Geometry of Constrained Random Walks and an Application to Frame Theory

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Key Ideas

- Polygons are closely related to tight frames.
- Symplectic geometry is a powerful tool for studying random polygons.
- Exploiting the connection to polygons gives new tools for synthesizing tight frames with nice properties.
- The connection between frame theory and symplectic geometry is under-explored but potentially fruitful.

Introduction

Random walks in $\mathbb{R}^3$ are classical objects in geometric probability which are commonly used in the polymer physics and biopolymers communities to model polymers like DNA in solution. Modifying the theory to apply to ring polymers requires a theory of **random polygons**, or loop random walks. Several recent breakthroughs have been made by thinking of random polygons as points in some nice conformation space and then exploiting the geometry – especially symplectic geometry – of the space.

There is a slightly surprsing connection between polygons and **frame theory**, the study of redundant bases useful for reconstructing lossy and/or noisy signals. In particular, polygons in $\mathbb{R}^3$ can be lifted via the Hopf map to tight frames in $\mathbb{C}^2$, producing algorithms and strategies for generating tight frames with nice properties. In general, symplectic geometry seems like a promising tool for understanding frames in $\mathbb{C}^d$ for any d.

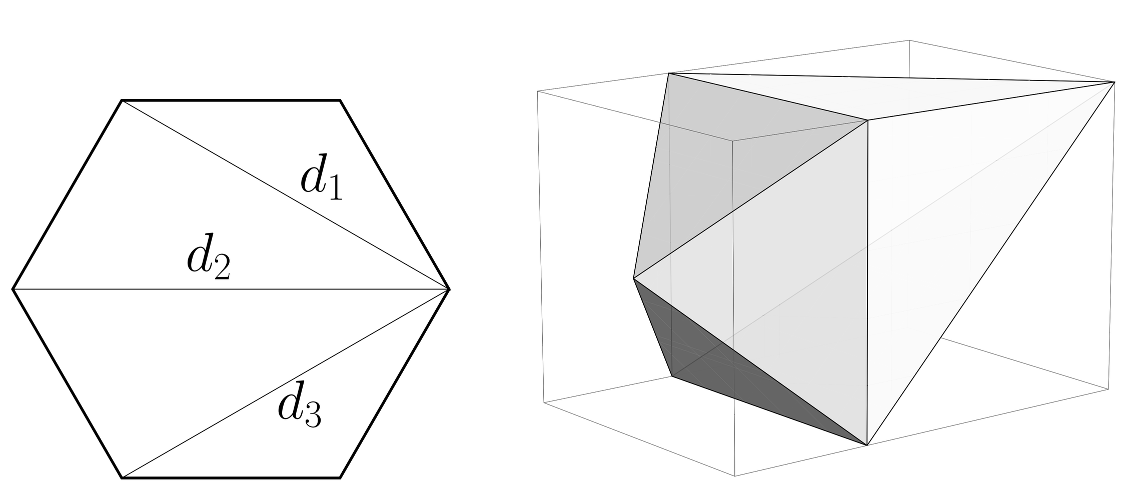

Figure 1: The three diagonals of a triangulation of a 6-gon and the corresponding convex polytope.

Polygons

An n-gon in $\mathbb{R}^3$ can be interpreted as a list of edge vectors $e_1, \ldots, e_n \in \mathbb{R}^3$ such that $e_1 + \ldots + e_n = 0$. Applying the Hopf map $(z, w) \mapsto \mathbf{i}(z\bar{z} - w\bar{w} + 2\bar{z}w\mathbf{j})$ columnwise gives a map $\mathbb{C}^{2\times n} \to (\mathbb{R}^3)^n$.

- The inverse image of the closed polygons under this map is the collection of $2 \times n$ complex matrices with orthogonal rows of the same norm: a cone over the Stiefel manifold $\mathrm{St}_2(\mathbb{C}^n)$.
- The cone parameter corresponds to the perimeter of the polygon, and $\mathrm{St}_2(\mathbb{C}^n)$ sits over polygons of perimeter 2.
- The Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$ sits over polygons modulo rotation.

Frames

A *frame* in a Hilbert space $\mathcal{H}$ is a redundant basis: a collection $\{\varphi_i\}_{i\in\mathcal{I}}$ of vectors in $\mathcal{H}$ with $A, B > 0$ so that

$$A\|x\|^2 \leq \sum_{i\in\mathcal{I}} |\langle x, \varphi_i\rangle|^2 \leq B\|x\|^2$$

for all $x \in \mathcal{H}$. The frame is *finite* if $\mathcal{I}$ is finite (which implies $\mathcal{H}$ is finite-dimensional), *tight* if we can take $A = B$, and *unit norm* if $\|\varphi_i\| = 1$ for all $i \in \mathcal{I}$. We abbreviate a **finite unit norm tight frame** as **FUNTF**.

- For frames in $\mathbb{C}^2$, the FUNTF condition is (1).
- FUNTFs are optimal for signal reconstruction in the presence of additive white Gaussian noise when each measurement has equal power.

Main Theorem

There exist explicit algorithms for simulating (synthesizing) random equilateral n-gons in $\mathbb{R}^3$ and length-n FUNTFs in $\mathbb{C}^2$ in expected $\Theta(n^{5/2})$ time.

