The Search for Higher Helicities

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Let V be a vector field on a compact domain $\Omega \subset \mathbb{R}^3$.

The *energy* of V is

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Suppose V is divergence-free. Then

$$V = \nabla \times W,$$

where W is the vector potential for V.

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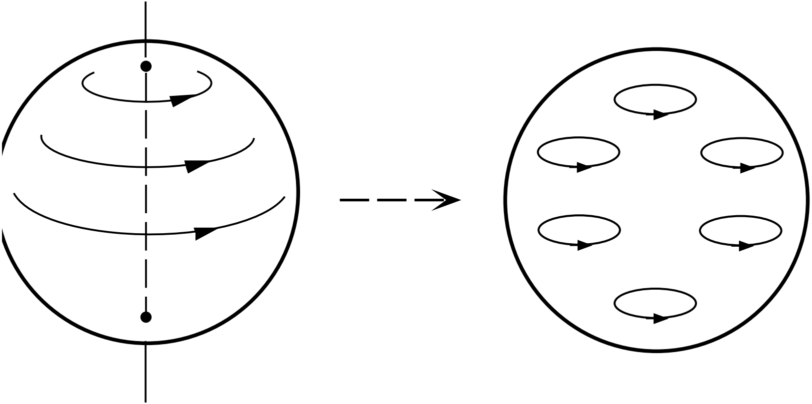
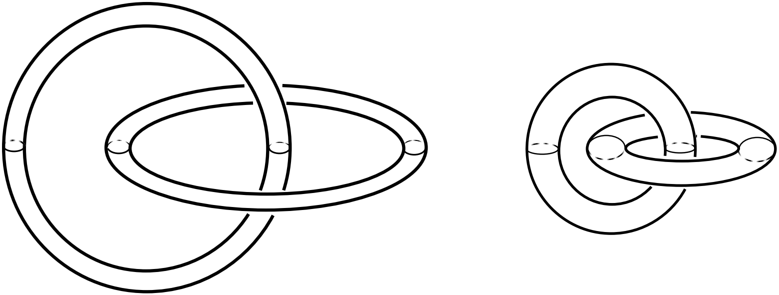
Mathematically, are there obstructions to making the field energy arbitrarily small via volume-preserving diffeomorphisms homotopic to the identity?

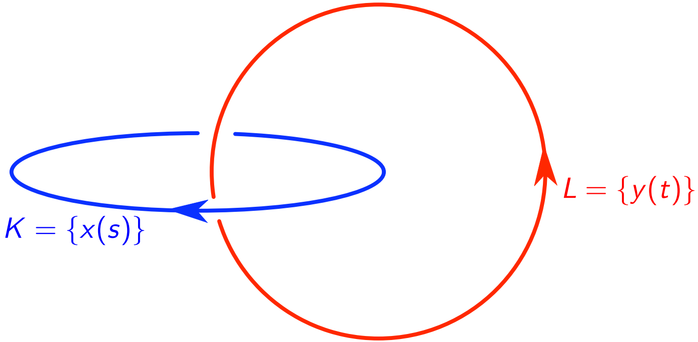
Answer: Not necessarily!



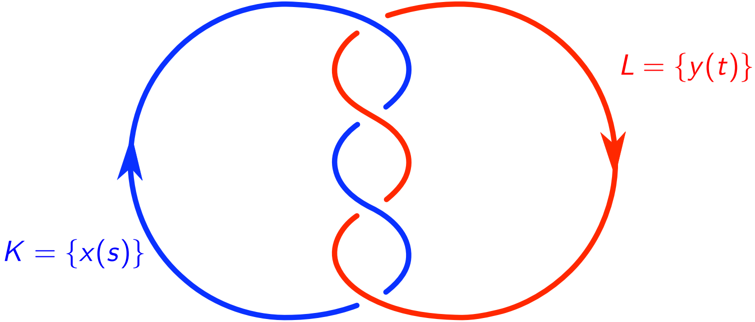
A field which can be relaxed to have arbitrarily small energy (Freedman).



Linked orbits prevent the energy from getting arbitrarily small.



$$1 = Lk(K, L) = \frac{1}{4\pi} \int_{K \times L} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x - y}{|x - y|^3} \, ds \, dt$$



$$2 = Lk(K, L) = \frac{1}{4\pi}\int_{K\times L} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x-y}{|x-y|^3}\, ds\, dt$$

The Gauss Linking Integral:

$$Lk(K,L) = \frac{1}{4\pi}\int_{K\times L} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x-y}{|x-y|^3}\, ds\, dt$$

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Important: the integrand is isometry-invariant.

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Definition (Woltjer)

The *helicity* of a divergence-free vector field V on a compact domain $\Omega \subset \mathbb{R}^3$ is

$$H(V) := \frac{1}{4\pi} \int_{\Omega \times \Omega} V(x) \times V(y) \cdot \frac{x - y}{|x - y|^3} \, \mathrm{dvol}_x \, \mathrm{dvol}_y .$$

Theorem

$H(V)$ is invariant under any volume-preserving diffeomorphism which is homotopic to the identity.

