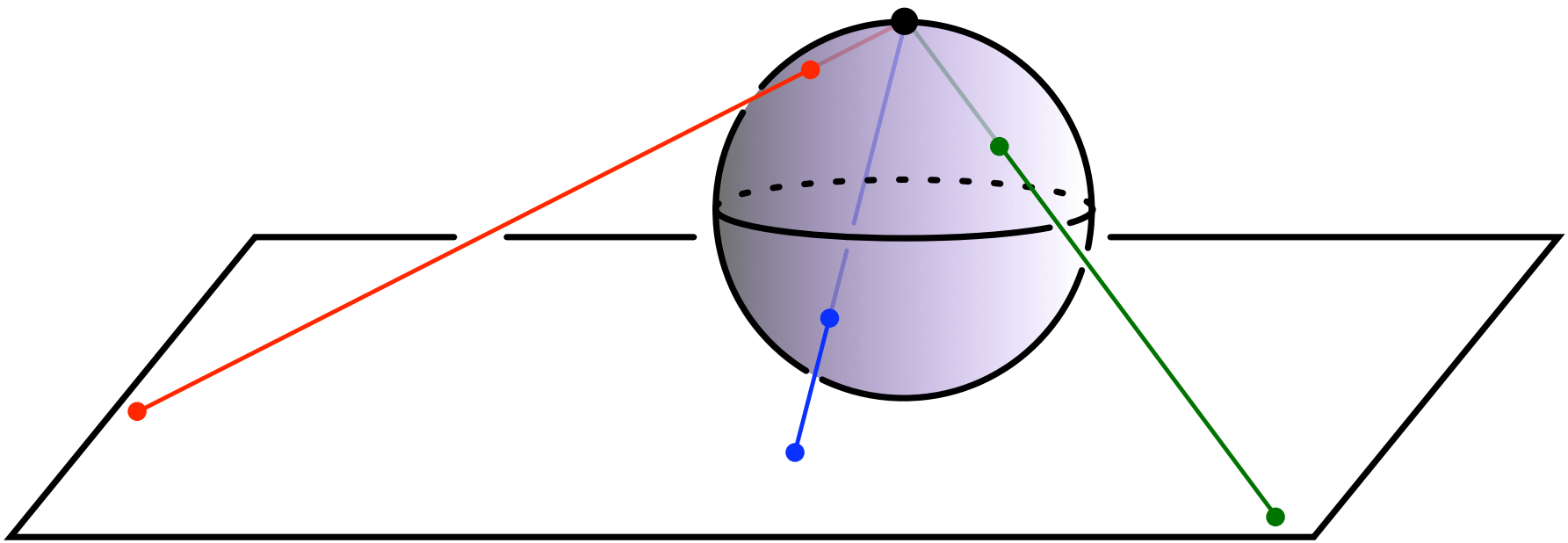

Visualizing the 3-sphere

Gluing:

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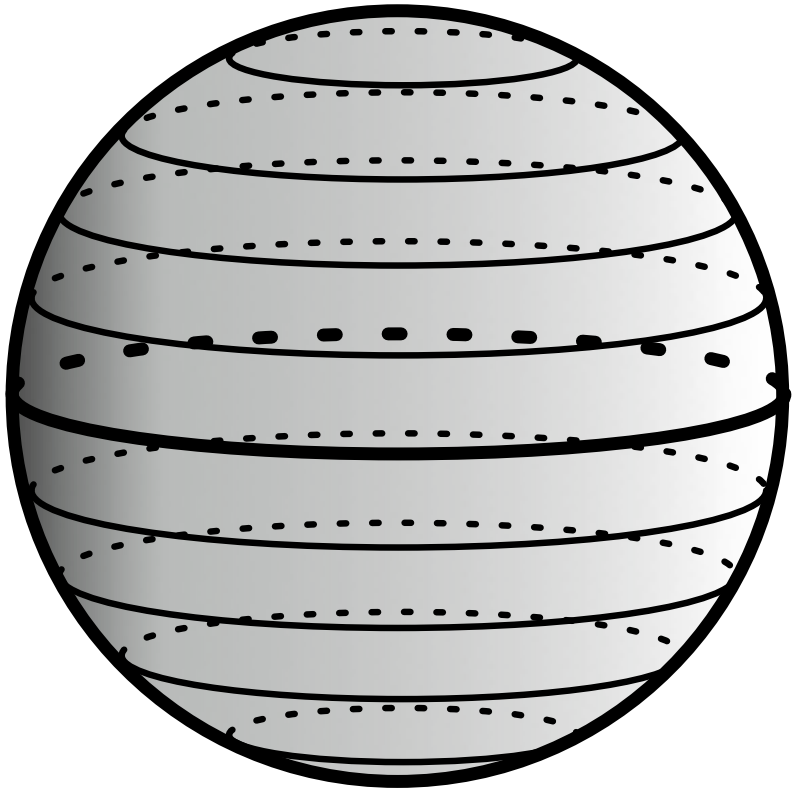

