The Geometry of Polygon Spaces

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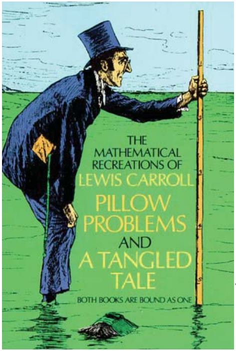
A pillow problem



57. (25, 80)

In a given Triangle describe three Squares, whose bases shall lie along the sides of the Triangle, and whose upper edges shall form a Triangle;

(1) geometrically; (2) trigonometrically. [27/1/91

58. (25, 83)

Three Points are taken at random on an infinite Plane. Find the chance of their being the vertices of an obtuse-angled Triangle. [20/1/84

What does it mean to take a random triangle?

Question

What does it mean to choose a random triangle?

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Statistician's Answer

> *The issue of choosing a "random triangle" is indeed problematic. I believe the difficulty is explained in large measure by the fact that there seems to be no natural group of transitive transformations acting on the set of triangles.*
>
> *–Stephen Portnoy, 1994*
> *(Editor, J. American Statistical Association)*

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Applied Mathematician's Answer

> *We will add a purely geometrical derivation of the picture of triangle space, delve into the linear algebra point of view, and connect triangles to random matrix theory.*
>
> *–Alan Edelman and Gil Strang, 2012*

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Differential Geometer's Answer

> *Pick by the measure defined by the volume form of the natural Riemannian metric on the manifold of 3-gons, of course. That ought to be a special case of the manifold of n-gons.*
>
> *But what's the manifold of n-gons?*
>
> *And what makes one metric natural?*
>
> *Don't algebraic geometers understand this?*
>
> *–Jason Cantarella to me, a few years ago*

(Seemingly) Trivial Observation 1

Given $z_1 = a_1 + \mathbf{i}b_1, \ldots, z_n = a_n + \mathbf{i}b_n$

$$\begin{aligned}\sum z_i^2 &= (a_1^2 - b_1^2) + \mathbf{i}(2a_1b_1) + \cdots + (a_n^2 - b_n^2) + \mathbf{i}(2a_nb_n)\\ &= (a_1^2 + \cdots + a_n^2) - (b_1^2 + \cdots + b_n^2) + \mathbf{i}\,2(a_1b_1 + \cdots + a_nb_n)\\ &= (|\vec{a}|^2 - |\vec{b}|^2) + \mathbf{i}\,2\left\langle \vec{a}, \vec{b} \right\rangle\end{aligned}$$

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&= (a_1^2 + \cdots + a_n^2) - (b_1^2 + \cdots + b_n^2) + \mathbf{i}\,2(a_1 b_1 + \cdots + a_n b_n) \\
&= (|\vec{a}|^2 - |\vec{b}|^2) + \mathbf{i}\,2\left\langle \vec{a}, \vec{b} \right\rangle
\end{aligned}$$

(Seemingly) Trivial Observation 2

Given $z_1 = a_1 + \mathbf{i}b_1, \ldots, z_n = a_n + \mathbf{i}b_n$

$$\begin{aligned}
|z_1^2| + \cdots + |z_n^2| &= |z_1|^2 + \cdots + |z_n|^2 \\
&= (a_1^2 + b_1^2) + \cdots + (a_n^2 + b_n^2) \\
&= (a_1^2 + \cdots + a_n^2) + (b_1^2 + \cdots + b_n^2) \\
&= |\vec{a}|^2 + |\vec{b}|^2
\end{aligned}$$

The algebraic geometer's answer, continued

Theorem (Hausmann and Knutson, 1997)

The space of closed planar n-gons with length 2, up to translation and rotation, is identified with the Grassmann manifold $G_2(\mathbb{R}^n)$ of 2-planes in $\mathbb{R}^n$.

The algebraic geometer's answer, continued

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Proof.

