FOUR ISOPERIMETRIC PROPERTIES OF HOMOGENEOUS SPHERICAL MEMBRANES

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ABSTRACT. In this talk I will demonstrate inequalities for four different types of inhomogeneous vibrating membranes which were proved by Hersch [4] using conformal techniques inspired by Szegö [9]. Specifically, the inequalities are lower bounds on the reciprocal sum of the first 3 non-trivial eigenvalues of the Laplacian of the membranes. It turns out that in each case the bound is achieved by a homogeneous spherical membrane, giving us the four isoperimetric properties promised in the title.

1. ISOPERIMETRIC PROPERTIES

The ancient isoperimetric problem was to determine the planar region with maximal area given a fixed perimeter. Of course, it was known to the Greeks that the circle enclosed maximal area for a given perimeter. In other words, for all regions R of fixed perimeter P,

$$\text{Area}(R) \leq \frac{P^2}{4\pi},$$

which is the original isoperimetric inequality, with equality achieved by a circle of radius $r = \frac{P}{2\pi}$.

Of course, the isoperimetric problem as stated can be extended to arbitrary dimensions by asking, say, what n-dimensional manifold M with boundary ∂M with fixed volume $\text{Vol}(\partial M)$ has maximal volume (where $\text{Vol}(\partial M)$ is $(n-1)$-dimensional volume and $\text{Vol}(M)$ is n-dimensional volume). But we needn't be so literal in our generalization of the original isoperimetric problem. For example, we might ask which closed surfaces of a fixed surface area have maximal electrostatic capacity, or maximal torsional rigidity, or minimal fundamental frequency. This last is what concerns us here.

The fundamental frequency of a homogeneous vibrating membrane with fixed boundary is given by the smallest (non-zero) eigenvalue of the Laplacian Δu with boundary condition $u = 0$. This problem has its roots in the work of Rayleigh [8], who conjectured that a circular region has the smallest non-zero eigenvalue (or, as he put it, "gravest fundamental tone") of all planar regions; this conjecture was later proved by Faber [2] and Krahn [6]. We can consider different boundary conditions; if instead we allow the boundary to be free (i.e. require that $\frac{\partial u}{\partial n} = 0$ on the boundary), then the membrane has eigenvalues $\mu_1 \leq \mu_2 \leq \ldots$. $\mu_1 = 0$ since $u = 1$ trivially satisfies the boundary condition. Szegö [9] proved that a circular membrane yields the maximum value of μ_2 among all membranes of fixed area and Weinberger [10] proved the analogous result in dimension n.

In this context, we will consider we will consider the Laplacian to be $\Delta u = \text{div grad} u$.

2. Closed Membranes

Proposition 2.1. *Let S be a closed surface of the topological type of a sphere which can be conformally mapped, except possibly in isolated points, to the sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2 = 1\}$. Let $\rho \geq 0$ be a density function on S. Then $M = \iint_S \rho dVol_S$ is the mass of S. Let Δ be the Laplace-Beltrami operator and consider the differential equation $\Delta u(p) + \mu\rho(p)u(p) = 0$ on S with eigenvalues $\mu_1 = 0 < \mu_2 \leq \mu_3 \leq \mu_4 \leq \ldots$. If we ignore the case where ρ has point masses, then*

$$\left[\frac{1}{\mu_2} + \frac{1}{\mu_3} + \frac{1}{\mu_4}\right]\frac{1}{M} \geq \frac{3}{8\pi}, \tag{1}$$

with equality precisely when ρ is constant and S is a sphere of radius r (then $\mu_2 = \mu_3 = \mu_4 = \frac{2}{r^2\rho}$ and $M = 4\pi r^2 \rho$). In other words, homogeneous spherical membranes minimize the above inequality.

Proof. Let $f : S \to S^2$ be the conformal map from S to the sphere in the statement of the proposition and, for $p \in S$, denote $f(p) = \hat{p}$. Let $\hat{\rho}(\hat{p}) = \rho(p)\frac{dVol_S}{dVol_{S^2}}$. To make things easier for ourselves, we will first consider the case where the center of mass $\hat{G}$ of S^2 coincides with the center $\hat{O}$ of the sphere; we will prove a lemma (Lemma 2.2 below) demonstrating that this is a valid assumption.

