HOMOTOPY STRING LINKS AND THE κ-INVARIANT

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Abstract. Koschorke introduced a map from the space of closed n-component links to the ordered configuration space of n-tuples of points in $\mathbb{R}^3$, and conjectured that this map separates homotopy links. The purpose of this paper is to construct an analogous map for string links, and to prove (1) this map in fact separates homotopy string links, and (2) Koschorke's original map factors through the map constructed here together with an analogue of Markov's closure map defined on the level of certain function spaces.

1. Introduction

Recall from [10, 11] that Koschorke's κ-invariant assigns to each link-homotopy class of n–component links an associated homotopy class of maps from a torus to a configuration space. The precise definition will be given below, but the basic appeal of Koschorke's construction is threefold.

First, one of Koschorke's primary motivations was to define an invariant which extended Milnor's $\overline{\mu}$-invariants [14, 15] to higher-dimensional links, and indeed he and others (e.g., [12, 16]) have given very precise descriptions of the higher-dimensional situation.

Second, despite its rather abstract appearance, the κ-invariant gives a natural way of defining numerical link-homotopy invariants: the κ-invariant of a link is a homotopy class of maps, so the homotopy periods of the map [7, 18, 17] are link-homotopy invariants of the link. Since link-homotopy invariants are otherwise relatively hard to come by, this is an important potential source for numerical link-homotopy invariants which may be applied to, for example, problems in plasma physics (cf. [8, 9, 5, 4]).

Finally, it remains an open question first posed by Koschorke whether the κ-invariant is a complete invariant of homotopy links (i.e. links up to link-homotopy). It is known to separate Borromean links [11, 3] as well as all 2- and 3-component links [5], but the situation remains unclear for links with 4 or more components.

If the κ-invariant really does separate all homotopy links, then it provides an alternative classification of homotopy links to that of Habegger and Lin [6]. Their classification was based on a Markov-type theorem for closures of homotopy string links, which suggests a strategy for answering Koschorke's question: first, define an analog $\check{\kappa}$ of κ for homotopy string links and show that it separates them, and then show that the "closure" map $\check{\kappa} \mapsto \kappa$ is compatible with Habegger and Lin's classification.

In this paper we carry out the first part of this plan, showing that $\check{\kappa}$ separates homotopy string links. Although it may not be explicitly apparent in what follows, we drew significant inspiration from the close connection between the κ-invariant and Milnor's $\overline{\mu}$-invariants, based on Habegger and Lin's observation that Milnor's invariants are naturally interpreted as integer invariants that separate string links.

The paper is structured as follows. In Section 2, we recall the definition of the κ-invariant, define $\check{\kappa}$, and give the precise statement of the main theorem. Section 3 is devoted to the translation between κ and $\check{\kappa}$. The key practical advantage of working with string links is that they form a group. In Section 4, we describe the relevant algebraic structures and show that $\check{\kappa}$ is actually a monoid homomorphism. We put all these pieces together to prove the main theorem in Section 5.

2. The κ-invariant

Fix an n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ of distinct points in $\mathbb{R}^3$ and consider the function space $\mathrm{link}_{\mathbf{a}}(\sqcup_{i=1}^n S^1; \mathbb{R}^3)$ of pointed n-component smooth link maps ($n \geq 2$), i.e. smooth immersions,

$$\begin{aligned} L\colon (S^1, s_1) \sqcup \ldots \sqcup (S^1, s_n) &\longrightarrow \mathbb{R}^3, \\ L_i(s_i) = a_i, \quad L_i(S^1) \cap L_j(S^1) &= \emptyset, \ i \neq j, \end{aligned} \tag{2.1}$$

where each component has a basepoint s_i mapped to a corresponding fixed point a_i in $\mathbb{R}^3$. Two link maps L and L' are *link homotopic* if there exists a homotopy $H\colon (\bigsqcup_{i=1}^n S^1) \times I \to \mathbb{R}^3$ connecting L and L' through pointed link maps. We denote the set of equivalence classes of pointed n-component link maps by $\mathrm{Link}(n)$, i.e.

$$\mathrm{Link}(n) = \pi_0(\mathrm{link}_{\mathbf{a}}(\sqcup_{i=1}^n S^1; \mathbb{R}^3)). \tag{2.2}$$

Recall, that for any X,

$$\mathrm{Conf}(X, n) := \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\},$$

and write

$$\mathrm{Conf}(n) := \mathrm{Conf}(\mathbb{R}^3, n).$$

The κ-invariant [10, 11] for classical links (i.e. 1-dimensional links in $\mathbb{R}^3$) is defined via the map

$$\begin{aligned} \mathrm{link}_{\mathbf{a}}(\sqcup_{i=1}^n S^1; \mathbb{R}^3) &\to \mathrm{map}_{\mathbf{a}}(\mathbb{T}^n, \mathrm{Conf}(n)), \\ L &\mapsto L_1 \times \ldots \times L_n, \end{aligned}$$

where $\mathrm{map}_{\mathbf{a}}(\mathbb{T}^n, \mathrm{Conf}(n))$ is the function space of continuous pointed maps from the n-torus $\mathbb{T}^n$ to the configuration space of n distinct points in $\mathbb{R}^3$. Define κ as the induced map on π_0:

$$\begin{aligned} \kappa\colon \mathrm{Link}(n) &\to [\mathbb{T}^n, \mathrm{Conf}(n)], \\ \kappa([L]) &= [L_1 \times \cdots \times L_n], \end{aligned} \tag{2.3}$$

where $[\mathbb{T}^n, \mathrm{Conf}(n)] = \pi_0(\mathrm{map}_{\mathbf{a}}(\mathbb{T}^n, \mathrm{Conf}(n))$.