Take an orthonormal frame $\vec{a}, \vec{b}$ for the plane, let

$$\vec{z} = (z_1 = a_1 + \mathbf{i}b_1, \ldots, z_n = a_n + \mathbf{i}b_n), \quad v_\ell = \sum_{j=1}^{\ell} z_j^2$$

By the observations, $v_0 = v_n = 0$, $\sum |v_{\ell+1} - v_\ell| = 2$. Rotating the frame a, b in their plane rotates the polygon in the complex plane. $\square$

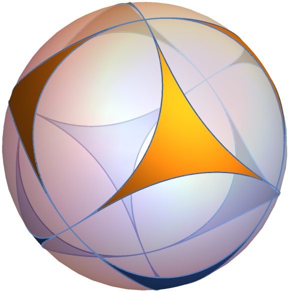
Theorem (with Cantarella and Deguchi)

The volume form of the standard Riemannian metric on $G_2(\mathbb{R}^n)$ defines the natural probability measure on closed, planar n-gons of length 2 up to translation and rotation. It has a (transitive) action by isometries given by the action of $SO(n)$ on $G_2(\mathbb{R}^n)$.

So *random triangles are points selected uniformly on* $\mathbb{RP}^2$ since

$$\text{random triangle} \to \text{random point in } G_2(\mathbb{R}^3) \simeq G_1(\mathbb{R}^3) = \mathbb{RP}^2.$$

Putting the pillow problem to bed



Acute triangles (gold) turn out to be defined by natural algebraic conditions on the sphere.

Proposition (with Cantarella, Chapman, and Needham, 2015)

The fraction of obtuse triangles is

$$\frac{3}{2} - \frac{\log 8}{\pi} \sim 83.8\%$$

Theorem (with Cantarella and Deguchi)

The volume form of the standard Riemannian metric on $G_2(\mathbb{C}^n)$ *defines the natural probability measure on closed space n-gons of length 2 up to translation and rotation. It has a (transitive) action by isometries given by the action of* $U(n)$ *on* $G_2(\mathbb{C}^n)$*.*

Proof.

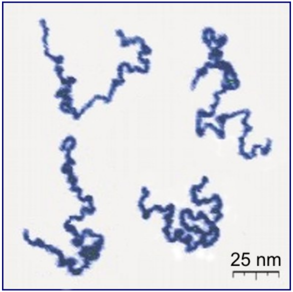
Again, we use an identification due to Hausmann and Knutson where

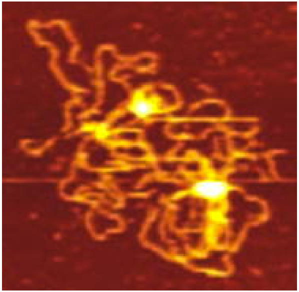
- instead of combining two real vectors to make one complex vector, we combine two complex vectors to get one quaternionic vector
- instead of squaring complex numbers, we apply the Hopf map to quaternions

□

Statistical Physics Point of View

A polymer in solution takes on an ensemble of random shapes, with topology (knot type!) as the unique conserved quantity.





Protonated P2VP
Roiter/Minko
Clarkson University

Plasmid DNA
Alonso-Sarduy, Dietler Lab
EPF Lausanne

Statistical Physics Point of View

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Physics Setup

> *Modern polymer physics is based on the analogy between a polymer chain and a random walk.*
> *—Alexander Grosberg, NYU.*

Definition

The total curvature of a space polygon is the sum of its turning angles.

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Theorem (with Cantarella, Grosberg, and Kusner)

The expected total curvature of a random n-gon of length 2 sampled according to the measure on $G_2(\mathbb{C}^n)$ is

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Corollary (with Cantarella, Grosberg, and Kusner)

At least $1/3$ of hexagons and $1/11$ of heptagons are unknots.

Responsible sampling algorithms

How can we sample and determine distributions of knot types?

Proposition (classical?)

The natural measure on $G_2(\mathbb{C}^n)$ is obtained by generating two random complex n-vectors with independent Gaussian coordinates and their span.

