Poincaré Duality Angles on Riemannian Manifolds with Boundary

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Realizing cohomology groups as spaces of differential forms

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Realizing cohomology groups as spaces of differential forms

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\[ H^p(M; \mathbb{R}) \]
\[ H^p(M, \partial M; \mathbb{R}) \]
de Rham’s Theorem

Suppose $M^n$ is a compact, oriented, smooth manifold. Then

$$H^p(M; \mathbb{R}) \cong \frac{C^p(M)}{E^p(M)},$$

where $C^p(M)$ is the space of closed $p$-forms on $M$ and $E^p(M)$ is the space of exact $p$-forms.
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$$\langle \omega, \eta \rangle := \int_M \omega \wedge \star \eta.$$
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Kodaira called this the space of harmonic $p$-fields on $M$. 
Hodge’s Theorem

If $M^n$ is a closed, oriented, smooth Riemannian manifold,

$$H^p(M; \mathbb{R}) \cong \mathcal{H}^p(M).$$
Define $i : \partial M \hookrightarrow M$ to be the natural inclusion.
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The $L^2$-orthogonal complement of the exact forms inside the space of closed forms is now:

$$\mathcal{H}_N^p(M) := \{ \omega \in \Omega^p(M) : d\omega = 0, \delta \omega = 0, i^* \ast \omega = 0 \}.$$
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Then

$$H^p(M; \mathbb{R}) \cong \mathcal{H}_N^p(M).$$
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$$H^p(M, \partial M; \mathbb{R}) \cong \mathcal{H}^p_D(M).$$

$$\mathcal{H}^p_D(M) := \{ \omega \in \Omega^p(M) : d\omega = 0, \delta \omega = 0, i^* \omega = 0 \}.$$
The concrete realizations of $H^p(M; \mathbb{R})$ and $H^p(M, \partial M; \mathbb{R})$ meet only at the origin:

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...but they are not orthogonal!
Definition (DeTurck–Gluck)

The *Poincaré duality angles* of the Riemannian manifold $M$ are the principal angles between the interior subspaces.
Guess
If $M$ is “almost” closed, the Poincaré duality angles of $M$ should be small.
Consider $\mathbb{CP}^2$ with its usual Fubini-Study metric. Let $p \in \mathbb{CP}^2$. Then define

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\( M_r \) has absolute cohomology in dimensions 0 and 2.

\( M_r \) has relative cohomology in dimensions 2 and 4.

Therefore, \( M_r \) has a single Poincaré duality angle \( \theta_r \) between \( \mathcal{H}^2_N(M_r) \) and \( \mathcal{H}^2_D(M_r) \).
Find harmonic 2-fields

So the goal is to find closed and co-closed 2-forms on $M_r$ which satisfy Neumann and Dirichlet boundary conditions.
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Find closed and co-closed $SU(2)$-invariant forms on $M_r$ satisfying Neumann and Dirichlet boundary conditions.
The Poincaré duality angle for $M_r$

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\cos \theta_r = \frac{1 - \sin^4 r}{1 + \sin^4 r}.
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Generalizes to $\mathbb{CP}^n - B_r(p)$. 
Consider

\[ N_r \defeq G_2 \mathbb{R}^n - \nu_r(G_1 \mathbb{R}^{n-1}) . \]
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**Theorem**

- As \( r \to 0 \), all the Poincaré duality angles of \( N_r \) go to zero.
- As \( r \) approaches its maximum value of \( \pi/2 \), all the Poincaré duality angles of \( N_r \) go to \( \pi/2 \).
Conjecture

If $M^n$ is a closed Riemannian manifold and $N^k$ is a closed submanifold of codimension $\geq 2$, the Poincaré duality angles of

$$ M - \nu_r(N) $$

go to zero as $r \to 0$. 

A question

What can you learn about the topology of $M$ from knowledge of $\partial M$?
Induce potentials on the boundary of a region and determine the conductivity inside the region by measuring the current flux through the boundary.
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Suppose $f$ is a potential on the boundary of a region $M \subset \mathbb{R}^3$. 
The Voltage-to-Current map

Suppose \( f \) is a potential on the boundary of a region \( M \subset \mathbb{R}^3 \).

Then \( f \) extends to a potential \( u \) on \( M \), where

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Then $f$ extends to a potential $u$ on $M$, where

$$\Delta u = 0, \quad u|_{\partial M} = f.$$ 

If $\gamma$ is the conductivity on $M$, the current flux through $\partial M$ is given by

$$(\gamma \nabla u) \cdot \nu = -\gamma \frac{\partial u}{\partial \nu}.$$
The Dirichlet-to-Neumann map

The map \( \Lambda_{\text{cl}} : C^\infty(\partial M) \to C^\infty(\partial M) \) defined by

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f \mapsto \frac{\partial u}{\partial \nu}
\]

is the classical *Dirichlet-to-Neumann map*. 
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**Theorem (Lee-Uhlmann)**

*If $M^n$ is a compact, analytic Riemannian manifold with boundary, then $M$ is determined up to isometry by $\Lambda_{\text{cl}}$.***
Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

\[ \Lambda : \Omega^p(\partial M) \to \Omega^{n-p-1}(\partial M) \]
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\[ \Lambda : \Omega^p(\partial M) \rightarrow \Omega^{n-p-1}(\partial M) \]

**Theorem (Belishev–Sharafutdinov)**

*The data \((\partial M, \Lambda)\) completely determines the cohomology groups of \(M\).*
Define the Hilbert transform $T := d\Lambda^{-1}$. 

Theorem

If $\theta_1, \ldots, \theta_k$ are the Poincaré duality angles of $M$ in dimension $p$, then the quantities $\left(-1\right)^{np + n + p} \cos^2 \theta_i$ are the non-zero eigenvalues of an appropriate restriction of $T^2$. 

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Belishev and Sharafutdinov posed the following question:

*Can the multiplicative structure of cohomologies be recovered from our data \((\partial M, \Lambda)\)? Till now, the authors cannot answer the question.*
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**Theorem**

*The mixed cup product*

\[
\cup : H^p(M; \mathbb{R}) \times H^q(M, \partial M; \mathbb{R}) \to H^{p+q}(M, \partial M; \mathbb{R})
\]

is completely determined by the data \((\partial M, \Lambda)\) when the relative class is restricted to come from the boundary subspace.
Some questions

- Poincaré duality angles for $G_4 \mathbb{R}^8 - \nu_r(G_3 \mathbb{R}^7)$? Other “Grassmann manifolds with boundary”?
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• Can the full mixed cup product be recovered from $(\partial M, \Lambda)$? What about other cup products?
Some questions

- Poincaré duality angles for $G_4\mathbb{R}^8 - \nu_r(G_3\mathbb{R}^7)$? Other “Grassmann manifolds with boundary”?
- What is the limiting behavior of the Poincaré duality angles as the manifold “closes up”?
- Can the full mixed cup product be recovered from $(\partial M, \Lambda)$? What about other cup products?
- Can the $L^2$ inner product on $\mathcal{H}_N^p(M)$ and $\mathcal{H}_D^p(M)$ be recovered from $(\partial M, \Lambda)$?
Thanks!