Homotopy, link homotopy, and (higher?) helicity

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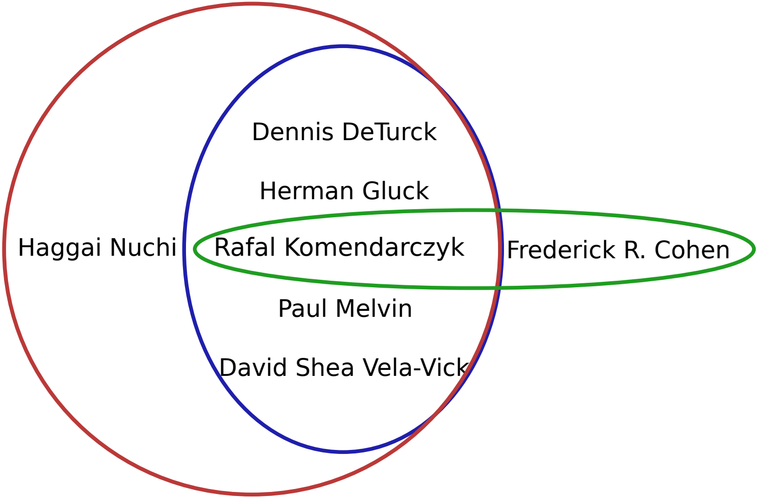
Joint work with



To find topological invariants of vector fields which provide lower bounds on the field energy.

Definition

The *helicity* of a vector field V supported on a compact domain $\Omega \subset \mathbb{R}^3$ is

$$H(V) := \frac{1}{4\pi} \int_{\Omega \times \Omega} V(\mathbf{x}) \times V(\mathbf{y}) \cdot \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{x} \, d\mathbf{y}.$$

Theorem (Woltjer, Moffatt, Arnol′d, ...)

$H(V)$ is invariant under any volume-preserving diffeomorphism which is homotopic to the identity.

Theorem (Arnol′d, ...)

$$|H(V)| \leq \lambda(\Omega) E(V),$$

where $\lambda(\Omega)$ is a positive constant depending on the shape and size of Ω.

Arnol′d and Khesin:

> *The dream is to define such a hierarchy of invariants for generic vector fields such that, whereas all the invariants of order* $\leq k$ *have zero value for a given field and there exists a nonzero invariant of order* $k+1$*, this nonzero invariant provides a lower bound for the field energy.*

Where does helicity come from?

$$H(V) = \frac{1}{4\pi} \int_{\Omega\times\Omega} V(\mathbf{x}) \times V(\mathbf{y}) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{x}\, d\mathbf{y}.$$

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If $L_1 \cup L_2$ is a two-component link in $\mathbb{R}^3$ such that $L_1 = \{\mathbf{x}(s) : s \in S^1\}$ and $L_2 = \{\mathbf{y}(t) : t \in S^1\}$, then the linking number between K and L is given by the Gauss linking integral

$$Lk(L_1, L_2) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \cdot \frac{\mathbf{x}(s) - \mathbf{y}(t)}{|\mathbf{x}(s) - \mathbf{y}(t)|^3} \, ds \, dt.$$

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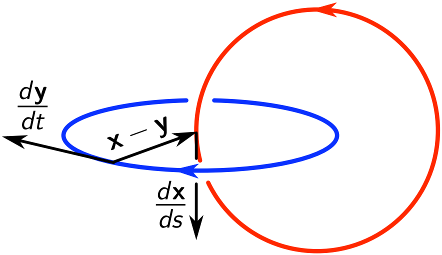