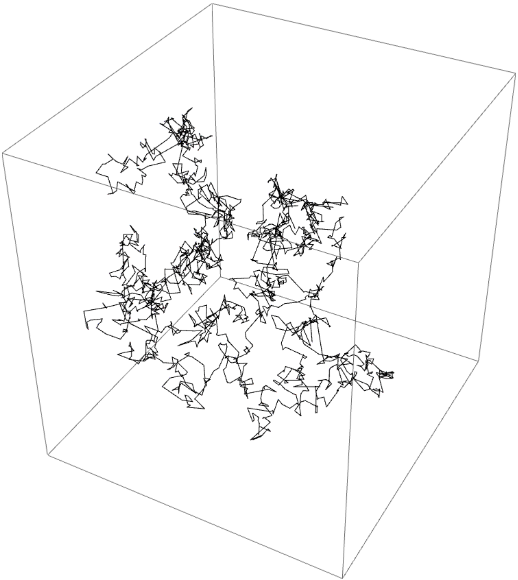
```
In[9]:= RandomComplexVector[n_] := Apply[Complex,
          Partition[#, 2] & /@ RandomVariate[NormalDistribution[], {1, 2 n}], {2}][[1]];

        ComplexDot[A_, B_] := Dot[A, Conjugate[B]];
        ComplexNormalize[A_] := (1 / Sqrt[Re[ComplexDot[A, A]]]) A;

        RandomComplexFrame[n_] := Module[{a, b, A, B},
            {a, b} = {RandomComplexVector[n], RandomComplexVector[n]};
            A = ComplexNormalize[a];
            B = ComplexNormalize[b - Conjugate[ComplexDot[A, b]] A];
           {A, B}
          ];
```

Using this, we can generate ensembles of random polygons . . .

Random 2,000-gons



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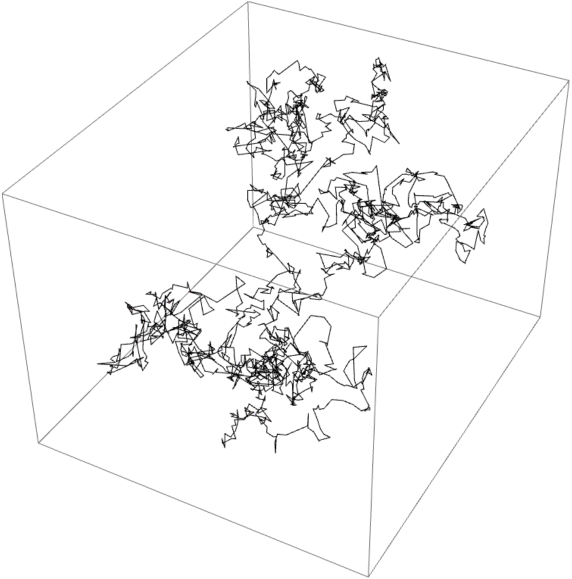

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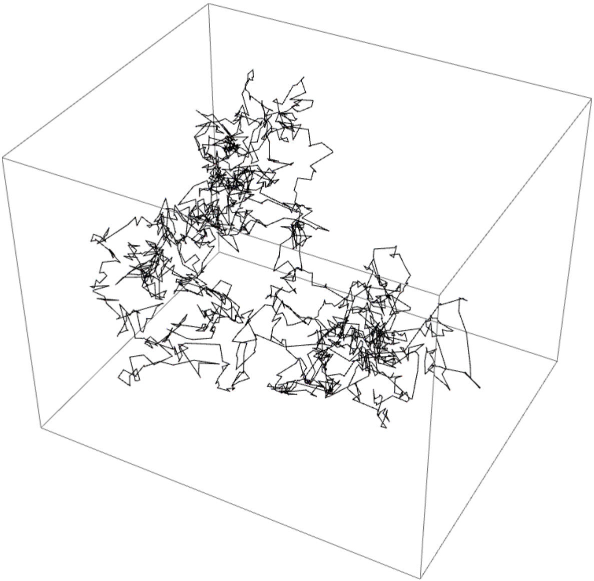