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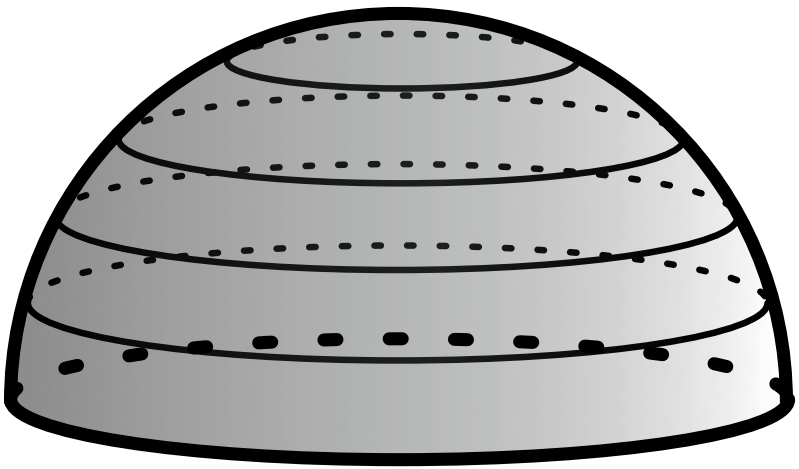

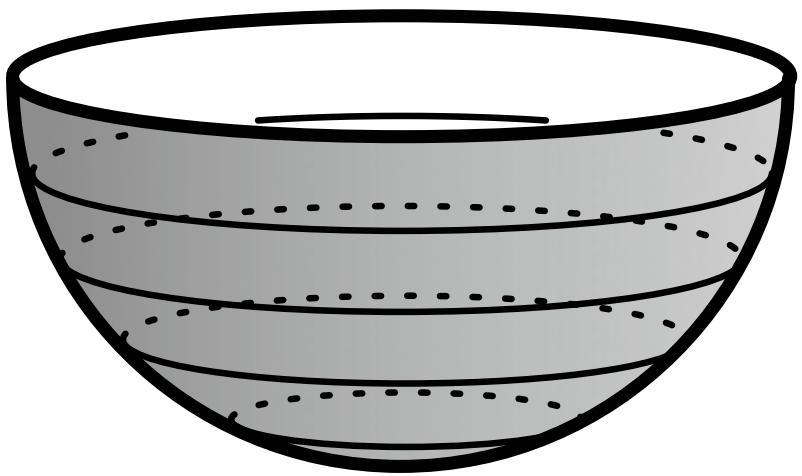

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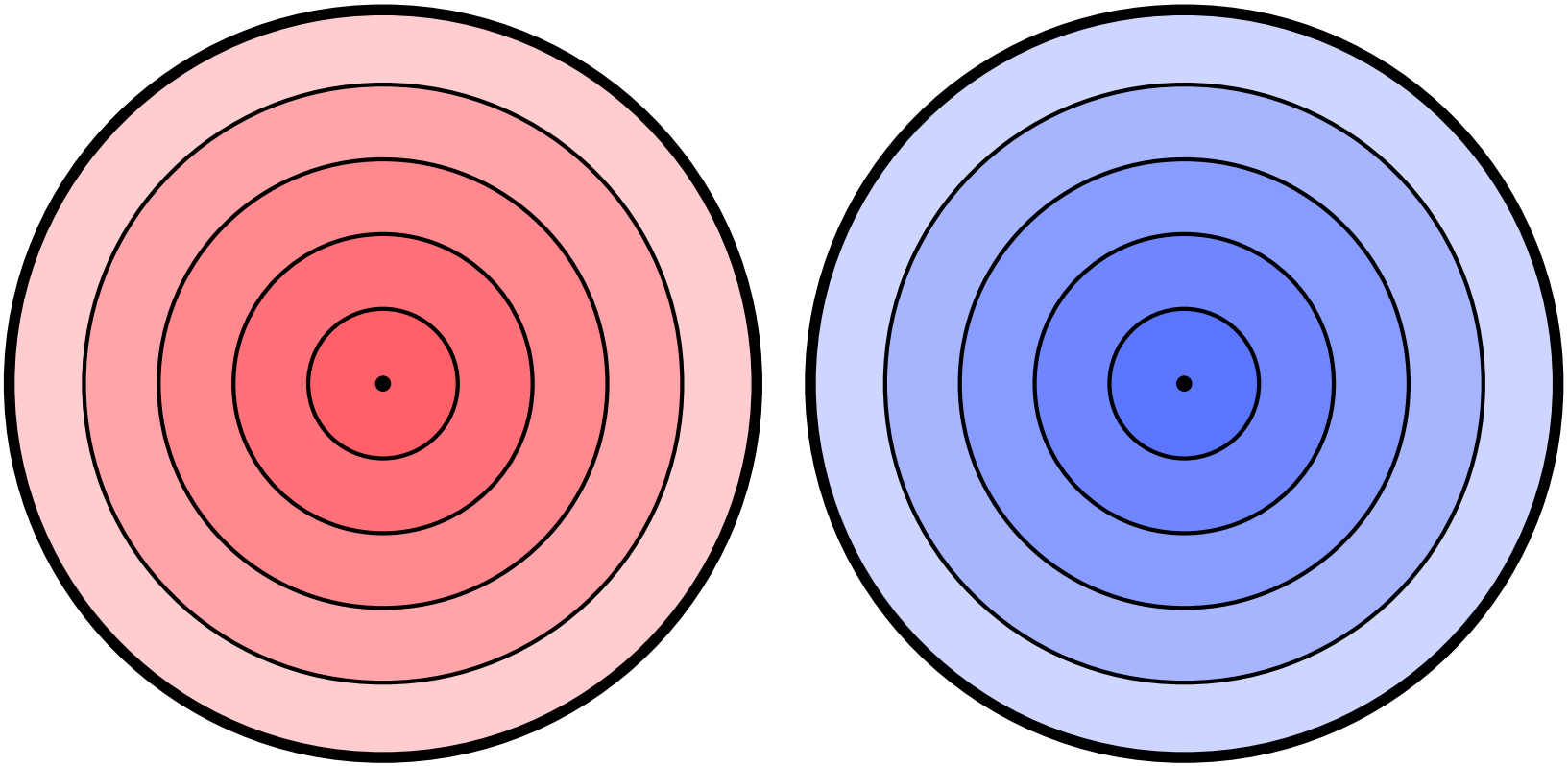