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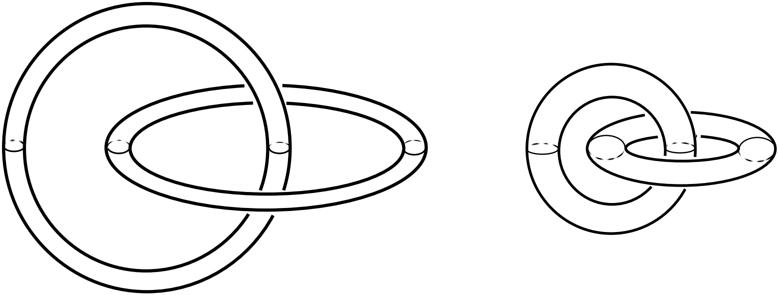
—Keith Moffatt



Linked orbits prevent the energy from getting arbitrarily small.

$$Lk(L_1, L_2) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \cdot \frac{\mathbf{x}(s) - \mathbf{y}(t)}{|\mathbf{x}(s) - \mathbf{y}(t)|^3} \, ds \, dt.$$

The integrand is invariant under the action of any orientation-preserving isometry of $\mathbb{R}^3$; i.e., if $h \in E^+(3)$, then the integrands for the links $L_1 \cup L_2$ and $h(L_1) \cup h(L_2)$ are identical functions of s and t.

The goal is to find "generalized Gauss linking integrals" which compute higher-order linking invariants and are *geometrically natural* (i.e. coordinate-independent) in the same sense.

The hope is that higher-order helicities can be defined by extending these integrals to vector fields.

Whence the Gauss linking integral?

The Gauss map of a two-component link $L = L_1 \cup L_2$ is

$$\begin{aligned} g_L : \ S^1 \times S^1 &\longrightarrow S^2 \\ (s,t) &\longmapsto \frac{\mathbf{x}(s) - \mathbf{y}(t)}{|\mathbf{x}(s) - \mathbf{y}(t)|} \end{aligned}$$

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The degree of g_L is equal to the linking number of L.

If ω is the area form on S^2 normalized to have area 1, then

$$\deg(g_L) = \int_{S^1 \times S^1} g_L^* \omega.$$

Expanding gives the Gauss linking integral.

Let Conf(n) be the configuration space of n distinct points in $\mathbb{R}^3$.

$$g_L : \; S^1 \times S^1 \hookrightarrow \mathrm{Conf}(2) \xrightarrow[\text{equiv.}]{\text{homotopy}} S^2$$

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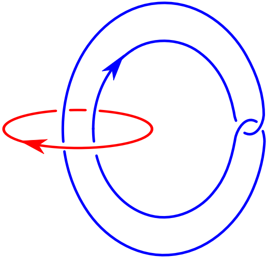
Key Idea: Interpret link homotopy invariants of links as homotopy invariants of maps.

Definition

A *link homotopy* of a link L is a deformation during which each component may cross itself, but distinct components must remain disjoint.



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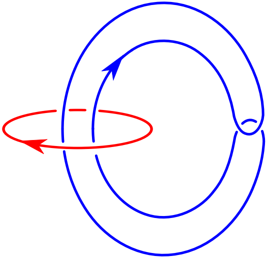
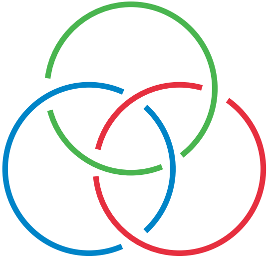
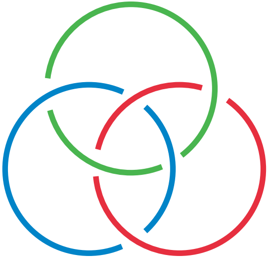




I want to consider *parametrized* n-component links $\{L_1, L_2, L_3, \ldots, L_n\}$, where

$$L_i : S^1 \to \mathbb{R}^3$$

have disjoint images.



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have disjoint images.

Let $LM(n)$ be the set of link homotopy classes of n-component links in $\mathbb{R}^3$.