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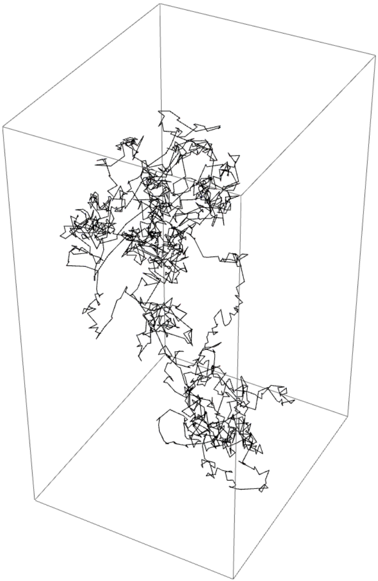

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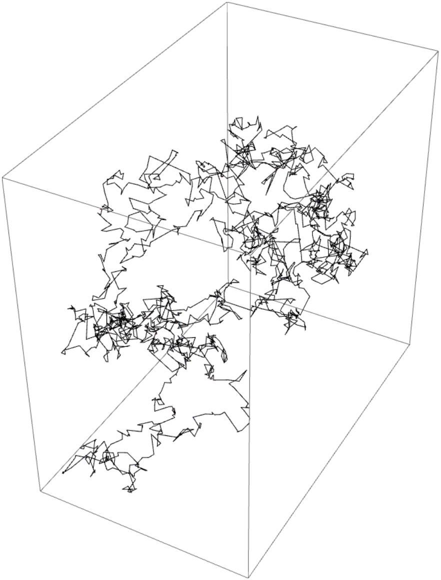

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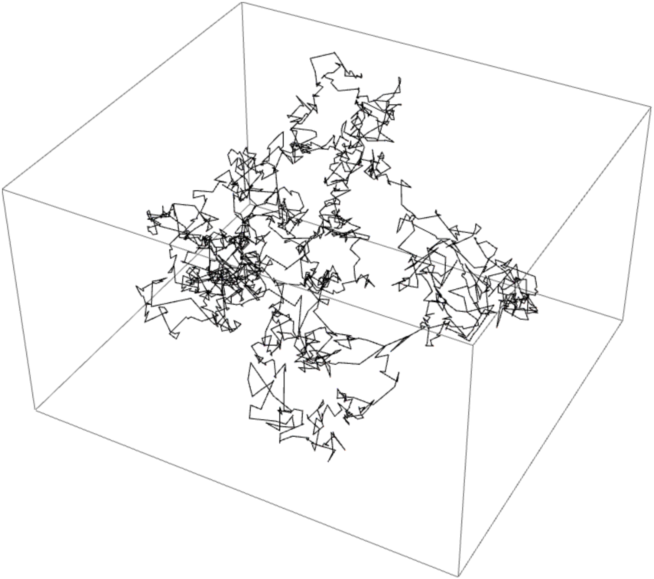

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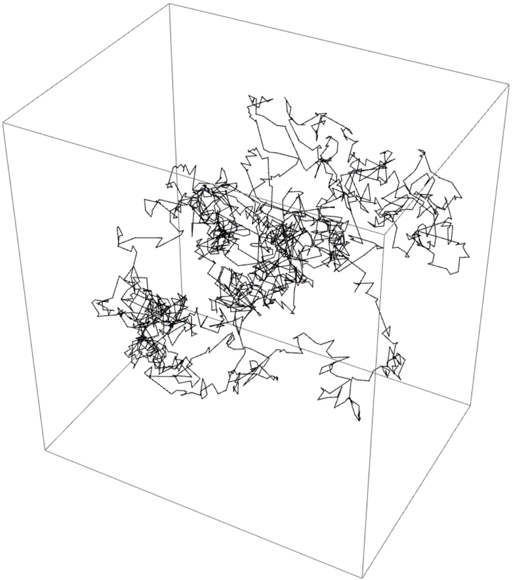