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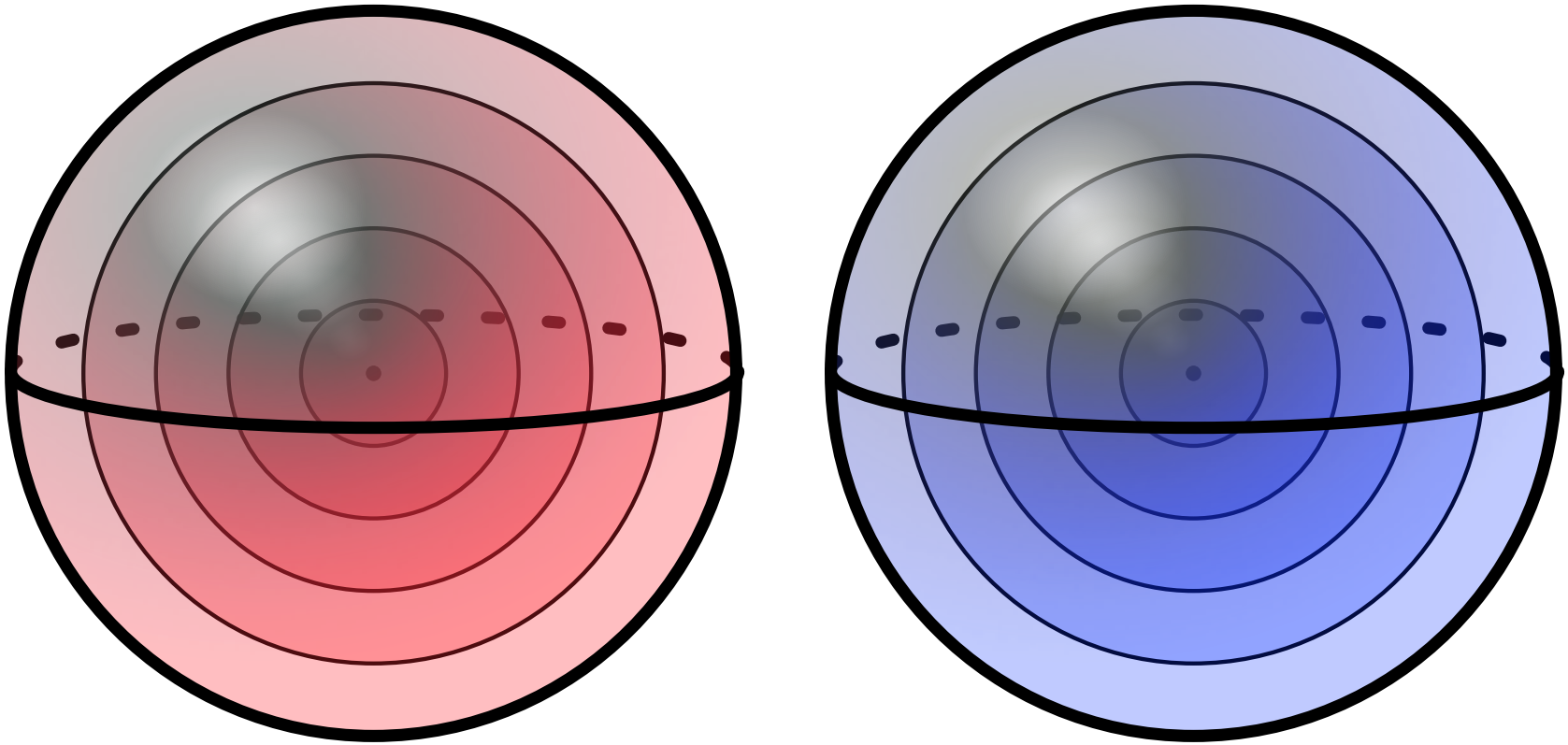