The evaluation map

Given an n-component link $L = \{L_1, \ldots, L_n\}$, there is a natural evaluation map

$$g_L : \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}} \longrightarrow \mathrm{Conf}(n)$$

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Here $\mathrm{Conf}(n)$ is the configuration space of n distinct points in $\mathbb{R}^3$:

$$\mathrm{Conf}(n) = \mathrm{Conf}_n \mathbb{R}^3 = \{(x_1, \ldots, x_n) \in (\mathbb{R}^3)^n : x_i \neq x_j \text{ for } i \neq j\}.$$

$$g_L : (S^1)^n \longrightarrow \mathrm{Conf}(n).$$

Link homotopies of L induce homotopies of g_L.

We get an induced map

$$\begin{aligned} \kappa : LM(n) &\longrightarrow [(S^1)^n, \mathrm{Conf}(n)] \\ L &\longmapsto [g_L] \end{aligned}$$

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Key Question: Is this map injective?

Conjecture (Koschorke)

The map $\kappa : LM(n) \to [(S^1)^n, \mathrm{Conf}(n)]$ *is injective for all* $n \geq 2$*.*

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If Koschorke's conjecture is true, then link homotopy invariants of links are homotopy invariants of the associated evaluation maps. Use this interpretation to find generalized Gauss integrals for higher-order linking invariants.

Theorem (Gauss)

Koschorke's conjecture is true for $n = 2$.

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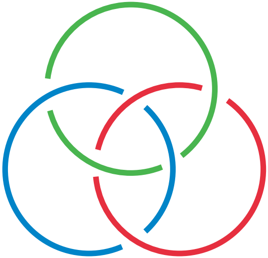
Let $BLM(n)$ be the set of *homotopy Brunnian n*-component links, meaning every $(n-1)$-component sublink is link homotopically trivial.



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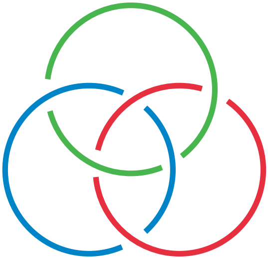
Theorem (Koschorke)

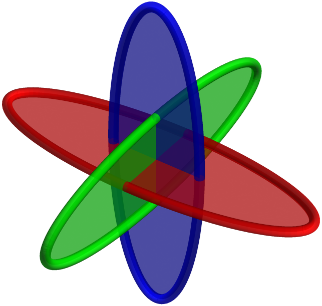
The restriction $\kappa : BLM(n) \to [(S^1)^n, \mathrm{Conf}(n)]$ *is injective for all* n.

$n = 3$



Three-component links were classified up to *link-homotopy* by Milnor:

- The pairwise linking numbers p, q, r.
- The *triple linking number* $\mu = \overline{\mu}_{123}$, which is an integer modulo $\gcd(p, q, r)$.



μ is defined algebraically, but Mellor and Melvin showed that it can also be interpreted geometrically in terms of how each link component pierces the Seifert surfaces of the other two components, plus a count of the triple point intersections of these surfaces.

Working in S^3 is easier...

$$S^1 \times S^1 \times S^1 \xrightarrow{\widetilde{g}_L} \mathrm{Conf}(S^3, 3) \xrightarrow[\text{equiv.}]{\text{hom.}} S^3 \times S^2 \xrightarrow{\quad \pi \quad} S^2.$$

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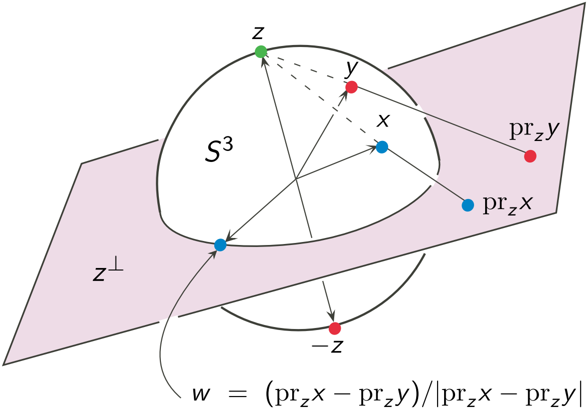