Theorem (Arnol'd)

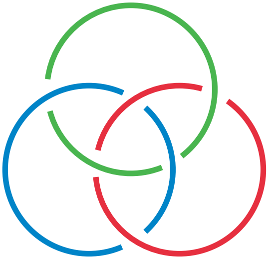
$$|H(V)| \leq R(\Omega)E(V)$$

where $R(\Omega)$ is a positive constant depending on the shape and size of Ω.

Arnol′d and Khesin:

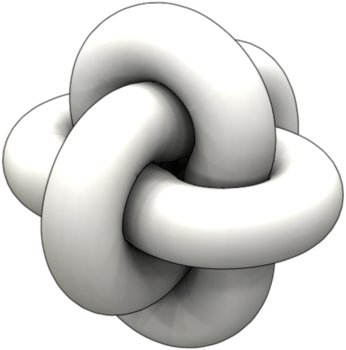
> *The dream is to define such a hierarchy of invariants for generic vector fields such that, whereas all the invariants of order* $\leq k$ *have zero value for a given field and there exists a nonzero invariant of order* $k+1$*, this nonzero invariant provides a lower bound for the field energy.*

Idea: Higher-order linking invariants



Three-component links were classified up to *link-homotopy* by Milnor:

- The pairwise linking numbers p, q, r.
- The *triple linking number* $\mu = \overline{\mu}_{123}$, which is an integer modulo $\gcd(p, q, r)$.



If V is supported on flux tubes, then μ provides a lower bound for the field energy (Monastyrsky–Retakh, Berger, etc.; see also Freedman–He).

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$$S^1 \times S^1 \hookrightarrow \mathrm{Conf}_2\mathbb{R}^3 \xrightarrow{\simeq} S^2$$

$$(s,t) \longmapsto (x(s), y(t)) \longmapsto \frac{x(s)-y(t)}{|x(s)-y(t)|}$$

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If f is the composition of the above maps

$$Lk(K,L) = \int_{S^1\times S^1} f^*\omega,$$

where ω is the standard area form on S^2 scaled to have area 1 instead of 4π.

Consider a three-component link in S^3:

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This determines a map

$$\begin{aligned} g: \quad \mathcal{L}(3) &\longrightarrow [S^1 \times S^1 \times S^1, S^2] \\ L &\longmapsto [g_L] \end{aligned}$$

from link-homotopy classes of 3-component links to homotopy classes of maps from the 3-torus to the 2-sphere.

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$[S^1 \times S^1 \times S^1, S^2]$ was classified by Pontryagin by:

- The degrees p, q, r on the 2-dimensional subtori.
- A "Hopf invariant" ν, which is an integer modulo $2\gcd(p, q, r)$.

Interpreting link-homotopy invariants as homotopy invariants

Theorem (DGKMSV)

The map $g : \mathcal{L}(3) \to [S^1 \times S^1 \times S^1, S^2]$ *is injective and maps*

$$p \mapsto p$$
$$q \mapsto q$$
$$r \mapsto r$$
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This implies that $\mathcal{L}(3) \to [S^1 \times S^1 \times S^1, \mathrm{Conf}_3\mathbb{R}^3]$ is injective, proving the $n = 3$ case of a conjecture of Koschorke.

A more symmetric version of g along with a modification of Whitehead's integral for the Hopf invariant of a map $S^3 \to S^2$ yields:

Theorem (DGKMSV)

$$\mu(L) = \frac{1}{2} \int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L,$$

where $\omega_L = g_L^* \omega$ *and* φ *is the fundamental solution of the Laplacian on* T^3.

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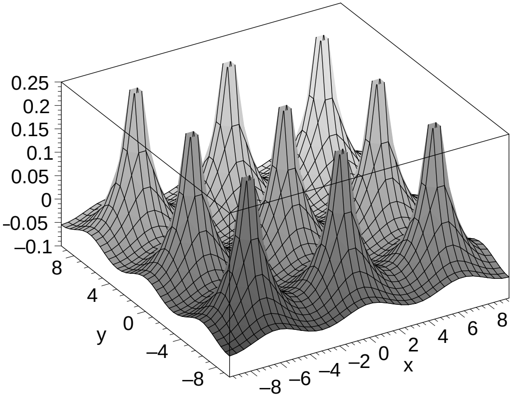
The fundamental solution of the Laplacian



$$\varphi(\mathbf{x}) = \frac{1}{8\pi^3} \sum_{\mathbf{n}\in\mathbb{Z}^3\setminus\{\mathbf{0}\}} \frac{e^{i\mathbf{n}\cdot\mathbf{x}}}{|\mathbf{n}|^2}.$$

Thanks!