Symplectic Geometry and Frame Theory

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Speaking loosely, a frame in a Hilbert space $\mathcal{H}$ is an overcomplete basis for $\mathcal{H}$. The overcompleteness of a frame allows for greater flexibility and greater robustness to data loss, both of which are of substantial importance in a variety of applications [4, 7, 8].

More precisely, a *frame* in $\mathbb{C}^d$ is a collection $F = \{f_j\}_{j=1}^N$ of vectors $f_j \in \mathbb{C}^d$ satisfying

$$a\|v\|^2 \leq \sum_{j=1}^{N} |\langle v, f_j\rangle|^2 \leq b\|v\|^2 \quad \forall\, v \in \mathbb{C}^d$$

for some numbers $0 < a \leq b$ called *frame bounds*. When $a = b$ the frame is *tight*, and when all the frame vectors have unit norm the frame is a *unit norm frame*. *(Finite) unit norm tight frames* (FUNTFs) are particularly interesting, providing optimal reconstructions in the context of measurements of equal power with additive white Gaussian noise [6]. Thinking of F as a $d \times n$ matrix with ith column f_i, the tight frame condition is equivalent to the *frame operator* FF^* being a multiple of the $d \times d$ identity matrix I_d, and the unit norm condition is equivalent to F^*F having 1's on the diagonal. Since $\operatorname{tr} FF^* = \operatorname{tr} F^*F$, it follows that the FUNTFs are precisely those frames for which

$$FF^* = \frac{N}{d} I_d \qquad \text{and} \qquad \left(\|f_1\|^2, \ldots, \|f_N\|^2\right) = (1, \ldots, 1). \tag{1}$$

A natural and surprisingly challenging question to ask is whether the space of FUNTFs with fixed d and N is connected; that the answer is "yes" is called the frame homotopy conjecture, posed by Larson in a 2002 REU and first appearing in the literature in Dykema and Strawn's 2006 paper [5]. This was recently proved by Cahill, Mixon, and Strawn [2].

Both of the conditions in (1) turn out to be natural to describe in the language of symplectic geometry, leading to a simple alternative proof of the frame homotopy conjecture. Briefly, a *symplectic manifold* is a pair (M, ω) where M is a smooth, even-dimensional manifold and $\omega \in \Omega^2(M)$ is a closed, non-degenerate 2-form on M; see [3] for a nice introduction to symplectic geometry. For example, $(\mathbb{R}^{2n}, \sum \mathrm{d}x_i \wedge \mathrm{d}y_i)$ is the standard example of a symplectic manifold; since $\mathbb{C}^{d\times N} \simeq \mathbb{R}^{2dN}$, the space of $d \times N$ complex matrices is also symplectic.

The action of a Lie group G on a symplectic manifold M is called *Hamiltonian* if there exists a *moment map* $\mu : M \to \mathfrak{g}^*$ such that

$$\omega_p(X_V, X) = D_p\mu(X)(V) \tag{2}$$

for all $p \in M$, $V \in \mathfrak{g}$, and $X \in T_pM$, where $X_V = \frac{d}{dt}\big|_{t=0} \exp(tV) \cdot p$ is the vector field on M induced by the infinitesimal transformation $V \in \mathfrak{g}$. By work of Marsden–Weinstein [9] and Meyer [10], the *symplectic reduction*

$$M \mathbin{/\!\!/}_{\mathcal{O}(\xi)} G := \mu^{-1}(\mathcal{O}(\xi))/G$$

is naturally a symplectic manifold, where $\xi \in \mathfrak{g}^*$ and $\mathcal{O}(\xi)$ is its coadjoint orbit.