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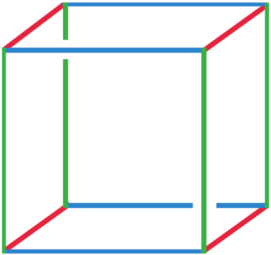
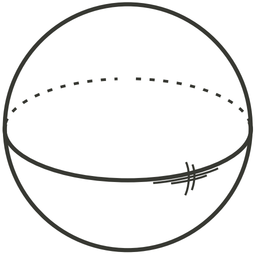
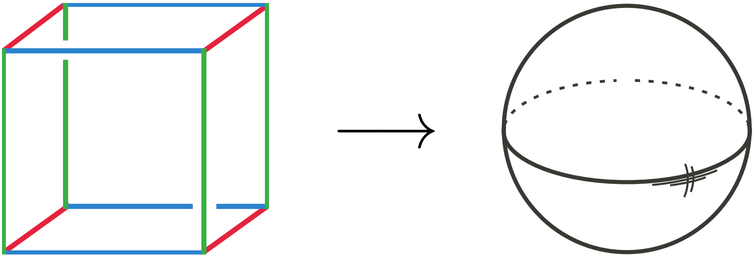
The composition induces a map

$$\begin{aligned} LM(3) &\longrightarrow [S^1 \times S^1 \times S^1, S^2] \\ L &\longmapsto [f_L] = [\pi \circ h \circ \widetilde{g}_L] \end{aligned}$$



$\longrightarrow$





Pontryagin showed that classes in $[S^1 \times S^1 \times S^1, S^2]$ are completely determined by:

- the degrees p, q, r on the 2-dimensional subtori and
- a "Hopf invariant" ν, which is an integer modulo $2\gcd(p, q, r)$.

Theorem (with DeTurck, Gluck, Komendarczyk, Melvin, and Vela-Vick)

The map $LM(3) \to [S^1 \times S^1 \times S^1, S^2]$ *is injective. In particular*

- *The degrees of the restrictions of* f_L *to the 2-dimensional subtori are equal to the pairwise linking numbers of* L
- $\nu(f_L) \equiv 2\mu(L) \mod 2\gcd(p, q, r)$.

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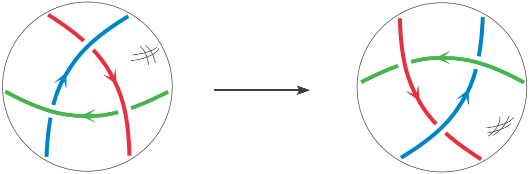
Corollary

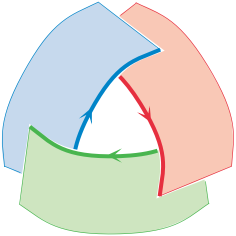
The "spherical Koschorke map" $\kappa_S : LM(3) \to [S^1 \times S^1 \times S^1, \mathrm{Conf}(S^3, 3)]$ *is injective.*

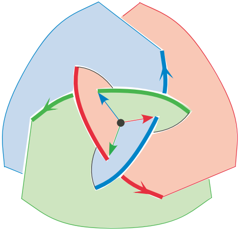
The **delta move** does not change pairwise linking numbers.



Murakami and Nakanishi (1989) proved that any two links with the same number of components and the same pairwise linking numbers are related by a sequence of delta moves.

The delta move increases μ by 1.





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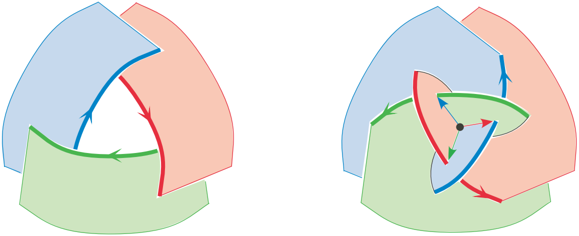

We developed a topological calculus for computing the Pontryagin ν-invariant of f_L, and used it to show that a delta move increases ν by 2.