Equilateral Polygons

Polygons with all unit edgelengths are traditionally used in polymer models. Lifting to $2 \times n$ matrices gives all $A \in \mathbb{C}^{2\times n}$ with

$$AA^* = \begin{bmatrix} n/2 & 0 \\ 0 & n/2 \end{bmatrix} \quad \text{and} \quad (A^*A)_{ii} = 1 \text{ for all } i. \tag{1}$$

- This is exactly the **FUNTF** condition, and the equilateral polygons are a torus quotient of FUNTF space.
- Simulating polygons reduces to generating random points in the convex polytope determined by triangle inequalities associated to any triangulation of an n-gon [1] (Figure 1).
- Changing coordinates and rejection sampling the hypercube produces the algorithm which is the first part of the Main Theorem [2].

Frames from Polygons

- The equilateral polygon sampler can be adapted to simulate random FUNTFs in $\mathbb{C}^2$ (second part of Main Theorem). Figure 2 shows the (curious) distribution of coherences of random length-4 FUNTFs in $\mathbb{C}^2$.
- Lifting symmetric polygons yields FUNTFs of low coherence; e.g., lifts of solutions to the Tammes problem produce FUNTFs of optimal or near-optimal coherence (Figure 3).

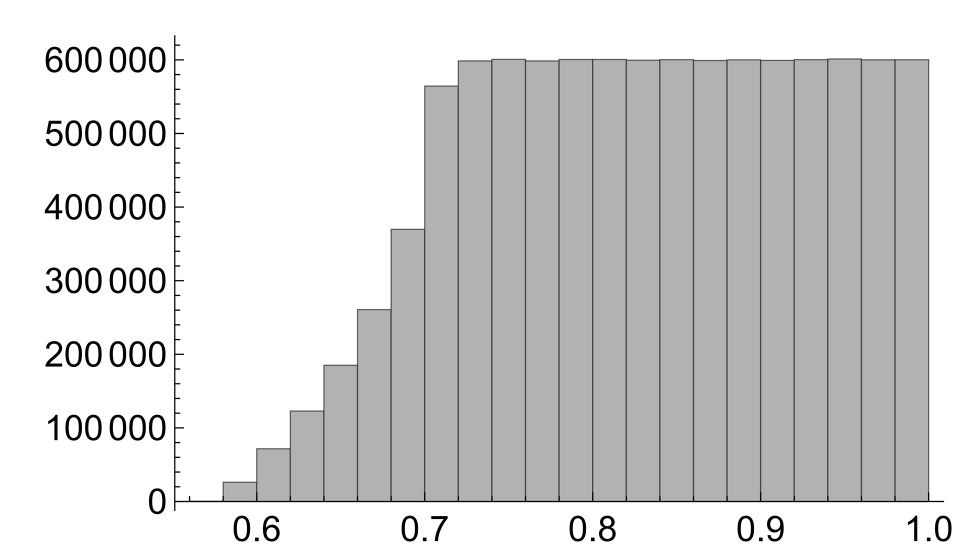

Figure 2: Coherences of 10m random length-4 FUNTFs in $\mathbb{C}^2$

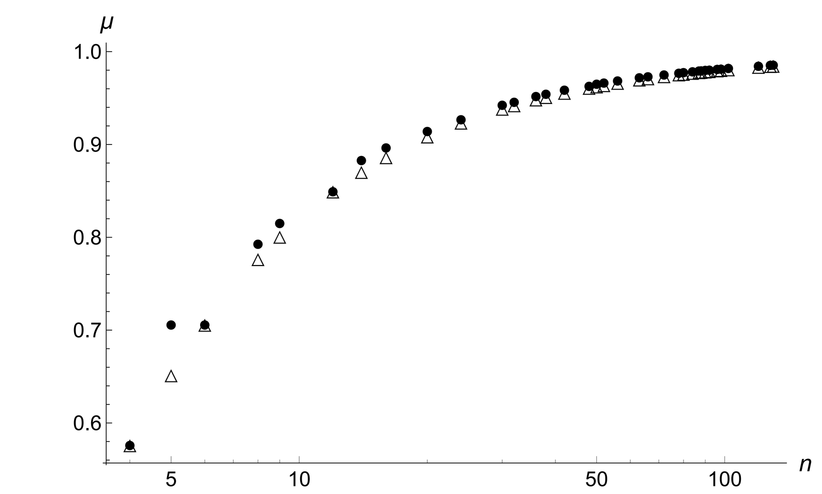

Figure 3: Coherences of lifted solutions to Tammes problem (●) compared to theoretical lower bound (△).

Looking Ahead

In general, FUNTF spaces are (almost) **toric symplectic** (cf. [3]), so action-angle coordinates should provide convenient almost-global coordinates for both computation and simulation. In particular, general FUNTFs in $\mathbb{C}^d$ are lifts of polygons in the Lie algebra $\mathfrak{su}(d)$ [4], so the foregoing story should generalize to FUNTFs in $\mathbb{C}^d$ for any d.

References

[1] Jason Cantarella and Clayton Shonkwiler.
The symplectic geometry of closed equilateral random walks in 3-space.
Ann. Appl. Probab., 26(1):549–596, 2016.

[2] Jason Cantarella, Bertrand Duplantier, Clayton Shonkwiler, and Erica Uehara.
A fast direct sampling algorithm for equilateral closed polygons.
J. Phys. A: Math. Gen., 49(27):275202, 2016.

[3] Tom Needham and Clayton Shonkwiler.
Symplectic geometry and connectivity of spaces of frames.
Preprint, 2018. arXiv:1804.05899 [math.FA].

[4] Hermann Flaschka and John J Millson.
Bending flows for sums of rank one matrices .
Canad. J. Math., 57(1):114–158, 2005.

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