Now, for $p \in S$, suppose $\hat{p} = (x,y,z)$ and, by abuse of notation, let $X(p) = x(\hat{p}) = x$, $Y(p) = y$ and $Z(p) = z$. Supposing that $\hat{G} = \hat{O}$, then

$$\iint_S \rho X dVol_S = \iint_{S^2} \hat{\rho} x dVol_{S^2} = 0$$

and, similarly, $\iint_S \rho Y dVol_S = 0$ and $\iint_S \rho Z dVol_S = 0$. Since the Dirichlet integral is invariant under conformal transformations,

$$\iint_S \langle \nabla X, \nabla Y \rangle dVol_S = \iint_{S^2} \langle \nabla x, \nabla y \rangle dVol_{S^2} = 0$$

and, similarly, $\iint_S \langle \nabla Y, \nabla Z \rangle dVol_S = 0$ and $\iint_S \langle \nabla Z, \nabla X \rangle dVol_S = 0$. Furthermore,

$$\iint_S \|\nabla X\|^2 dVol_S = \iint_{S^2} \|\nabla x\|^2 dVol_{S^2} = \frac{8\pi}{3}$$

by a simple calculation and, similarly, $\iint_S \|\nabla Y\|^2 dVol_S = \iint_S \|\nabla Z\|^2 dVol_S = \frac{8\pi}{3}$. Finally, note that $X^2+Y^2+Z^2 = x^2+y^2+z^2 = 1$. Therefore, the linear space $L(X,Y,Z)$ spanned by X, Y and Z admits the following variational characterization due to an earlier result of Hersch [3]:

$$\begin{aligned}\frac{1}{\mu_2}+\frac{1}{\mu_3}+\frac{1}{\mu_4} &\geq \frac{\iint \rho X^2 dVol_S}{\iint \|\nabla X\|^2}+\frac{\iint \rho Y^2 dVol_S}{\iint \|\nabla Y\|^2}+\frac{\iint \rho Y^2 dVol_S}{\iint \|\nabla Y\|^2}\\ &= \frac{3}{8\pi}\iint_S \rho(X^2+Y^2+Z^2)dVol_S\\ &= \frac{3}{8\pi}\iint_S \rho dVol_S\\ &= \frac{3M}{8\pi}.\end{aligned}$$

Note that if S is the sphere S^2 and ρ is a constant, then the coordinate functions x, y and z are eigenfunctions of the Laplacian, so the inequality above becomes an equality. This proves the desired result provided we can show that our assumption about the center of mass was a legitimate one, which is the content of the following lemma. □

Lemma 2.2. *Let $\widetilde{\rho} \geq 0$ be a density on the sphere S^2 (without point masses). Then there exists a Möbius transformation $g : S^2 \to S^2$ such that $\hat{\rho}(g(p)) = \widetilde{\rho}(p)$ has center of mass $\hat{G}$ in the center $\hat{O}$ of the sphere.*

Proof. This proof is inspired by a method of Szegö [9]. Let p be a point of S^2 and let $0 < t \leq 1$. Let $H_{p,t}$ be the Möbius transformation induced by the homothety $\zeta \mapsto t\zeta$ on T_pS^2; that is,

$$H_{p,t}(q) = \begin{cases}\gamma_q(t) & q \neq -p\\ q & q = -p\end{cases}$$

where $\gamma_q : [0,1] \to S^2$ is the geodesic from p to q. So long as $t \neq 1$, the only fixed points of $H_{p,t}$ are p and its antipode; $H_{p,1}$ is the identity. Denote by $\widetilde{\rho}_{p,t}$ the new density function induced by the application of $H_{p,t}$ to $\widetilde{\rho}$ and by $\widetilde{G}_{p,t}$ the center of mass of $\widetilde{\rho}_{p,t}$. Then certainly $\widetilde{\rho}_{p,1} = \widetilde{\rho}$ and $\widetilde{G}_{p,1} = \widetilde{G}$; as $t \to 0$, $\widetilde{G}_{p,t} \to p$. On the other hand, the total mass is always conserved: $\widetilde{M}_{p,t} = \widetilde{M} = M$. Consider the surface