Let $H : \mathrm{Conf}(3) \to \mathrm{Conf}(S^3, 3)$ be the map induced by inverse stereographic projection.

Then H is an embedding and we can define the composition

$$S^1 \times S^1 \times S^1 \xrightarrow{g_L} \mathrm{Conf}(3) \xrightarrow{H} \mathrm{Conf}(S^3, 3) \xrightarrow{\pi \circ h} S^2.$$

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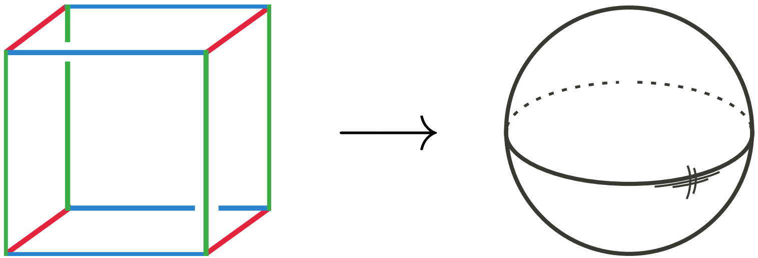
Corollary

The Koschorke map $\kappa : LM(3) \to [S^1 \times S^1 \times S^1, \mathrm{Conf}(n)]$ *is injective.*



When the pairwise linking numbers are all zero, the map $f_L : S^1 \times S^1 \times S^1 \to S^2$ is null-homotopic on the 2-skeleton. Then $\nu(f_L)$ is an integer and equals the Hopf invariant of the map $S^3 \to S^2$ obtained by collapsing the 2-skeleton.

Whitehead's integral formula for the Hopf invariant

- $f : S^3 \to S^2$ smooth.
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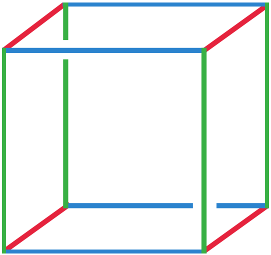
- ω the normalized area form on S^2 ($\int_{S^2} \omega = 1$).
- $f^*\omega$ is closed and, therefore, exact.
- Then
$$\mathrm{Hopf}(f) = \int_{S^3} d^{-1}(f^*\omega) \wedge f^*\omega,$$
where $d^{-1}(f^*\omega)$ is any primitive for $f^*\omega$.

If L has pairwise linking numbers all zero, then $\omega_L = f_L^* \omega$ is an exact 2-form on $S^1 \times S^1 \times S^1$.



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Whitehead's integral transplanted to $S^1 \times S^1 \times S^1$:

$$\nu(f_L) = \int_{(S^1)^3} d^{-1}(\omega_L) \wedge \omega_L$$

for any primitive $d^{-1}(\omega_L)$ of ω_L.

An integral formula for the μ invariant

If L has pairwise linking numbers all zero, then $\omega_L = f_L^*\omega$ is an exact 2-form on $S^1 \times S^1 \times S^1$.

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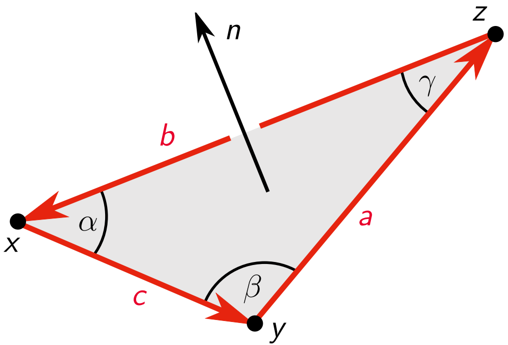
$$\nu(f_L) = \int_{(S^1)^3} d^{-1}(\omega_L) \wedge \omega_L$$

for any primitive $d^{-1}(\omega_L)$ of ω_L.