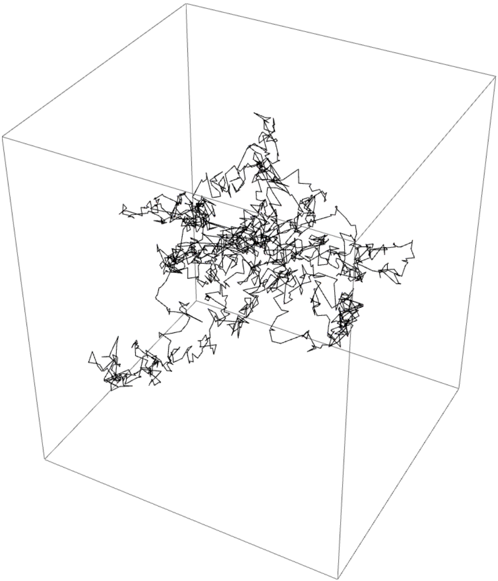

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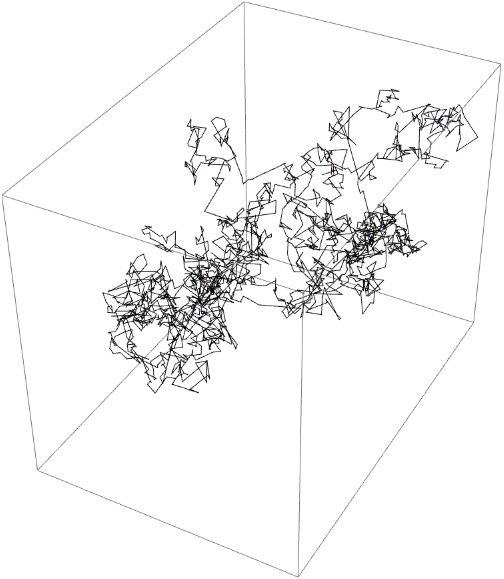

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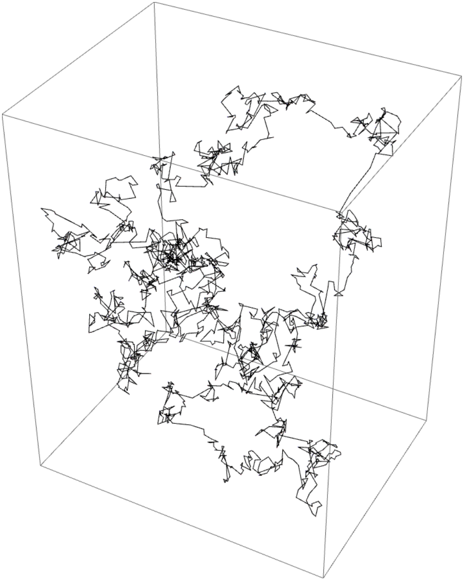

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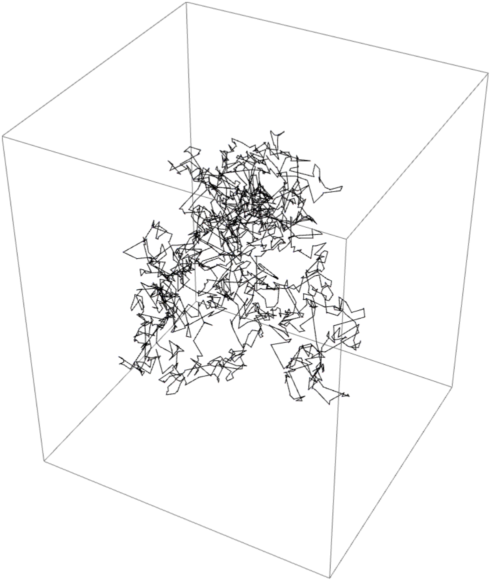