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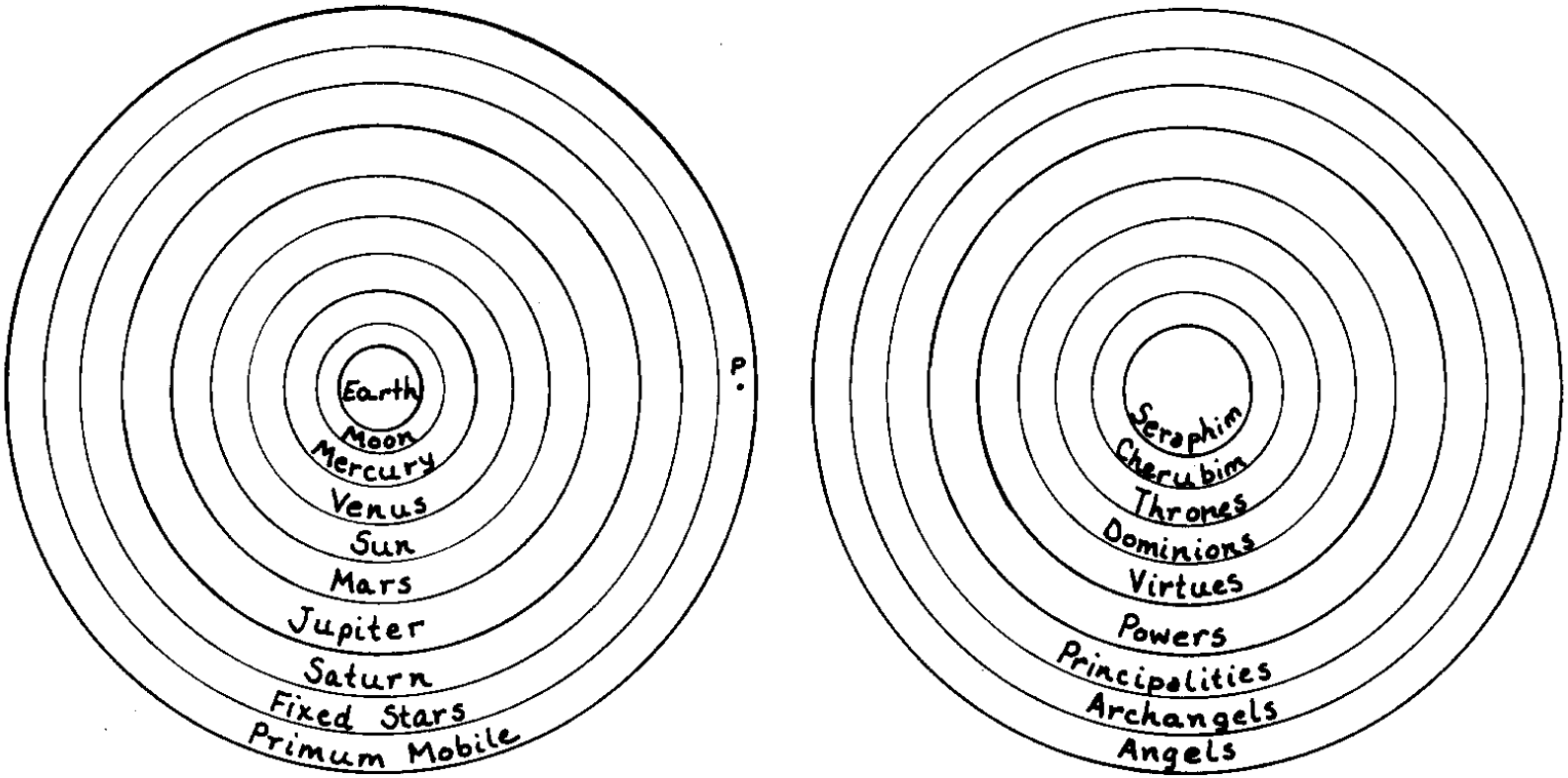

Living on the 3-sphere

If you stand at the north pole and your friend moves away from you, what do you see?