Note that we use pointed maps in (2.1)–(2.3) mainly for convenience (see Appendix A). In his works [10, 11, 12] Koschorke introduced the following central question concerning the κ-invariant.

Question 2.1 (Koschorke [11])**.** Is the κ-invariant injective and therefore a complete invariant of n-component classical links up to link homotopy?

In [11] Koschorke showed that κ is injective on link-homotopy classes of so-called Borromean link maps BLink(n) [11, Theorem 6.1 and Corollary 6.2]. More recently, a positive answer to the above question has been given when $n = 3$ [5]. See also [16] for the modern treatment of the subject via the Goodwillie calculus.

In the current paper, we study an analog of the map in (2.3) in the case Link(n) is replaced by the group of homotopy string links $\mathcal{H}(n)$, $n \geq 2$. Recall that homotopy string links were introduced by Habegger and Lin [6] to address the classification problem for closed homotopy links.

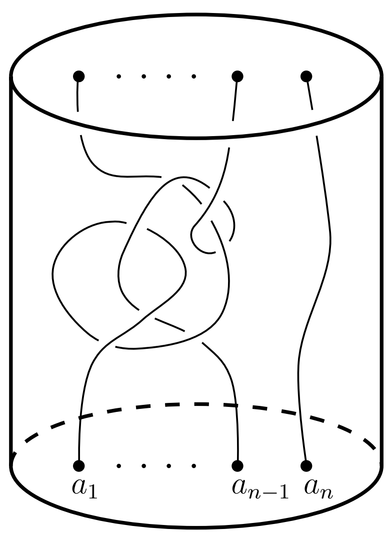

FIGURE 1. n-component string link.

Following Habegger and Lin, consider a cylinder $\mathcal{C} = D \times I$ in $\mathbb{R}^3$, where D is the unit disk and $I = [0, 1]$. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an n-tuple of marked points in D. A *string link* σ is a smooth immersion of n disjoint copies of the unit interval $I = I_i \cong [0, 1]$ into $\mathcal{C}$, i.e.

$$\sigma\colon I_1 \sqcup \ldots \sqcup I_n \longrightarrow \mathcal{C}, \qquad \sigma_i(I_i) \cap \sigma_j(I_j) = \emptyset,\ i \neq j,$$

where the ith component satisfies

$$\sigma_i(0) = \sigma|_{I_i}(0) = (a_i, 0) \qquad \text{and} \qquad \sigma_i(1) = \sigma|_{I_i}(1) = (a_i, 1). \tag{2.4}$$

Here $\sigma_i : I \longrightarrow \mathcal{C}$ denotes the ith strand of σ. Further, for technical reasons we assume that each strand σ_i satisfies the following condition

$$\begin{gathered}\sigma_i\big((0,1)\big) \text{ is contained in the interior of } \mathcal{C} \text{ and each} \\ \text{strand meets } D \times \{0\} \text{ and } D \times \{1\} \text{ orthogonally.}\end{gathered} \tag{2.5}$$

Denote the resulting function space by $\mathrm{slink}_{\mathbf{a}}(\sqcup_{i=1}^n I_i, \mathcal{C})$. Then

$$\mathcal{H}(n) = \pi_0(\mathrm{slink}_{\mathbf{a}}(\sqcup_{i=1}^n I_i, \mathcal{C}))$$

is the set (in fact, group) of link-homotopy classes of string links, a.k.a. homotopy string links. For related constructions of the function spaces of string links see the recent work [13].

Remark 2.2. Notice that $\mathcal{H}(n)$ is the analog of $\mathrm{Link}(n)$: the former denotes link-homotopy classes of string links, while the latter denotes link-homotopy classes of closed links.

Further, let us define an appropriate subspace $\mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$ of maps from the n-cube I^n into the configuration space of n points in $\mathcal{C}$. Let $f \in \mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$ with $f = (f_1, \ldots, f_n)$ if and only if it satisfies the following conditions

(endpoints) for all $i = 1, \ldots, n$,

$$f_i|_{I_1\times\ldots\times I_{i-1}\times\{0\}\times I_{i+1}\times\ldots\times I_n} = (a_i, 0), \quad f_i|_{I_1\times\ldots\times I_{i-1}\times\{1\}\times I_{i+1}\times\ldots\times I_n} = (a_i, 1), \tag{2.6}$$

(periodicity) for all $i = 1, \ldots, n$ and $k \neq i$,

$$f_k|_{I_1\times\ldots\times I_{i-1}\times\{0\}\times I_{i+1}\times\ldots\times I_n} = f_k|_{I_1\times\ldots\times I_{i-1}\times\{1\}\times I_{i+1}\times\ldots\times I_n}. \tag{2.7}$$

(support) the image of the interior of I^n under f is in the interior of $\mathrm{Conf}(\mathcal{C}, n)$

$$f(\mathrm{int}(I^n)) \subset \mathrm{int}(\mathrm{Conf}(\mathcal{C}, n)). \tag{2.8}$$

Due to condition (2.6) we say that maps in $\mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$ are *based at* $\mathbf{a} = (a_1, \ldots, a_n)$.