$$\widetilde{G}_t = \{\widetilde{G}_{p,t} | p \in S^2\}$$

which, in general, has self-intersections. For t close to zero, $\widetilde{G}_t$ contains $\hat{O}$, the center of S^2. However, as $t \to 1$, $\widetilde{G}_t$ closes down on the point $\widetilde{G}$ (for $t = 1$, $\widetilde{G}_t$ *is* the point $\widetilde{G}$). Obviously, if $\widetilde{G} = \hat{O}$, the desired transformation is the identity, so we let g simply be the identity. Otherwise, there exists a t_0 such that $\hat{O} \in \widetilde{G}_t$, so $\hat{O} = \widetilde{G}_{p_0,t_0}$ for some $p_0 \in S^2$. Then $g := \widetilde{G}_{p_0,t_0}$ is the desired transformation. □

Corollary 2.3. *The following inequalities are an immediate consequence of Proposition 2.1:*

$$\left[\frac{1}{\mu_2}+\frac{2}{\mu_3}\right]\frac{1}{M}\geq\frac{3}{8\pi}, \tag{2}$$

$$\mu_2 M\leq 8\pi\approx 25.133. \tag{3}$$

If $\rho=1$, then M is simply the surface area $A(S)$ of the membrane, so $\mu_2 A(S)\leq 8\pi$. Hence, we see that for homogeneous membranes of a fixed surface area A, the sphere has maximal μ_2, so this is an isoperimetric property of homogeneous spherical membranes.

Example: if S is a regular tetrahedron with constant density ρ, then $\mu_2 M=\frac{4\pi^2}{\sqrt{3}}\approx 22.793$.

3. Jordan Domains

Proposition 3.1. *Let J be a Jordan domain with density $\rho\geq 0$. Let λ_1 be the first eigenvalue of J as a membrane with fixed boundary (i.e. with boundary condition $u|_{\partial S}=0$) and let $\mu_1=0<\mu_2\leq\mu_3\leq\dots$ be the eigenvalues of the free membrane (i.e. with boundary condition $\frac{\partial u}{\partial n}\big|_{\partial S}=0$). Then, excluding the case of point masses,*

$$\left[\frac{1}{\lambda_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}\right]\frac{1}{M}=\frac{3}{4\pi} \tag{4}$$

with equality when ρ is constant and J is a hemisphere of radius r (then $\lambda_1=\mu_2=\mu_3=\frac{2}{r^2\rho}$ and $M=2\pi r^2\rho$).

Proof. As before, we want to make things easier for ourselves using a conformal transplantation of J, this time to the hemisphere $\hat{J}=\{(x,y,z)\in\mathbb{R}^3|x^2+y^2+z^2=1,z>0\}$. We would like this transplantation to be such that the center of mass $\hat{G}$ of $\hat{J}$ lies on the z-axis. That such a conformal map exists is the content of the following lemma:

Lemma 3.2. *Given a density $\widetilde{\rho}\geq 0$ on the hemisphere $\hat{J}$ without point masses with center of mass $\widetilde{G}$, there exists a conformal map $H_{p_0,t_0}:\hat{J}\to\hat{J}$ with p_0 on the equator E of $\hat{J}$ such that the density $\hat{\rho}$ induced by the application of H_{p_0,t_0} to $\widetilde{\rho}$ has center of mass $\hat{G}$ on the z-axis.*

