Poincaré Duality Angles on Riemannian Manifolds with Boundary

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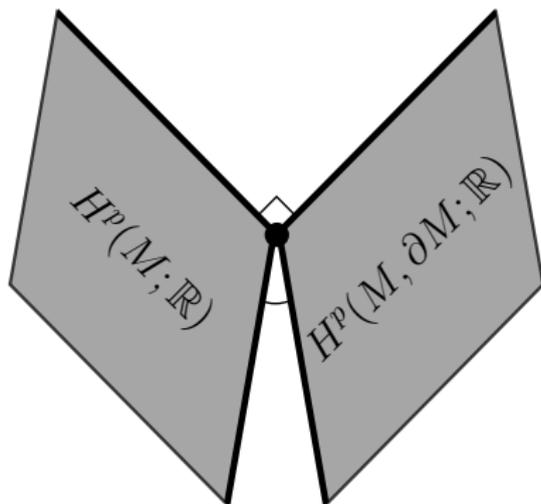
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Realizing cohomology groups as spaces of differential forms

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de Rham's Theorem

Suppose M^n is a compact, oriented, smooth manifold. Then

$$H^p(M; \mathbb{R}) \cong \mathcal{C}^p(M)/\mathcal{E}^p(M),$$

where $\mathcal{C}^p(M)$ is the space of closed p-forms on M and $\mathcal{E}^p(M)$ is the space of exact p-forms.

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Kodaira called this the space of *harmonic p-fields* on M.

Hodge's Theorem

If M^n is a closed, oriented, smooth Riemannian manifold,

$$H^p(M; \mathbb{R}) \cong \mathcal{H}^p(M).$$

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Then

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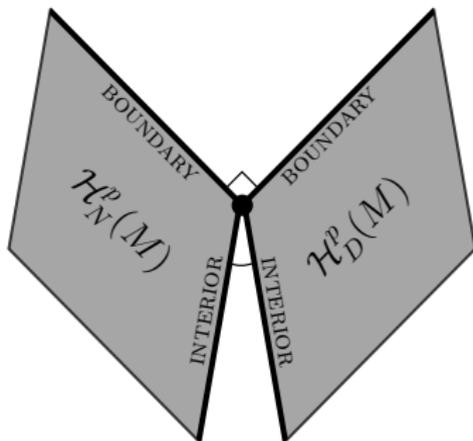
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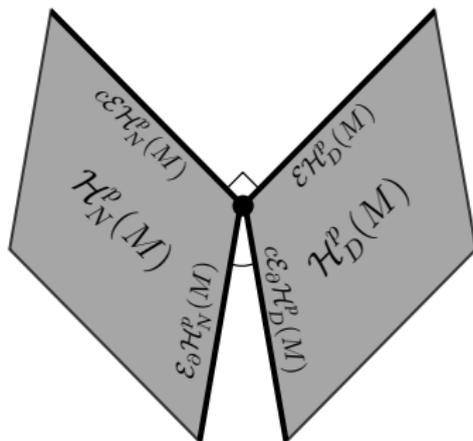
Interior subspace of $\mathcal{H}_D^p(M)$:

$$\star \mathcal{E}_\partial \mathcal{H}_N^{n-p}(M) = c\mathcal{E}_\partial \mathcal{H}_D^p(M)$$
$$= \{\eta \in \mathcal{H}_D^p(M) : i^* \star \eta = d\psi, \psi \in \Omega^{n-p-1}(\partial M)\}.$$

Definition (DeTurck–Gluck)