Next, we can define the analog of the κ-map in (2.3) for homotopy string links. We start with the map

$$\begin{gathered}\mathrm{slink}_{\mathbf{a}}(\sqcup_{i=1}^n I_i; \mathcal{C}) \longrightarrow \mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n)), \\ \sigma \longmapsto \sigma_1 \times \ldots \times \sigma_n,\end{gathered}$$

and define $\breve{\kappa}$ to be the induced map on π_0,

$$\begin{aligned}\breve{\kappa} : \mathcal{H}(n) &\longrightarrow \mathcal{M}(n), \\ \breve{\kappa}([\sigma]) &= [\sigma_1 \times \cdots \times \sigma_n],\end{aligned} \tag{2.9}$$

where $\mathcal{M}(n) = \pi_0(\mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$. It is straightforward to show that, thanks to properties (2.4) and (2.5), $\breve{\kappa}$ is well defined. The main theorem of this paper can now be stated:

Main Theorem. *The map $\breve{\kappa}$ separates homotopy string links.*

Equivalently, the main theorem says that $\breve{\kappa}$ is injective.

Remark 2.3. The map that induces $\check{\kappa}$ is closely related to the Goodwillie–Weiss emebedding calculus. In fact, the space $\mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$ is a subspace of $T_{(1,1,\ldots,1)}\,\mathrm{slink}_{\mathbf{a}}(\sqcup_{i=1}^n I_i, \mathcal{C})$, the $(1, 1, ..., 1)$-stage of the multivariable Taylor tower for the space of string links. Definitions of the latter space can be found in [16, Sections 6.3, 6.4]. Removing the periodicity condition in the definition of $\mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$ produces a space which one can show is equivalent to $T_{(1,1,\ldots,1)}\,\mathrm{slink}_{\mathbf{a}}(\sqcup_{i=1}^n I_i, \mathcal{C})$. Thus our result says that this $(1, 1, ..., 1)$-stage of the tower separates homotopy string links.

3. Transition between κ and $\check{\kappa}$

There is a relation between the κ-invariant of Koschorke defined in (2.3) and the $\check{\kappa}$ of (2.9) for string links, via the Markov "closure" operation: $\widehat{\cdot} : \mathcal{H}(n) \longrightarrow \mathrm{Link}(n)$. In Habegger–Lin [6] the closure $\widehat{\sigma}$ of a given string link $\sigma \in \mathcal{H}(n)$ is obtained by adding "trival" closing strands outside of the embedded cylinder $\mathcal{C}$. For our purposes we use an equivalent but different method of closing a string link. Consider an immersion

$$b : \mathcal{C} \longmapsto \mathcal{T} \subset \mathbb{R}^3,$$

which "bends" the cylinder $\mathcal{C}$ into a solid torus $\mathcal{T}$ embedded in $\mathbb{R}^3$. Under b the bottom disk $D_0 = D \times \{0\}$ of $\mathcal{C}$ is identified with the top disk $D_1 = D \times \{1\}$ into a cross-sectional disk in $\mathcal{T}$. The endpoints of strands are glued under b, i.e. $(a_i, 0) \sim (a_i, 1)$, and the resulting n-tuple of points is again denoted by $\mathbf{a} = (a_1, \ldots, a_n)$. Clearly, for any string link σ in $\mathcal{C}$, $b(\sigma)$ is a closed link, with each i-component based at a_i and contained in $\mathcal{T} \subset \mathbb{R}^3$, therefore we obtain a map

$$b\colon \mathrm{slink}_{\mathbf{a}}(\sqcup_{i=1}^n I_i, \mathcal{C}) \longrightarrow \mathrm{link}_{\mathbf{a}}(\sqcup_{i=1}^n S^1_i, \mathbb{R}^3)$$

equivalent (modulo link homotopy) to the usual Markov closure map $\widehat{\cdot}$. Since b is an embedding on $\mathrm{int}(\mathcal{C})$ it descends to a map of configuration spaces

$$\hat{b}\colon \mathrm{Conf}(\mathrm{int}(\mathcal{C}), n) \longrightarrow \mathrm{Conf}(\mathcal{T}, n) \subset \mathrm{Conf}(\mathbb{R}^3, n).$$

Thus, thanks to conditions (2.6) and (2.8), it gives a well defined map between the respective function spaces*

$$\hat{b}\colon \mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n)) \longrightarrow \mathrm{map}_{\mathbf{a}}(\mathbb{T}^n, \mathrm{Conf}(n)),$$

defined by $\hat{b}(f) = b \circ f$. Note that $\hat{b}(f)$ is a well defined map on $\mathbb{T}^n$ thanks to (2.6) and (2.7). The definition of maps b, $\hat{b}$, κ, and $\check{\kappa}$ immediately implies that $\hat{b} \circ \check{\kappa} = \kappa \circ b$ on the function

*We use the same symbols b and $\hat{b}$ for these various maps, as it should be obvious from the context which one is applied.

spaces and therefore (after applying $\pi_0(\,\cdot\,)$) we obtain the diagram

$$\begin{array}{ccc} \mathcal{H}(n) & \xrightarrow{\check{\kappa}} & \mathcal{M}(n) \\ \downarrow b & & \downarrow \hat{b} \\ \mathrm{Link}(n) & \xrightarrow{\kappa} & [\mathbb{T}^n, \mathrm{Conf}(\mathbb{R}^3, n)] \end{array} \tag{3.1}$$

4. Relevant algebraic structures

4.1. **Group structure on $\mathcal{H}(n)$.** Following [6], $\mathcal{H}(n)$ has a group structure where the multiplication is defined by "stacking" string links in a vertical fashion. In order to give a precise formula on representatives, which is needed later, first define two maps $l, u\colon \mathcal{C} \to \mathcal{C}$, lower and upper, by rescaling the cylinder to its lower and upper half respectively. Specifically, with $\mathcal{C} = D \times I$,

$$\begin{aligned} l, u\colon D \times I &\longrightarrow D \times I, \\ l(x,s) &= (x, \tfrac{1}{2}s), \\ u(x,s) &= (x, \tfrac{1}{2}s + \tfrac{1}{2}). \end{aligned} \tag{4.1}$$

For $\sigma, \sigma' : I_1 \sqcup \ldots \sqcup I_n \longmapsto \mathcal{C}$, the product $\sigma * \sigma'$ is given on each factor by

$$(\sigma * \sigma')_i(t_i) = \begin{cases} u \circ \sigma'_i(2t_i - 1), & t_i \in [\frac{1}{2}, 1], \\ l \circ \sigma_i(2t_i), & t_i \in [0, \frac{1}{2}], \end{cases} \tag{4.2}$$

i.e. the usual product of paths $l \circ \sigma$ and $u \circ \sigma'$ in $\mathcal{C}$. The result satisfies (2.4) and (2.5), so it is a well defined element of $\mathrm{slink}_{\mathbf{a}}(\sqcup_{i=1}^n I_i, \mathcal{C})$.