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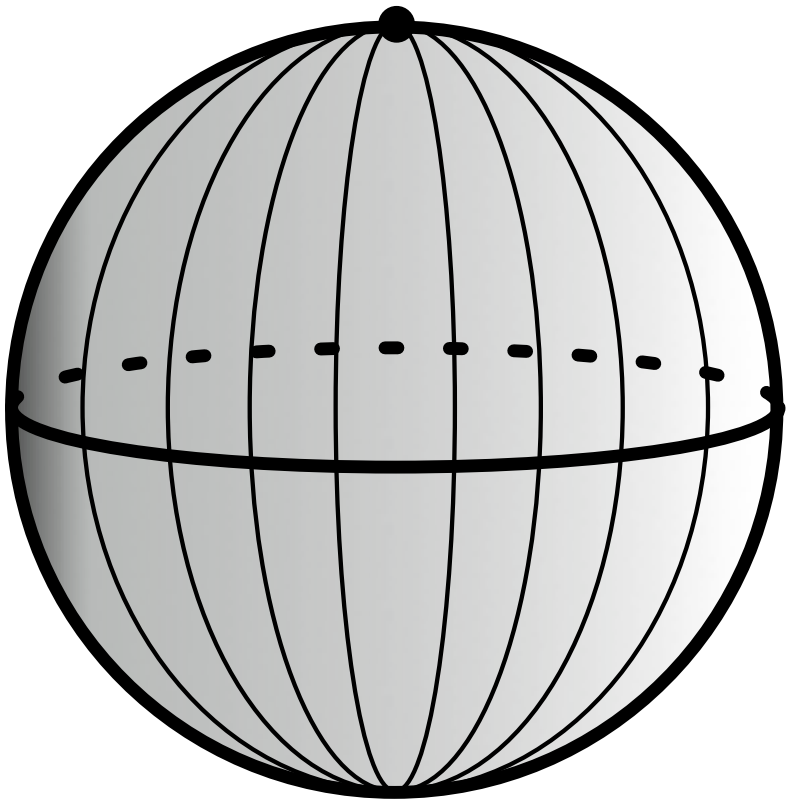

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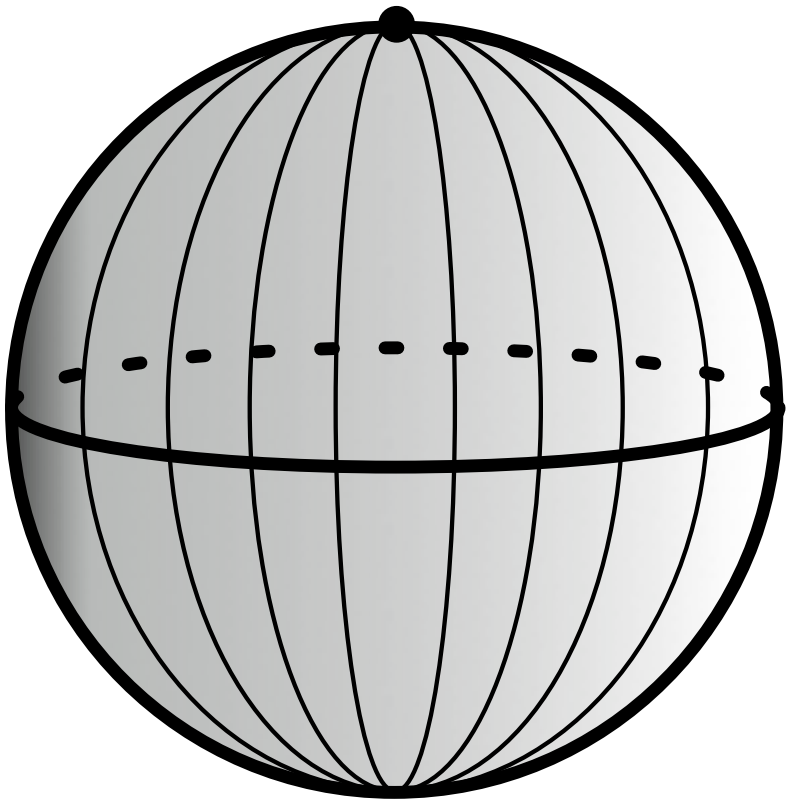

A Linking Integral for the 3-sphere

Given $K = \{\mathbf{x}(s)\}$ and $L = \{\mathbf{y}(t)\}$, two closed curves in the 3-sphere, we want to find an integral formula for $Lk(K, L)$.