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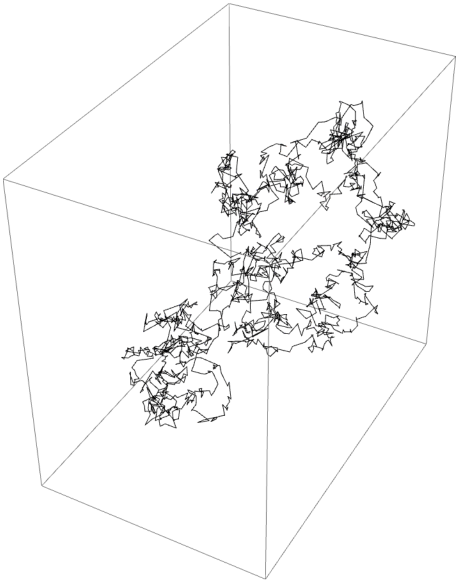

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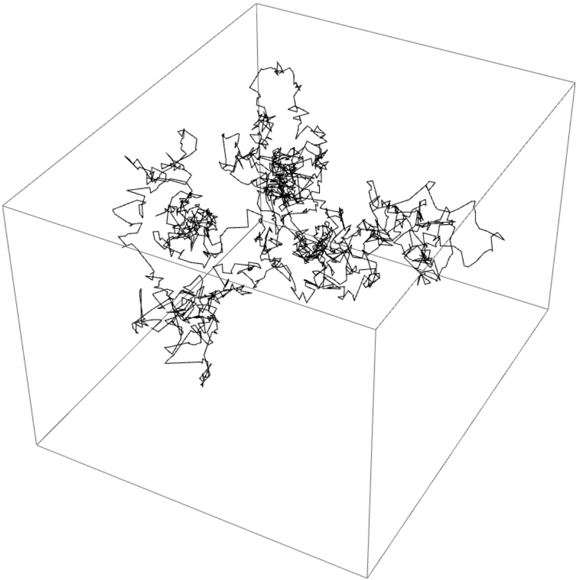

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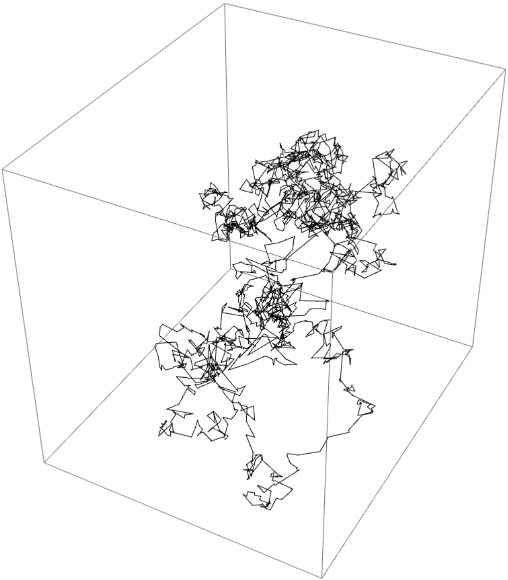