In fact, Habegger and Lin showed that $\mathcal{H}(n)$ is an extension of groups in the split short exact sequence

$$1 \longrightarrow \mathcal{K}_i(n-1) \longrightarrow \mathcal{H}(n) \xrightarrow{\delta_i} \mathcal{H}_i(n-1) \longrightarrow 1. \tag{4.3}$$

Here $\mathcal{H}_i(n-1)$ is the copy of $\mathcal{H}(n-1)$ which is the image of the map δ_i which deletes the ith strand; i.e., $\mathcal{H}_i(n-1)$ consists of all $(n-1)$-component string links based at $\mathbf{a}_{\hat{i}} = (a_1, \ldots, \widehat{a_i}, \ldots, a_n)$. The normal subgroup $\mathcal{K}_i(n-1)$ is isomorphic to the *reduced free group* $RF(n-1)$ on $n-1$ generators.

Building on Habegger and Lin's construction, consider the homomorphism

$$\delta = \prod_{i=1}^n \delta_i\colon \mathcal{H}(n) \longrightarrow \prod_{i=1}^n \mathcal{H}_i(n-1), \tag{4.4}$$

where δ_i is as defined in (4.3). Then

$$\ker \delta = \mathcal{K}_1(n-1) \cap \mathcal{K}_2(n-1) \cap \ldots \cap \mathcal{K}_n(n-1).$$

The string links representing elements of $\ker\delta$ have a natural geometric meaning: they are precisely the string links which become link-homotopically trivial after removing *any* of their components, which we call† *Borromean string links* and denote by $\mathcal{BH}(n)$; i.e.,

$$\mathcal{BH}(n) := \ker\delta. \tag{4.5}$$

The structure of $\mathcal{BH}(n)$ is well understood, in particular $\mathcal{BH}(n)$ is isomorphic to a direct product of $(n-2)!$ copies of integers, generated by length $(n-1)$ iterated commutators in $RF(n-1)$, (see Lemma 2.2 in [3]). The subspace of $\mathrm{BLink}(n)$ of Borromean links in $\mathrm{Link}(n)$, i.e. links which become trivial after removing any one of the components, is the image of $\mathcal{BH}(n)$ under the closure map.

We summarize needed results of [3]; note that (ii) has also been proved by Koschorke [11].

Theorem 4.1.

(i) *The closure* $b:\mathcal{H}(n)\longrightarrow \mathrm{Link}(n)$ *restricted to* $\mathcal{BH}(n)$ *is one-to-one; i.e., b induces an isomorphism of* $\mathcal{BH}(n)$ *with* $\mathrm{BLink}(n)$*.*

(ii) κ *given in* (2.3) *is one-to-one on* $\mathrm{BLink}(n)$*.*

As a corollary we obtain:

Lemma 4.2. *The map* $\check{\kappa}$ *is one-to-one on* $B\mathcal{H}(n)$*.*

Proof. The claim follows from diagram (3.1) where both b and κ are one-to-one when restricted to $\mathcal{BH}(n)$ and $\mathrm{BLink}(n)$ respectively. □

4.2. **Monoid structure on $\mathcal{M}(n)$.** In this section we will introduce a multiplication on $\mathrm{map}_{\mathbf{a}}(I^n,\mathrm{Conf}(\mathcal{C},n))$ which turns $\mathcal{M}(n)$ into a monoid and makes $\check{\kappa}$ into a monoid homomorphism.

Let $f,g\in \mathrm{map}_{\mathbf{a}}(I^n,\mathrm{Conf}(\mathcal{C},n))$ with $f=(f_1,\dots,f_n)$ and $g=(g_1,\dots,g_n)$. Subdivide the unit cube

$$I^n = I_1\times\ldots\times I_n,\quad I_k=[0,1],$$

with coordinates $(t_1,\dots,t_n)$, into 2^n subcubes simply by splitting each edge into half. Label the first half $[0,\frac{1}{2}]\subset I_k$ by f and the second half $[\frac{1}{2},1]\subset I_k$ by g. Then any subcube is determined by an n-tuple $\mathbf{z}=(z_1,z_2,\dots,z_n)$, where each z_k is either f or g. Such a subcube will further be denoted by

$$I(\mathbf{z}) = I(z_1,z_2,\dots,z_n) = I(z_1)\times\ldots\times I(z_n).$$

Now, we construct maps $R_f,R_g\colon I^n\longrightarrow I^n$. The kth coordinate $(R_f)_k\colon I^n\to I$ of R_f is given by

$$u_{f,k}=(R_f)_k(t_1,\dots,t_n)=\begin{cases}2t_k & \text{if } t_k\le\frac{1}{2}\\ 1 & \text{if } t_k\ge\frac{1}{2}.\end{cases} \tag{4.6}$$

†Some authors use the term *Brunnian links*.