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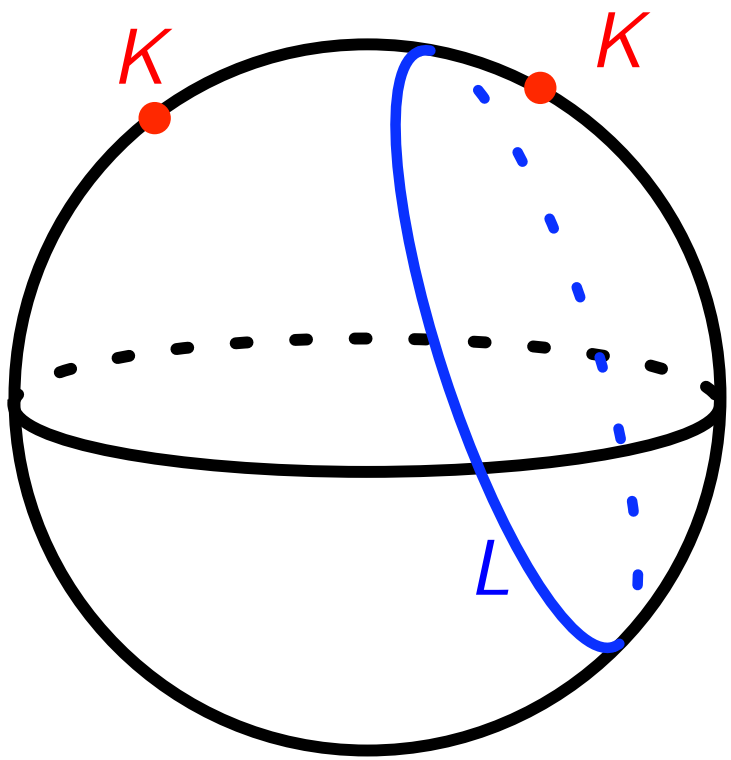
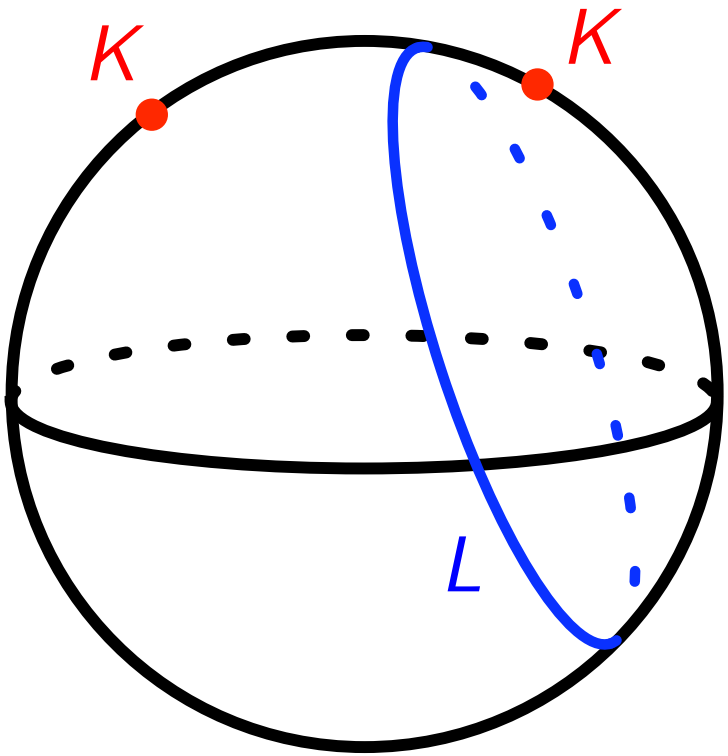
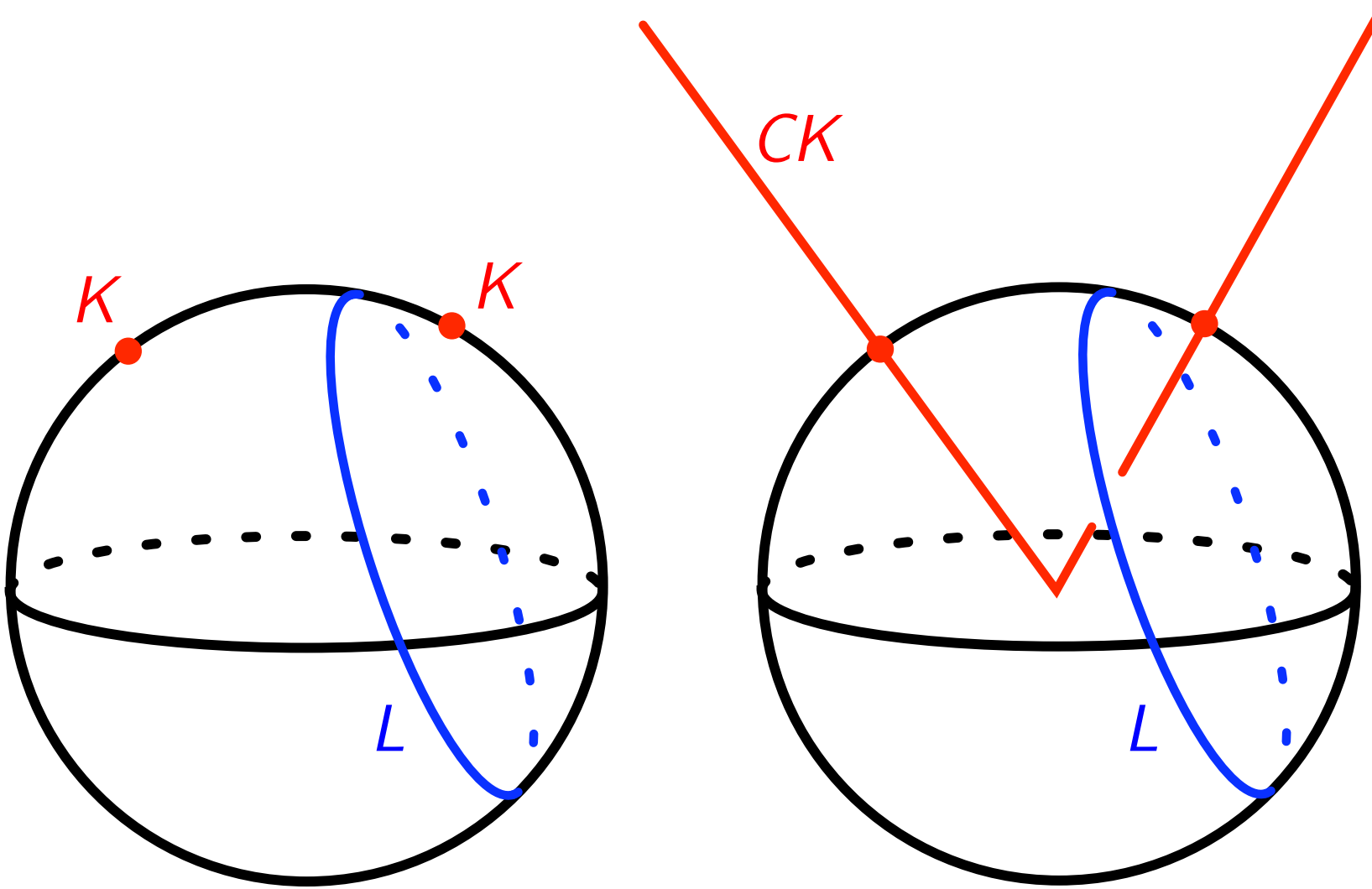