The *Poincaré duality angles* of the Riemannian manifold M are the principal angles between the interior subspaces.
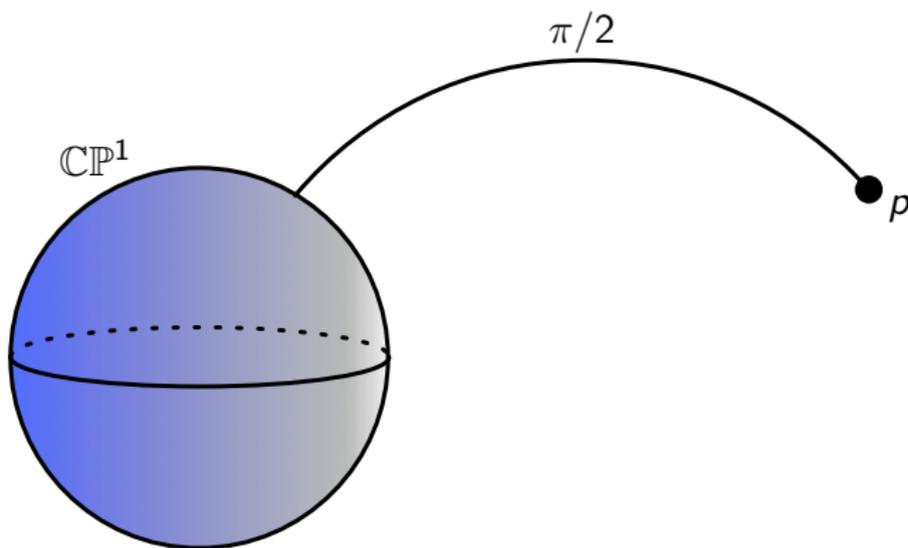
Guess
If M is "almost" closed, the Poincaré duality angles of M should be small.

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M_r has absolute cohomology in dimensions 0 and 2.

M_r has relative cohomology in dimensions 2 and 4.

Therefore, M_r has a single Poincaré duality angle θ_r between $\mathcal{H}^2_N(M_r)$ and $\mathcal{H}^2_D(M_r)$.

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Find closed and co-closed $SU(2)$-invariant forms on M_r satisfying Neumann and Dirichlet boundary conditions

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Generalizes to $\mathbb{CP}^n - B_r(p)$.

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Theorem

- *As $r \to 0$, all the Poincaré duality angles of N_r go to zero.*
- *As r approaches its maximum value of $\pi/2$, all the Poincaré duality angles of N_r go to $\pi/2$.*

Conjecture

If M^n is a closed Riemannian manifold and N^k is a closed submanifold of codimension ≥ 2, the Poincaré duality angles of

$$M - \nu_r(N)$$

go to zero as $r \to 0$.

What can you learn about the topology of M from knowledge of ∂M?

Induce potentials on the boundary of a region and determine the conductivity inside the region by measuring the current flux through the boundary.

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If γ is the conductivity on M, the current flux through ∂M is given by

$$(\gamma \nabla u) \cdot \nu = -\gamma \frac{\partial u}{\partial \nu}$$

The map $\Lambda_{\mathrm{cl}} : C^\infty(\partial M) \to C^\infty(\partial M)$ defined by

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Theorem (Lee-Uhlmann)

If M^n is a compact, analytic Riemannian manifold with boundary, then M is determined up to isometry by Λ_{cl}.

Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

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Theorem (Belishev–Sharafutdinov)

The data $(\partial M, \Lambda)$ completely determines the cohomology groups of M.

Define the *Hilbert transform* $T := d\Lambda^{-1}$.

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Theorem
If $\theta_1, \ldots, \theta_k$ are the Poincaré duality angles of M in dimension p, then the quantities

$$(-1)^{np+n+p} \cos^2 \theta_i$$

are the non-zero eigenvalues of an appropriate restriction of T^2.

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Theorem
The mixed cup product

$$\cup : H^p(M; \mathbb{R}) \times H^q(M, \partial M; \mathbb{R}) \to H^{p+q}(M, \partial M; \mathbb{R})$$

is completely determined by the data $(\partial M, \Lambda)$ when the relative class is restricted to come from the boundary subspace.

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- Can the L^2 inner product on $\mathcal{H}_N^p(M)$ and $\mathcal{H}_D^p(M)$ be recovered from $(\partial M, \Lambda)$?

Thanks!