and the kth coordinate $(R_g)_k\colon I^n \to I$ of R_g is given by

$$u_{g,k} = (R_g)_k(t_1,\ldots,t_n) = \begin{cases} 0 & \text{if } t_k \leq \frac{1}{2} \\ 2t_k - 1 & \text{if } t_k \geq \frac{1}{2}. \end{cases} \tag{4.7}$$

Given this framework, we can define the product $f \cdot g$ of f and g on each $I(\mathbf{z})$ as follows: the kth coordinate is

$$\big(f\cdot g\big)_k\big|_{I(\mathbf{z})}(t_1,\ldots,t_n) = \begin{cases} u\circ g_k(R_g(t_1,\ldots,t_n)) & \text{if } z_k = g, \\ l\circ f_k(R_f(t_1,\ldots,t_n)) & \text{if } z_k = f. \end{cases} \tag{4.8}$$

where the maps u and l were defined in (4.1).
The following lemma verifies that $\cdot$ is a well-defined binary operation on $\mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$.

Lemma 4.3. *For any $f, g \in \mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$, $f\cdot g$ is an element of $\mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$.*

Proof. Consider subcubes $I(\mathbf{z})$ and $I(\mathbf{z}')$ which share a boundary face, i.e. $\mathbf{z}$ and $\mathbf{z}'$ differ in just one coordinate. Suppose $z_j \neq z_j'$, and $z_i = z_i'$ for $i \neq j$. It suffices to check for each k that

$$\big(f\cdot g\big)_k|_{I(\mathbf{z})} = \big(f\cdot g\big)_k|_{I(\mathbf{z}')} \quad \text{along } I(\mathbf{z}) \cap I(\mathbf{z}'). \tag{4.9}$$

By possibly swapping the labels $\mathbf{z}$ and $\mathbf{z}'$, we may assume that $z_j = f$ and $z_j' = g$.

We have the $I(z_j)$ factor of $I(\mathbf{z})$ equal to $[0, \frac{1}{2}]$, and $I(z_j)$ factor of $I(\mathbf{z}')$ equal to $[\frac{1}{2}, 1]$, and all other factors are equal. From (4.8), the kth coordinate of $f \cdot g$, for $k \neq j$ is given by

$$\big(f\cdot g\big)_k|_{I(\mathbf{z})}\big|_{t_j=\frac{1}{2}} = h_k(u_{z_k,1},\ldots,u_{z_k,n}), \tag{4.10}$$

$$\big(f\cdot g\big)_k|_{I(\mathbf{z}')}\big|_{t_j=\frac{1}{2}} = h_k(u_{z_k,1},\ldots,u_{z_k,n}), \tag{4.11}$$

where we set $h_k = l \circ f_k$ if $z_k = f$ and $h_k = u \circ g_k$ if $z_k = g$. Recall that $u_{f,i}$ and $u_{g,i}$ are defined in (4.6) and (4.7). Since the expressions (4.10) and (4.11) are equal, (4.9) holds for $k \neq j$. In the case $k = j$ we obtain

$$\big(f\cdot g\big)_j|_{I(\mathbf{z})}\big|_{t_j=\frac{1}{2}} = l\circ f_j(u_{f,1},\ldots,u_{f,j-1},1,u_{f,j+1},\ldots,u_{f,n}) = l(a_j,1),$$
$$\big(f\cdot g\big)_j|_{I(\mathbf{z}')}\big|_{t_j=\frac{1}{2}} = u\circ g_j(u_{g,1},\ldots,u_{g,j-1},0,u_{g,j+1},\ldots,u_{g,n}) = u(a_j,0),$$

and by definition $l(a_j, 1) = (a_j, \frac{1}{2}) = u(a_j, 0)$ which verifies (4.9) for $k = j$. The image of I^n under $f \cdot g$ is in $\mathrm{Conf}(\mathcal{C}, n)$ thanks to the assumptions (2.6) and (2.8) on f and g. ☐

Proposition 4.4. $(\mathcal{M}(n), \cdot)$ *is a monoid with identity.*

Proof. We first check that the multiplication $\cdot$ on $\mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$ is associative up to homotopy. We compare the products $\mu(0) := (f \cdot g) \cdot h$ and $\mu(1) := f \cdot (g \cdot h)$. The formula for each is most easily expressed by subdivisions of the unit cube into 3^n subcubes, albeit different subdivisions.

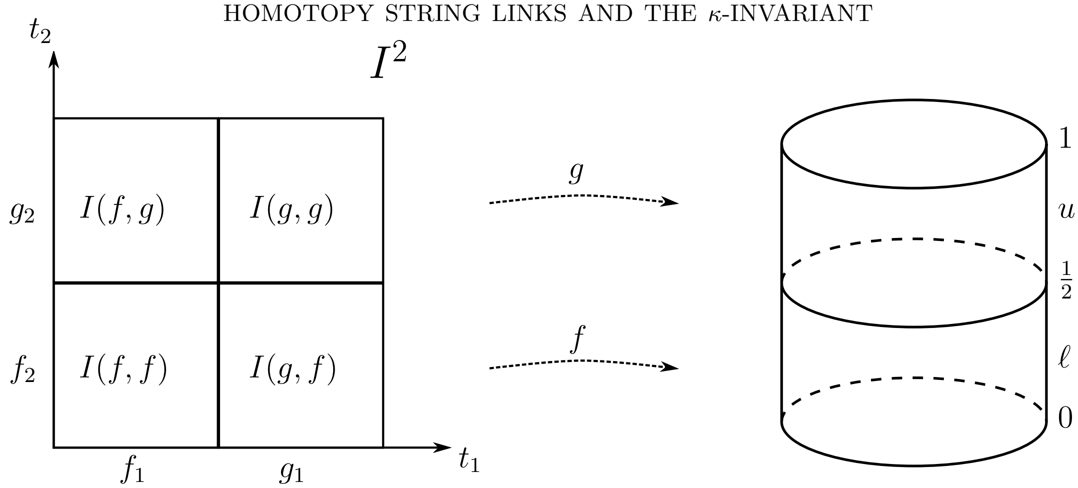

FIGURE 2. Product defined in (4.8) for $n = 2$.