Therefore, by the above theorem,

$$\mu(L) = \frac{1}{2}\nu(f_L) = \frac{1}{2}\int_{(S^1)^3} d^{-1}(\omega_L) \wedge \omega_L.$$

Define $F : \mathrm{Conf}(3) \to \mathbb{R}^3 - \{\vec{0}\}$ by



$$F(x,y,z) = \left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|} \right) + (\sin\alpha + \sin\beta + \sin\gamma)\vec{n}.$$

$$\begin{aligned} g_L : S^1 \times S^1 \times S^1 &\longrightarrow \mathrm{Conf}(3) \\ (s,t,u) &\longmapsto (L_1(s), L_2(t), L_3(u)) \end{aligned}$$

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For 3-component link L, define the *generalized Gauss map* $F_L : S^1 \times S^1 \times S^1 \to S^2$ by

$$F_L := \frac{F \circ g_L}{|F \circ g_L|}.$$

F_L is homotopic to f_L, so

$$\mu(L) = \frac{1}{2}\nu(f_L) = \frac{1}{2}\nu(F_L) = \int_{S^1 \times S^1 \times S^1} d^{-1}(\omega_L) \wedge \omega_L$$

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where $\omega_L = F_L^* \omega$.

The geometrically natural choice of $d^{-1}(\omega_L)$ is the primitive with smallest energy, which can be obtained by convolving ω_L with the fundamental solution φ of the scalar Laplacian and taking the exterior co-derivative:

$$d^{-1}(\omega_L) = \delta(\varphi * \omega_L).$$

Proposition

The fundamental solution of the scalar Laplacian on the 3-torus is given by the Fourier series

$$\varphi(s,t,u) = \frac{1}{8\pi^3} \sum_{(n_1,n_2,n_3)\neq\vec{0}} \frac{e^{i(sn_1+tn_2+un_3)}}{n_1^2+n_2^2+n_3^2}.$$

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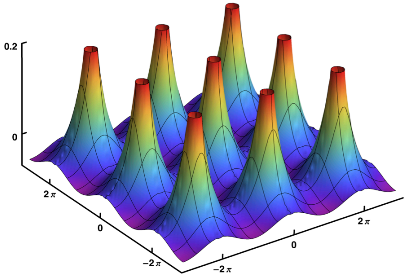


Theorem (with DeTurck, Gluck, Komendarczyk, Melvin, Nuchi, and Vela-Vick)

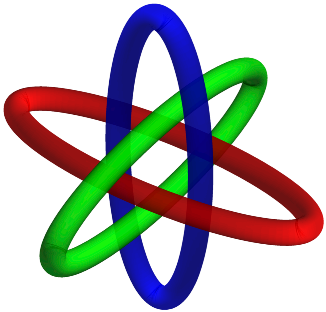
For a three-component link L in $\mathbb{R}^3$,

$$\begin{aligned}\mu(L) &= \frac{1}{2}\int_{S^1\times S^1\times S^1} \delta(\varphi * \omega_L)\wedge\omega_L \\ &= 8\pi^3 \sum_{\vec{n}\in\mathbb{Z}^3-\{\vec{0}\}} \vec{a}_{\vec{n}}\times\vec{b}_{\vec{n}}\bullet\frac{\vec{n}}{|\vec{n}|^2},\end{aligned}$$

where $\vec{a}_{\vec{n}}$ and $\vec{b}_{\vec{n}}$ are the real and imaginary parts of the Fourier coefficients of ω_L.

The integrand is invariant under orientation-preserving isometries of $\mathbb{R}^3$.

$$L_1(s) = (2\cos s, 7\sin s, 0)$$
$$L_2(t) = (0, 2\cos t, 7\sin t)$$
$$L_3(u) = (7\sin u, 0, 2\cos u)$$



Approximating Fourier coefficients by splitting the s, t, and u intervals into 256 subintervals and for $-64 \leq n_1, n_2, n_3 \leq 64$, yields the approximation

$$\mu(L) \simeq -0.99999997.$$

The moral of the preceding story is that, for $n = 2$ and 3, interpreting link homotopy invariants of links as homotopy invariants of maps to configuration space yields geometric linking integrals.