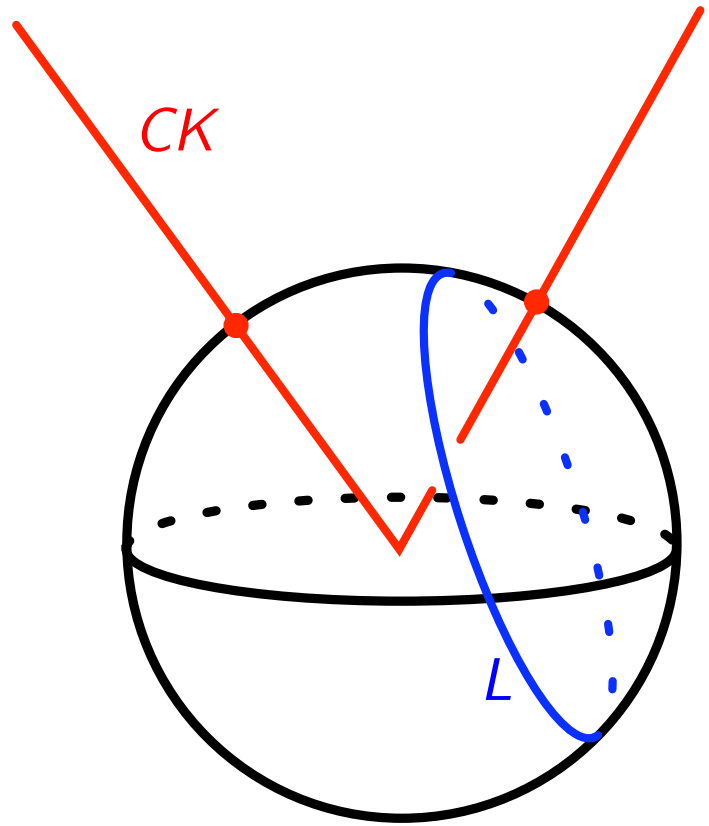

K
K
L


K
K
L
CK
L

Bumping up the dimension of K

We bump up the dimension of K by taking the half-infinite cone from the origin:

$$CK := \{\tau \mathbf{x} \,|\, \mathbf{x} \in K, \tau \in [0, \infty)\}$$



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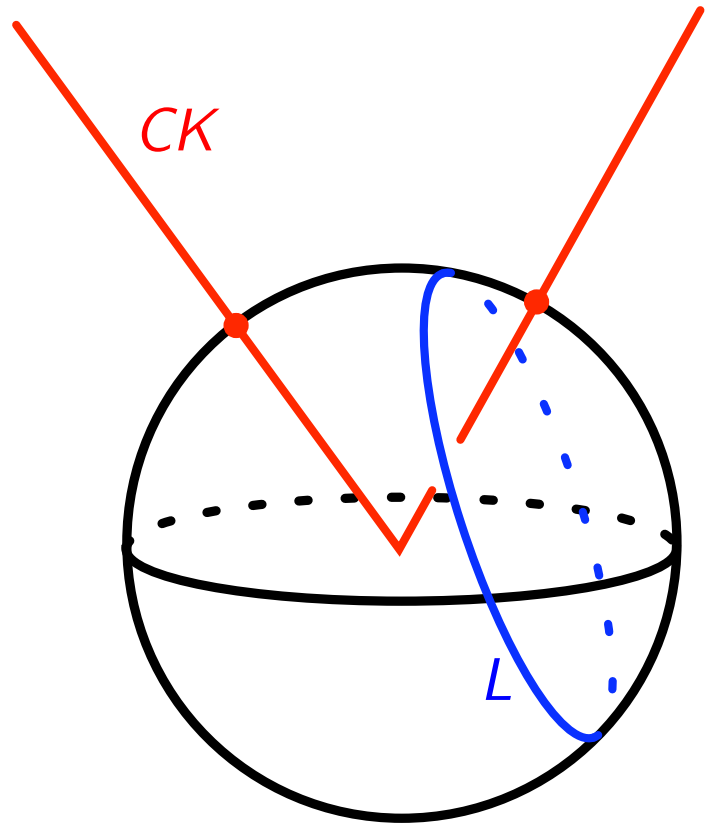

Then $Lk_{S^3}(K, L) = Lk_{\mathbb{R}^4}(CK, L)$, which we can compute using the linking integral in $\mathbb{R}^{n+1}$.

The Linking Integral for S^3

$$Lk(K,L) = \frac{1}{2\pi^2} \int_{CK \times L} \frac{-1}{|\tau\mathbf{x} - \mathbf{y}|^4} [\tau\mathbf{x} - \mathbf{y}, d(\tau\mathbf{x}), d\mathbf{y}]$$

$$\begin{aligned} Lk(K,L) &= \frac{1}{2\pi^2}\int_{CK\times L}\frac{-1}{|\tau\mathbf{x}-\mathbf{y}|^4}\,[\tau\mathbf{x}-\mathbf{y},d(\tau\mathbf{x}),d\mathbf{y}] \\ &= \frac{1}{2\pi^2}\int_{K\times L}\left(\int_{\tau=0}^{\infty}\frac{\tau}{|\tau\mathbf{x}-\mathbf{y}|^4}\,d\tau\right)[\mathbf{x},d\mathbf{x},\mathbf{y},d\mathbf{y}] \end{aligned}$$

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&= \frac{1}{2\pi^2}\int_{K\times L}\left(\int_{\tau=0}^{\infty}\frac{\tau}{|\tau\mathbf{x}-\mathbf{y}|^4}\,d\tau\right)[\mathbf{x},\,d\mathbf{x},\,\mathbf{y},\,d\mathbf{y}] \\
&= \frac{1}{2\pi^2}\int_{K\times L}\frac{\Omega(\alpha)}{\sin^3\alpha}\,[\mathbf{x},\,d\mathbf{x},\,\mathbf{y},\,d\mathbf{y}]
\end{aligned}
$$

$$Lk(K,L) = \frac{1}{2\pi^2} \int_{K\times L} \frac{\Omega(\alpha)}{\sin^3 \alpha} [\mathbf{x}, d\mathbf{x}, \mathbf{y}, d\mathbf{y}]$$

Here $\alpha(s,t)$ is the spherical distance from $\mathbf{x}(s)$ to $\mathbf{y}(t)$ and

$$\Omega(\alpha) = \int_{\theta=\alpha}^{\pi} \sin(\theta - \alpha) \sin\theta \, d\theta.$$

The same techniques work for spheres of arbitrary dimension:

Theorem (Kuperberg, DeTurck–Gluck, ___–Vela-Vick)

Let K^k, L^ℓ be disjoint, oriented, closed submanifolds of S^n with $k + \ell = n - 1$. Then

$$Lk(K, L) = \frac{1}{Vol\ S^n} \int_{K \times L} \frac{\Omega_{k,\ell}(\alpha)}{\sin^n \alpha} [x, dx, y, dy]$$

where

$$\Omega_{k,\ell}(\alpha) = \int_{\theta=\alpha}^{\pi} \sin^k(\theta - \alpha) \sin^\ell \theta d\theta$$

and $\alpha(x, y)$ is the geodesic distance in S^n from $x \in K$ to $y \in L$.