For $\mu(0)$, we label $[0, \frac{1}{4}] \subset I_k$ by f, $[\frac{1}{4}, \frac{1}{2}] \subset I_k$ by g, and $[\frac{1}{2}, 1] \subset I_k$ by h. Thus, as above, an n-tuple $\mathbf{z} = (z_1, ..., z_n)$ with each $z_k \in \{f, g, h\}$ determines a particular subcube $I(\mathbf{z}, 0) = I(z_1, 0) \times \ldots \times I(z_n, 0)$.

For $\mu(1)$, we label $[0, \frac{1}{2}] \subset I_k$ by f, $[\frac{1}{2}, \frac{3}{4}] \subset I_k$ by g, and $[\frac{3}{4}, 1] \subset I_k$ by h. Again, an n-tuple $\mathbf{z} = (z_1, ..., z_n)$ with each $z_k \in \{f, g, h\}$ determines a subcube $I(\mathbf{z}, 1) = I(z_1, 1) \times \ldots \times I(z_n, 1)$.

To get a homotopy between these subdivisions we let $s \in [0, 1]$ and for such s, subdivide I as

$$I(f,s) \cup I(g,s) \cup I(h,s) := \left[0, \frac{1}{4} + \frac{1}{4}s\right] \cup \left[\frac{1}{4} + \frac{1}{4}s, \frac{1}{2} + \frac{1}{4}s\right] \cup \left[\frac{1}{2} + \frac{1}{4}s, 1\right]. \qquad (4.12)$$

We now define "rescaling" maps $R_{f,s},\ R_{g,s},\ R_{h,s}$.

The ith coordinate of $R_{f,s}$ is the map $(R_{f,s})_i \colon I^n \to I$ given by the projection $(t_1, ..., t_n) \mapsto t_i$ followed by the map $I \to I$ which scales and translates $I(f, s)$ to I; the latter map then necessarily takes points that are in $I(g, s)$ or $I(h, s)$ to 1.

Similarly the ith coordinate of $R_{g,s}$ is the map $(R_{g,s})_i \colon I^n \to I$ given by the projection $(t_1, ..., t_n) \mapsto t_i$ followed by the map $I \to I$ which scales and translates $I(g, s)$ to I; this map takes points in $I(f, s)$ to 0 and points in $I(h, s)$ to 1.

Finally, the ith coordinate of $R_{h,s}$ is the map $(R_{h,s})_i \colon I^n \to I$ given by the projection $(t_1, ..., t_n) \mapsto t_i$ followed by the map $I \to I$ which scales and translates $I(h, s)$ to I, taking points that are in $I(f, s)$ or $I(g, s)$ to 0.

For $z \in \{f, g, h\}$, let $Q_{z,s} \colon I \to I(z, s)$ be the map which scales and translates I to $I(z, s)$. Notice that if ι_i denotes the inclusion $I \hookrightarrow I^n$ of the ith factor, then

$$(R_{z,s})_i \circ \iota_i|_{I(z,s)} = (Q_{z,s})^{-1}.$$

Recall that for any map $\mu \in \mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$, we write μ_k to denote μ followed by the projection to the kth configuration point. Since the configuration space is a subspace of the

Cartesian product, the $\mu_1, ..., \mu_n$ completely determine μ. Now for any $s \in [0,1]$, define

$$\big(\mu(s)\big)_k\big|_{I(\mathbf{z})} = \begin{cases} (Q_{f,s} \times id_{D^2}) \circ f_k \circ R_{f,s} & \text{if } z_k = f, \\ (Q_{g,s} \times id_{D^2}) \circ g_k \circ R_{g,s} & \text{if } z_k = g, \\ (Q_{h,s} \times id_{D^2}) \circ h_k \circ R_{h,s} & \text{if } z_k = h. \end{cases}$$

Checking continuity is similar to the proof of Lemma 4.3. For each $s \in [0,1]$ this defines an element $\mu(s) \in \mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$, and thus a homotopy from $\mu(0) = (f \cdot g) \cdot h$ to $\mu(1) = f \cdot (g \cdot h)$. Thus, at the level of π_0 our multiplication is associative.

It remains to check that $(\mathcal{M}(n), \cdot)$ has an identity element. Let e be the image of the standard trivial link in $\mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$. In other words,

$$e(t_1, ..., t_n) = ((a_1, t_1), ..., (a_n, t_n))$$

and e represents the homotopy class $\check{\kappa}(\mathbf{1}) \in \mathcal{M}(n)$, where $\mathbf{1} \in \mathcal{H}(n)$ is the identity element. We claim e is the identity in $\mathcal{M}(n)$. We first check that $e \cdot f$ is homotopic to f for any f.