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Conjecture (Koschorke)

The map $\kappa : LM(n) \to [(S^1)^n, \mathrm{Conf}(n)]$ *is injective for all* n.

Theorem (Koschorke, with Cohen and Komendarczyk)

The restriction $\kappa : BLM(n) \to [(S^1)^n, \mathrm{Conf}(n)]$ *is injective. Moreover,*

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- *the image of $BLM(n)$ is contained in a copy of $\pi_n(\mathrm{Conf}(n)) \cong \pi_{n-1}(\Omega\,\mathrm{Conf}(n))$ and is a free, rank $(n-2)!$ module generated by iterated Samelson products of "Ptolemaic epicycles";*

Milnor invariants as homotopy invariants

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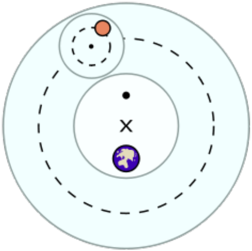
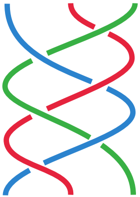
- *for any $L \in BLM(n)$, the Milnor invariants of L are the coefficients of the expansion of $\kappa(L)$ in the above basis.*

A Ptolemaic epicycle



$$\begin{aligned} S^2 &\longrightarrow \mathrm{Conf}(n) \\ \xi &\longmapsto (x_1, \ldots, \underbrace{x_j}_{i\text{th}}, \ldots, \underbrace{x_j + \xi}_{j\text{th}}, \ldots, x_n) \end{aligned}$$

Habegger and Lin introduced the group of homotopy string links $\mathscr{H}(n)$.



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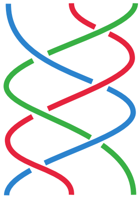

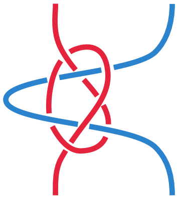

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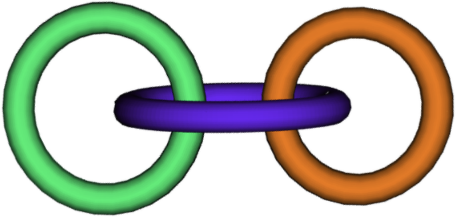
The basic idea is to use knowledge about the composition to make conclusions about κ.

$$\mathscr{H}(n) \twoheadrightarrow LM(n) \xrightarrow{\kappa} [(S^1)^n, \mathrm{Conf}(n)]$$

Let $CLM(n)$ be the homotopy classes of n-component links with *some* trivial $(n-1)$-component sublink.



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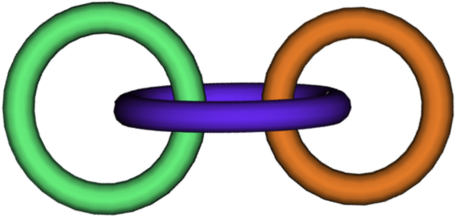

Theorem (with Cohen and Komendarczyk)

The restriction of κ to $CLM(n)$ is injective.

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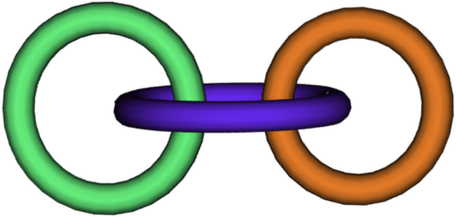

Theorem (with Cohen and Komendarczyk)

The restriction of κ to $CLM(n)$ is injective.

Moreover, the image of $CLM(n)$ is contained in the Fox torus homotopy group $\tau_n(\mathrm{Conf}(n))$.

- Prove Koschorke's conjecture (or find a counterexample!)

Future directions

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- Use these homotopy interpretations of Milnor invariants to find geometric integral formulas for them.
- Higher helicities?

Thanks!