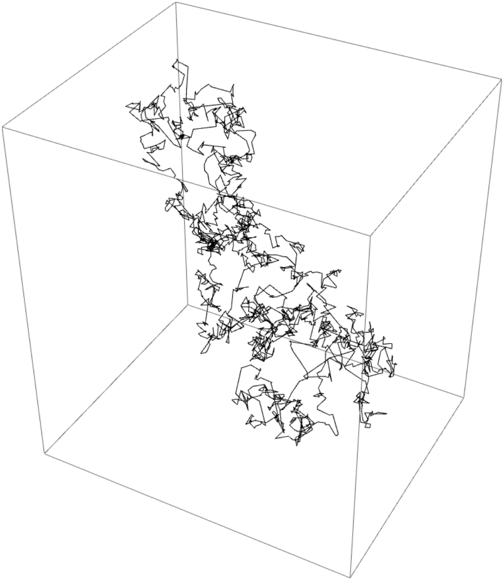

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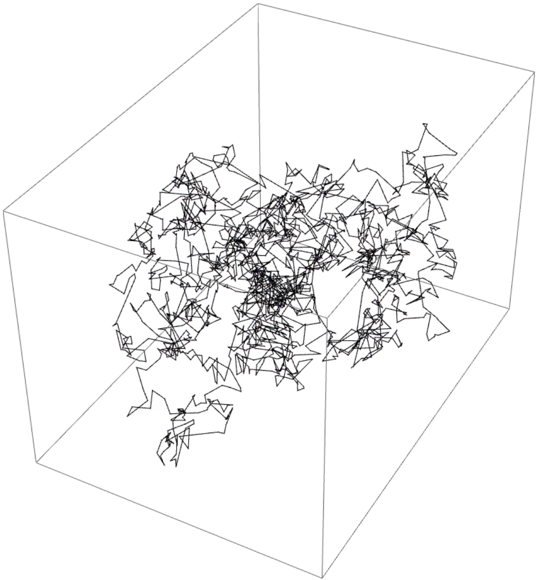

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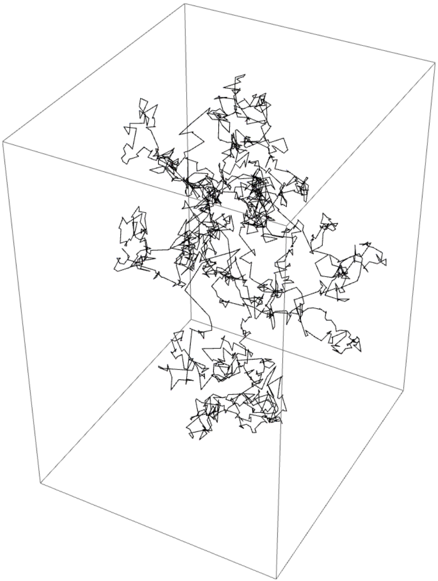

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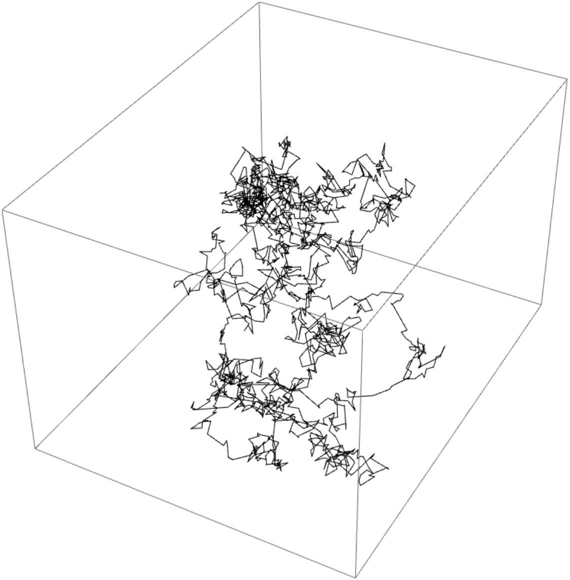