As before, we can subdivide I^n into 2^n cubes, and we do this for each $s \in (0,1]$. For such an s, we subdivide I as

$$I(e,s) \cup I(f,s) := [0, s/2] \cup [s/2, 1]. \tag{4.13}$$

We label the first half of I_k by e and the second half of I_k by f. For $z \in \{e, f\}$, define $R_{z,s}$ and $Q_{z,s}$ as in the proof of homotopy-associativity, but using the subdivision (4.13) instead of (4.12). For each $s \in (0,1]$ we then define a map $\nu(s) \in \mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$ by

$$\big(\nu(s)\big)_k\big|_{I(\mathbf{z})} = \begin{cases} (Q_{e,s} \times id_{D^2}) \circ e_k \circ R_{e,s} & \text{if } z_k = e \\ (Q_{f,s} \times id_{D^2}) \circ f_k \circ R_{f,s} & \text{if } z_k = f. \end{cases}$$

The map $\nu(1)$ is precisely $e \cdot f$ and the limit of $\nu(s)$ as $s \to 0$ is f, so $e \cdot f$ is homotopic to f.

A similar argument shows that $f \cdot e$ is also homotopic to f. Thus e is an identity up to homotopy in $\mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$, or equivalently, the identity element in $\mathcal{M}(n)$. □

Remark 4.5. (a) Our multiplication on the mapping space is quite similar to a multiplication defined in [2, Section 4.2] on the Taylor tower $T_n\mathrm{Emb}(I, I^3)$ for the space of long knots $\mathrm{Emb}(I, I^3)$. That paper uses a model $T_n\mathrm{Emb}(I, I^3)$, which is a certain space of maps from an n-simplex to a configuration space of n points in $\mathbb{R}^3$. This model closely resembles the targets of κ and $\check{\kappa}$.

(b) As with the multiplication in [2], our multiplication defined above can be parametrized by an action of the little intervals operad. Defining such an action is a relatively straightforward generalization of our proof of homotopy-associativity. (Note that the image of the trivial string link plays the same role as the constant loop in the based loop space ΩX of a space X.)

Lemma 4.6. $\check{\kappa} : \mathcal{H}(n) \longrightarrow \mathcal{M}(n)$ *a monoid homomorphism.*

Proof. The proof follows immediately from the comparison of the two product formulas: one for string links given in (4.2) and one for maps in (4.8). ☐

5. Proof of the Main Theorem

First, let us define an analog of the projection δ_i from (4.3) for the monoid $\mathcal{M}(n)$. For any $i = 1, \ldots, n$, we have the obvious projection

$$p_i\colon (\mathrm{Conf}(\mathcal{C}, n), \mathbf{a}) \longrightarrow (\mathrm{Conf}(\mathcal{C}, n-1), \mathbf{a}_{\widehat{i}})$$
$$p_i(x_1, \ldots x_n) = (x_1, \ldots, \widehat{x}_i, \ldots, x_n),$$

defined by "forgetting" the ith coordinate; in particular, the basepoint $\mathbf{a} = (a_1, \ldots, a_n)$ is mapped to $\mathbf{a}_{\widehat{i}} = (a_1, \ldots, \widehat{a}_i, \ldots, a_n)$. Given any $f \in \mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n))$, we obtain $p_i \circ f\colon I^n \to \mathrm{Conf}(\mathcal{C}, n-1)$. We further restrict $p_i \circ f = p_i \circ f(t_1, \ldots, t_n)$ to the i-th face $I(i)$ of I^n, i.e. $I(i) = I_1 \times \ldots \times I_{i-1} \times \{0\} \times I_{i+1} \times \ldots \times I_n$. That is, we precompose by the inclusion $j_i\colon I(i) \longmapsto I^n$ to get a map

$$\widetilde{\psi}_i(f) = p_i \circ f \circ j_i\colon I^{n-1} \longrightarrow \mathrm{Conf}(\mathcal{C}, n-1)$$

satisfying properties (2.6)–(2.8) with the basepoint $\mathbf{a}_{\widehat{i}}$. The above yields a well-defined map on function spaces

$$\widetilde{\psi}_i\colon \mathrm{map}_{\mathbf{a}}(I^n, \mathrm{Conf}(\mathcal{C}, n)) \longrightarrow \mathrm{map}_{\mathbf{a}_{\widehat{i}}}(I^{n-1}, \mathrm{Conf}(\mathcal{C}, n-1)).$$

and therefore descends to a map of monoids

$$\psi_i := \pi_0(\widetilde{\psi}_i)\colon \mathcal{M}(n) \longrightarrow \mathcal{M}_i(n-1),$$

where $\mathcal{M}_i(n-1)$ denotes $\pi_0(\mathrm{map}_{\mathbf{a}_{\widehat{i}}}(I^{n-1}, \mathrm{Conf}(\mathcal{C}, n-1)))$. It can be shown that ψ_i is a monoid homomorphism, but we have not used this fact in the following argument, and thus we leave the proof to the reader. An analog of the map δ from (4.4) for $\mathcal{M}(n)$ can be now defined as

$$\psi := \psi_1 \times \ldots \times \psi_n\colon \mathcal{M}(n) \longrightarrow \prod_{i=1}^{n} \mathcal{M}_i(n-1).$$

Consider the following diagram

$$\begin{array}{ccc}
B\mathcal{H}(n) & \xrightarrow{\ \check{\kappa}|_{B\mathcal{H}(n)}\ } & B\mathcal{M}(n) \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \jmath} \\
\mathcal{H}(n) & \xrightarrow{\ \check{\kappa}\ } & \mathcal{M}(n) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \psi} \\
\prod_{i=1}^{n} \mathcal{H}_i(n-1) & \xrightarrow{\ (\check{\kappa})^n\ } & \prod_{i=1}^{n} \mathcal{M}_i(n-1)
\end{array} \tag{5.1}$$

where $B\mathcal{M}(n)$ denotes the image of $B\mathcal{H}(n)$ in $\mathcal{M}(n)$ under $\check{\kappa}$, and ι and $\jmath$ are inclusion monomorphisms. The fact that diagram (5.1) commutes follows directly from definitions; in fact, it is derived as π_0 of a commuting diagram of maps on the respective function spaces. By Lemma 4.2, the top map $\check{\kappa}\big|_{B\mathcal{H}(n)}$ in (5.1) is an isomorphism.