Proof. Again, the methodology here is inspired by Szegö [9]. We let $H_{p,t}$ be the restriction of the map defined in Lemma 2.2 to the hemisphere, with corresponding density $\widetilde{\rho}_{p,t}$ and center of mass $\widetilde{G}_{p,t}$. Then, as $t\to 0$, $\widetilde{G}_{p,t}\to p\in E$. Consider the curve

$$\widetilde{G}_t=\{\widetilde{G}_{p,t}|p\in E\}.$$

Then as $t\to 0$, the curve $\widetilde{G}_t$ has winding number 1 with respect to the z-axis. If $\widetilde{G}$ already lies on the z-axis, then the desired transformation is the identity; otherwise, for t sufficiently close to 1, $\widetilde{G}_t$ has winding number 0 about the z-axis. Thus, there exists $t_0 \in (0,1)$ such that $\widetilde{G}_{t_0}$ intersects the z-axis in a point $\hat{G} = \widetilde{G}_{p_0,t_0}$. Then H_{p_0,t_0} is the desired transformation. □

With this lemma in hand, we are now free to assume that the center of mass of $\hat{J}$ lies on the z-axis. With notation as in Proposition 2.1, this implies that

$$\iint_J \rho X dVol_J = \iint_J \rho Y dVol_J = 0, \quad \iint_J \langle \nabla X, \nabla Y \rangle dVol_J = 0,$$

$$\iint_J \|\nabla X\|^2 dVol_J = \iint_J \|\nabla Y\|^2 dVol_j = \iint_J \|\nabla Z\|^2 dVol_j = \frac{4\pi}{3}$$

and

$$X^2 + Y^2 + Z^2 \equiv x^2 + y^2 + z^2 \equiv 1.$$

Since $z = 0$ on the boundary E of $\hat{J}$, $\lambda_1 \leq \frac{\iint_J \|\nabla Z\|^2 dVol_J}{\iint_J \rho Z^2 dVol_J}$ and, by the variational characterization in [3],

$$\frac{1}{\mu_2} + \frac{1}{\mu_3} \geq \frac{\iint_J \rho X^2 dVol_J}{\iint_J \|\nabla X\|^2 dVol_J} + \frac{\iint_J \rho Y^2 dVol_J}{\iint_J \|\nabla Y\|^2 dVol_J}.$$

Hence,

$$\begin{aligned}\frac{1}{\lambda_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} &\geq \frac{\iint_J \rho X^2 dVol_J}{\iint_J \|\nabla X\|^2 dVol_J} + \frac{\iint_J \rho Y^2 dVol_J}{\iint_J \|\nabla Y\|^2 dVol_J} + \frac{\iint_J \rho Z^2 dVol_J}{\iint_J \|\nabla Z\|^2 dVol_J} \\ &= \frac{3}{4\pi} \iint_J \rho dVol_J \\ &= \frac{3M}{4\pi}.\end{aligned}$$

(4) follows directly. Note that on $\hat{J}$ with $\rho \equiv 1$, z is the eigenfunction of λ_1 and x and y are eigenfunctions of $\mu_2 = \mu_3$. □

4. 2-sided membranes

Proposition 4.1. *Let B be a Jordan domain with two fixed points on its boundary and let a and b be the "sides" of B. Let $\rho \geq 0$ be the density on B. Let λ_a be the first eigenvalue of the Laplacian on B with a fixed, let λ_b be the first eigenvalue with b fixed and let $\mu_1 = 0 < \mu_2 \leq \dots$ be the eigenvalues of the free membrane. Excluding the case of point masses,*

$$\left[\frac{1}{\lambda_a} + \frac{1}{\lambda_b} + \frac{1}{\mu_2}\right] \frac{1}{M} \geq \frac{3}{2\pi} \tag{5}$$

with equality when ρ is constant and B is an "orange slice" (quarter sphere) given by $x^2 + y^2 + z^2 = r^2$, $x > 0$, $y > 0$ (then $\lambda_a = \lambda_b = \mu_2 = \frac{2}{r^2 \rho}$ and $M = \pi r^2 \rho$).