There is a natural action of $U(d) \times U(1)^N$ on the space $\mathbb{C}^{d\times N}$ of $d \times N$ complex matrices, where $U(d)$ acts by multiplication on the left and $U(1)^N$ acts by multiplication on the right by a diagonal unitary matrix. In fact the scalar matrices in $U(d)$ and $U(1)^N$ have the same effect, so there is some redundancy in this action. Taking the quotient of $U(1)^N$ by the subgroup of scalar matrices produces an effective action of $U(d)\times U(1)^{N-1}$. Since $\mathfrak{u}(d)^* \simeq \mathcal{H}(d)$, the $d\times d$ Hermitian matrices, and $\left(\mathfrak{u}(1)^{N-1}\right)^* = (\mathfrak{u}(1)^*)^{N-1} \simeq \mathbb{R}^{N-1}$, the corresponding moment map is a map from $\mathbb{C}^{d\times N}$ to $\mathcal{H}(d) \times \mathbb{R}^{N-1}$ which turns out to be given by

$$\mu : F \mapsto \left(FF^*, \left(-\frac{1}{2}\|f_1\|^2, \ldots, -\frac{1}{2}\|f_{N-1}\|^2\right)\right).$$

Therefore, the FUNTFs are simply the level set $\mu^{-1}\left(\frac{N}{d}I_d, \left(-\frac{1}{2}, \ldots, -\frac{1}{2}\right)\right)$ and, while this space is not itself symplectic, its quotient

$$\mathcal{Q}_{d,N} := \mu^{-1}\left(\frac{N}{d}I_d, \left(-\frac{1}{2}, \ldots, -\frac{1}{2}\right)\right)/(U(d)\times U(1)^{N-1}) = C^{d\times N} \mathbin{/\!\!/}_{\left(\frac{N}{d}I_d, -\frac{\vec{1}}{2}\right)} U(d)\times U(1)^{N-1}$$

is symplectic. Performing the reduction in stages yields

$$\mathcal{Q}_{d,N} \simeq \left(\mathbb{C}^{d\times N} \mathbin{/\!\!/}_{\frac{N}{d}I_d} U(d)\right) \mathbin{/\!\!/}_{-\frac{\vec{1}}{2}} U(1)^{N-1} \simeq \mathrm{Gr}_d(\mathbb{C}^n) \mathbin{/\!\!/}_{-\frac{\vec{1}}{2}} U(1)^{N-1},$$

where $\mathrm{Gr}_d(\mathbb{C}^n)$ is the Grassmannian of d-dimensional linear subspaces of $\mathbb{C}^n$. But $\mathrm{Gr}_d(\mathbb{C}^n)$ is connected, and a theorem of Atiyah [1] implies that symplectic reductions of connected manifolds by tori are connected, so $\mathcal{Q}_{d,N}$ is connected. Since $\mathcal{Q}_{d,N}$ is the quotient of the space of FUNTFs by a connected group, this gives a simple symplectic proof of the frame homotopy conjecture:

Theorem 1 ([2, 11])**.** *The space of length-N FUNTFs in $\mathbb{C}^d$ is path-connected for all $N \geq d \geq 1$.*

This approach generalizes to spaces of frames with arbitrary prescribed frame operator and arbitrary prescribed frame vector norms. Specifically, if S is a positive-definite Hermitian $d \times d$ matrix and $\vec{r} = (r_1, \ldots, r_N)$ with $r_i > 0$ for all i, let

$$\mathcal{F}_S^{d,N}(\vec{r}) = \{F \in \mathbb{C}^{d\times N} | FF^* = S, \|f_i\|^2 = r_i\}$$

be the space of frames with frame operator S and frame vector norms determined by $\vec{r}$. Then a suitable generalization of the above argument yields the following generalized frame homotopy theorem:

Theorem 2 ([11]). *For any $N \geq d \geq 1$, any S, and any admissible $\vec{r}$, $\mathcal{F}_S^{d,N}(\vec{r})$ is path-connected.*

This only scratches the surface of a potentially fruitful connection between frame theory and symplectic geometry: the symplectic machinery should naturally generalize to fusion frames and seems well-adapted to other frame theory questions like the Paulsen problem, phase recovery, the existence of maximal equiangular tight frames, and the problem of uniformly sampling random FUNTFs.

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