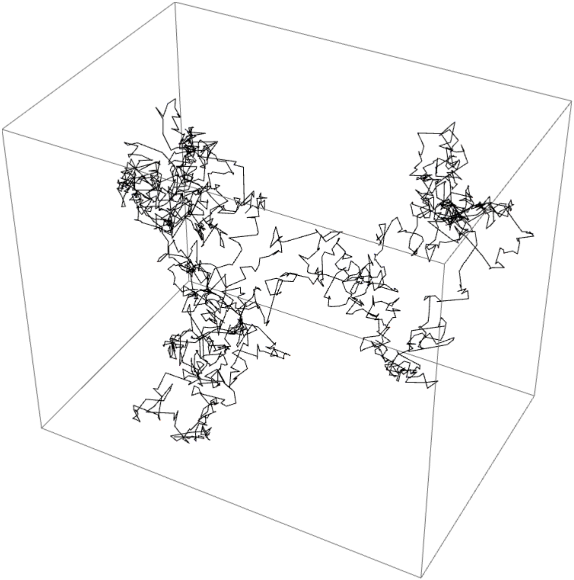

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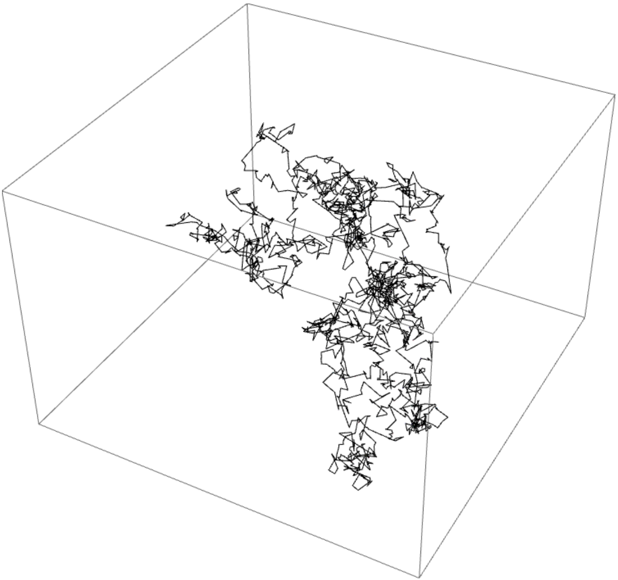

Physicists model polymers with *equilateral* random walks; i.e., walks consisting of n unit-length steps. The moduli space of such walks up to translation is just $\underbrace{S^2(1) \times \ldots \times S^2(1)}_{n}$.

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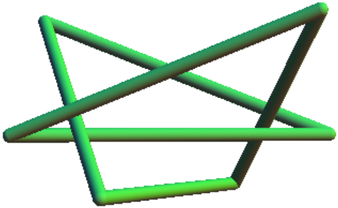
Let ePol(n) be the submanifold of *closed* equilateral random walks (or *random equilateral polygons*): those walks which satisfy both $\|\vec{e}_i\| = 1$ for all i and

$$\sum_{i=1}^{n} \vec{e}_i = \vec{0}.$$



Theorem (Kapovich–Millson, Hausmann–Knutson, Howard–Manon–Millson)

The complex analytic space $\mathrm{ePol}(n)/\mathrm{SO}(3)$ *is toric and Kähler, and arises as a symplectic reduction in two different ways:*

1. *as the reduction* $(S^2)^n /\!\!/ \mathrm{SO}(3)$ *of the product of spheres, or*
2. *(after scaling) as the reduction* $G_2(\mathbb{C}^n) /\!\!/ U(1)^{n-1}$ *of the complex Grassmannian.*

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This is precisely the symplectic analogue of the Gelfand–MacPherson correspondence of GIT quotients,

$$G_2(\mathbb{C}^n) /\!\!/ (\mathbb{C}^*)^{n-1} \simeq (\mathbb{CP}^1)^n /\!\!/ PSL(2, \mathbb{C}),$$

and $\mathrm{ePol}(n)/\mathrm{SO}(3)$ is homeomorphic to either of the above varieties.

Algorithm (with Cantarella)

A Markov chain algorithm on $\mathrm{ePol}(n)/\mathrm{SO}(3)$ *which converges to the correct distribution. Steps take* $O(n^2)$ *time. Generalizes to other fixed edgelength spaces and various confinement regimes.*

Algorithm (with Cantarella and Uehara)

An unbiased sampling algorithm which generates a uniform point on $\mathrm{ePol}(n)/\mathrm{SO}(3)$ *in* $O(n^3)$ *time.*

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At least $1/2$ *of all equilateral hexagons are unknots.*

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Compare to the experimental observation that (with 95% confidence) between 1.1 and 1.5 in 10,000 hexagons are knotted.

...?

The moduli space of *planar* equilateral polygons is not symplectic, though it still arises as a GIT quotient

$$G_2(\mathbb{R}^n) /\!\!/ (\mathbb{R}^*)^{n-1} \simeq (\mathbb{RP}^1)^n /\!\!/ PSL(2, \mathbb{R}).$$

- How can we use this to sample equilateral planar polygons?
- Can we extract natural coordinates on equilateral planar polygons?

Thank you for listening!

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References

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`http://arxiv.org/a/shonkwiler_c_1`