Proof. Again, we conformally map B onto a spherical membrane $\hat{B}$. Let

$$\hat{B} = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1, x > 0, y > 0\}$$

with sides $\hat{a}, \hat{b}$ the meridians bounding $\hat{B}$. If we let $\hat{N}$ be the north pole of $\hat{B}$ and $\hat{S}$ the south pole, then, as $t \to 0$, $\widetilde{G}_{\hat{N},t} \to \hat{N}$. As $t \to \infty$, $\widetilde{G}_{\hat{N},t} \to \hat{S}$. Hence, there exist $t_0 \in (0, \infty)$ and $p_0 \in \hat{B}$ such that $\widetilde{G}_{p_0,t_0}$ lies in the xy-plane. Then $\widetilde{H}_{p_0,t_0} : \hat{B} \to \hat{B}$ is a conformal transformation with center of mass on the xy-plane. Then, with terminology as in Proposition 2.1,

$$\iint_B \rho Z dVol_B = 0, \iint_B \|\nabla X\|^2 dVol_B = \iint_B \|\nabla Y\|^2 dVol_B = \iint_B \|\nabla Z\|^2 dVol_B = \frac{2\pi}{3}$$

and $X^2 + Y^2 + Z^2 = x^2 + y^2 + z^2 = 1$. Since $X = 0$ on a and $Y = 0$ on b, $\lambda_a \leq \frac{\iint_B \|\nabla X\|^2 dVol_B}{\iint_B \rho X^2 dVol_B}$ and $\lambda_b \leq \frac{\iint_B \|\nabla Y\|^2 dVol_B}{\iint_B \rho Y^2 dVol_B}$ and, by the variational characterization of μ_2,

$$\mu_2 \leq \frac{\iint_B \|\nabla Z\|^2 dVol_B}{\iint_B \rho Z^2 dVol_B}.$$

Therefore,

$$\begin{aligned}
\frac{1}{\lambda_a} + \frac{1}{\lambda_b} + \frac{1}{\mu_2} &\geq \frac{\iint_B \rho X^2 dVol_B}{\iint_B \|\nabla X\|^2 dVol_B} + \frac{\iint_B \rho Y^2 dVol_B}{\iint_B \|\nabla Y\|^2 dVol_B} + \frac{\iint_B \rho Z^2 dVol_B}{\iint_B \|\nabla Z\|^2 dVol_B} \\
&= \frac{3}{2\pi} \iint_B \rho dVol_B \\
&= \frac{3M}{2\pi}.
\end{aligned}$$

(5) follows directly. □

5. 3-sided Jordan Domains

Proposition 5.1. *Let T be a Jordan domain with 3 fixed points on its boundary, with corresponding "sides" a, b and c. Let $\rho \geq 0$ be the density of T and let λ_a, λ_b and λ_c be the eigenvalues with a, b and c fixed, respectively. Then*

$$\left[\frac{1}{\lambda_a} + \frac{1}{\lambda_b} + \frac{1}{\lambda_c}\right] \frac{1}{M} \geq \frac{3}{\pi}, \tag{6}$$

with equality when ρ is constant and T is an equilateral spherical triangle (then $\lambda_a = \lambda_b = \lambda_c = \frac{2}{r^2\rho}$ and $M = \frac{\pi r^2 \rho}{2}$).

This proposition follows from a conformal transplantation argument essentially similar to those presented above; see [5].

6. Conclusion

Just a few comments on the above to wrap things up. First, in many cases (virtually all of the citations I found) we're interested in homogeneous membranes. In such cases, we can let $\rho = 1$; then what we called the mass M above is simply the surface area $A(S)$ of the membrane S. Then, for example, the inequality (3) simply tells us that $\mu_2 A(S) \leq 8\pi$, with equality if and only if S is isometric to a constant curvature metric on S^2.

Second, note that the above arguments rely heavily on conformal arguments and so do not generalize well to higher dimensions. Yang and Yau [12] did obtain a generalization for surfaces of genus g, namely that

$$\mu_2 A(S) \leq 8\pi(1+g).$$

Li and Yau [7] give another argument for the above results which also yields

$$\mu_2 A(S) \leq 12\pi$$

for S homeomorphic to $\mathbb{RP}^2$. For a manifold M of dimension $n \geq 3$, the results above would suggest we should look for bounds on $\mu_2^{n/2} V(M)$ that depend at most on the topology of M, but it turns out that no such bound exists when M is diffeomorphic to S^3 [11]. See Chavel [1] for a more thorough summary.

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