Let us reason inductively with respect to n. All 2-component string links are Borromean, so $\check{\kappa} = \jmath \circ \check{\kappa}\big|_{B\mathcal{H}(2)}$ is a monomorphism by Lemma 4.2. This takes care of the base case $n = 2$.

For $n > 2$, assume that $\check{\kappa}$ is a monomorphism on $\mathcal{H}(n-1)$. Then, after an appropriate choice of basepoints, we can conclude that the bottom map $(\check{\kappa})^n$ is also a monomorphism. Suppose $x, y \in \mathcal{H}(n)$ so that $\check{\kappa}(x) = \check{\kappa}(y)$. Since $\check{\kappa}$ is a monoid homomorphism, we obtain

$$\check{\kappa}(\mathbf{1}) = \check{\kappa}(x \cdot x^{-1}) = \check{\kappa}(x) \cdot \check{\kappa}(x^{-1}) = \check{\kappa}(y) \cdot \check{\kappa}(x^{-1}) = \check{\kappa}(y \cdot x^{-1}),$$

where $\mathbf{1}$ denotes the identity of $\mathcal{H}(n)$. From the diagram

$$(\check{\kappa})^n \circ \delta(\mathbf{1}) = \psi \circ \check{\kappa}(\mathbf{1}) = \psi \circ \check{\kappa}(y \cdot x^{-1}) = (\check{\kappa})^n \circ \delta(y \cdot x^{-1}).$$

Since $(\check{\kappa})^n$ is a monomorphism, we get

$$\mathbf{1} = \delta(\mathbf{1}) = \delta(y \cdot x^{-1}),$$

implying $y \cdot x^{-1} \in \ker(\delta)$ and therefore, by (4.5), $y \cdot x^{-1} \in \mathcal{BH}(n)$. The composition $\jmath \circ \check{\kappa}\big|_{B\mathcal{H}(n)}$ is a monomorphism, thus $y \cdot x^{-1} = \mathbf{1}$ implying $x = y$. Hence, $\check{\kappa}$ is a monomorphism on $\mathcal{H}(n)$. □

Appendix A. About the basepoints

In Section 2 we considered pointed homotopy classes in the domain and codomain of the κ-invariant (2.3). Here, we intend to show that the basepoint-free version of κ is equivalent to the pointed one.

Let $s_i \in S^1$ for each $i = 1, \ldots, n$ and let $\mathrm{link}(\sqcup_{i=1}^n S^1; \mathbb{R}^3)$ denote the function space of link maps $L : (S^1, s_1) \sqcup \ldots \sqcup (S^1, s_n) \longrightarrow \mathbb{R}^3$ (defined in (2.1)) without the basepoint condition $L_i(s_i) = a_i$. With $\mathbf{s} = (s_i)_i$ fixed we have a well defined evaluation map

$$ev_{\mathbf{s}} \colon \mathrm{link}(\sqcup_{i=1}^n S^1; \mathbb{R}^3) \longmapsto \mathrm{Conf}(n).$$

By standard results—see e.g. [1]—$ev_{\mathbf{s}}$ is a fibration, with fiber $\mathrm{link}_{\mathbf{a}}(\sqcup_{i=1}^n S^1; \mathbb{R}^3)$. The map induced from fiber inclusion in the long exact sequence of the fibration

$$j_* : \pi_0(\mathrm{link}_{\mathbf{a}}(\sqcup_{i=1}^n S^1; \mathbb{R}^3)) \longrightarrow \pi_0(\mathrm{link}(\sqcup_{i=1}^n S^1; \mathbb{R}^3)),$$

can be shown to be surjective.‡ The map j_* is also injective since $j_*(L) = j_*(L')$ if and only if L and L' differ by an element in $\pi_1(\mathrm{Conf}(n))$ (c.f. [1, p. 132]), which is the trivial group. A completely analogous argument applies to the spaces $\mathrm{map}_{\mathbf{a}}(\mathbb{T}^n, \mathrm{Conf}(n))$ and

‡One may easily change the basepoint of a link after an isotopy.

$\mathrm{map}(\mathbb{T}^n, \mathrm{Conf}(n))$, and therefore we have the following bijections

$$\pi_0(\mathrm{link}(\sqcup_{i=1}^n S^1; \mathbb{R}^3)) \cong \pi_0(\mathrm{link}_{\mathbf{a}}(\sqcup_{i=1}^n S^1; \mathbb{R}^3)),$$
$$\pi_0(\mathrm{map}(\mathbb{T}^n, \mathrm{Conf}(n)) \cong \pi_0(\mathrm{map}_{\mathbf{a}}(\mathbb{T}^n, \mathrm{Conf}(n)).$$

The above maps fit into the diagram

$$\begin{array}{ccc}
\pi_0(\mathrm{link}_{\mathbf{a}}(\sqcup_{i=1}^n S^1; \mathbb{R}^3)) & \xrightarrow{\kappa} & \pi_0(\mathrm{map}_{\mathbf{a}}(\mathbb{T}^n, \mathrm{Conf}(n)) \\
\downarrow \cong & & \downarrow \cong \\
\pi_0(\mathrm{link}(\sqcup_{i=1}^n S^1; \mathbb{R}^3)) & \xrightarrow{\kappa_{\mathrm{free}}} & \pi_0(\mathrm{map}(\mathbb{T}^n, \mathrm{Conf}(n))
\end{array}$$

and hence the pointed κ-invariant is injective if and only if the free κ-invariant is injective.

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