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Symplectic Geometry and Connectivity of Spaces of Frames

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Abstract Frames provide redundant, stable representations of data which have important applications in signal processing. We introduce a connection between symplectic geometry and frame theory and show that many important classes of frames have natural symplectic descriptions. Symplectic tools seem well-adapted to addressing a number of important questions about frames; in this paper we focus on the frame homotopy conjecture posed in 2002 and recently proved by Cahill, Mixon, and Strawn, which says that the space of finite unit norm tight frames is connected. We give a simple symplectic proof of a generalization of the frame homotopy conjecture, showing that spaces of complex frames with arbitrary prescribed norms and frame operators are connected. To spark further investigation, we also suggest a number of fundamental questions in frame theory which seem amenable to a symplectic approach.

Keywords Frame theory · Tight frames · Symplectic geometry

1 Introduction

Speaking loosely, a frame in a Hilbert space $\mathcal{H}$ is an overcomplete dictionary for $\mathcal{H}$. The overcompleteness of a frame allows for both greater flexibility and greater robustness to data loss than a basis would, both of which are of substantial importance in a variety of applications.

Frames have a long history in the signal processing community, having been introduced by Duffin and Schaeffer in 1952 [23], though they were rel-

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atively neglected until Daubechies, Grossmann, and Meyer's pioneering work on wavelets in the 1980s [22]. In the 21st century, an interest in *finite frames* (when $\mathcal{H} = \mathbb{R}^k$ or $\mathbb{C}^k$) led to an explosion of theoretical work and new applications; see [19,37,38] for an introduction.

While some of the modern work on finite frames leverages the algebraic geometry of frame varieties [9], we know of no previous attempts to use symplectic geometry to study frames: the goal of this paper is to show that spaces of complex frames are closely related to nice symplectic manifolds, where some very powerful results from symplectic geometry apply. As an application, we prove a double generalization of the *frame homotopy conjecture* – which says that the space of unit-norm tight frames is path-connected – but this application of the symplectic machinery only scratches the surface of a potentially fruitful connection between frame theory and symplectic geometry, so we hope that this paper will inspire others to explore this connection further.

In order to describe the setting and our results, we recall some definitions. Our results are solely about frames in finite-dimensional complex vector spaces, so we state our definitions in that setting.

A (finite) *frame* in $\mathbb{C}^k$ is a collection $F = \{f_j\}_{j=1}^N$ of vectors $f_j \in \mathbb{C}^k$ so that for all $v \in \mathbb{C}^k$ we have

$$a\|v\|^2 \leq \sum_{j=1}^{N} |\langle v, f_j\rangle|^2 \leq b\|v\|^2 \tag{1}$$

for some numbers $0 < a \leq b$ called *frame bounds*. Throughout the paper, we use $\langle \cdot, \cdot \rangle$ to denote the standard Hermitian product on $\mathbb{C}^k$ for any k and $\|\cdot\|$ its induced norm. Let $\mathcal{F}^{N,k}$ denote the space of frames of N vectors in $\mathbb{C}^k$. Since the parameters N and k will for the most part be fixed throughout the paper, we typically shorten our notation to $\mathcal{F} = \mathcal{F}^{N,k}$. Identifying a frame $F = \{f_j\}_{j=1}^N \in \mathcal{F}$ with the $k \times N$ matrix with columns given by the vectors f_j represented in the standard basis, $\mathcal{F}$ can be viewed as an open, dense subset of $\mathbb{C}^{k\times N}$.

When $a = b$ in (1), the frame satisfies a scaled Parseval identity

$$\sum_{j=1}^{N} |\langle v, f_j\rangle|^2 = a\|v\|^2,$$

and hence such frames are particularly useful in signal reconstruction problems and are called *a-tight* (or just *tight*). Tight frames are natural from the symplectic perspective, though this is most easily seen through the following alternative characterization: any frame $F \in \mathcal{F}$ has an associated *analysis operator*

$$\begin{aligned} \mathbb{C}^k &\to \mathbb{C}^N \\ v &\mapsto (\langle v, f_1\rangle, \ldots, \langle v, f_N\rangle), \end{aligned} \tag{2}$$

and the adjoint *synthesis operator*

$$\begin{aligned} \mathbb{C}^N &\to \mathbb{C}^k \\ (z_1, \ldots, z_N) &\mapsto \sum_{j=1}^{N} z_j f_j. \end{aligned}$$

Again thinking of the frame F as a $k \times N$ matrix, the analysis operator is given by $v \mapsto F^* v$ and the synthesis operator by $v \mapsto Fv$. In turn, composing these defines the *frame operator* $\mathbb{C}^k \to \mathbb{C}^k$ given by $v \mapsto FF^* v$. It is straightforward to show that a frame F is a-tight if and only if its frame operator is $a\mathrm{Id}_k$, where Id_k denotes the identity map on $\mathbb{C}^k$. We use the notation $\mathcal{F}_S = \mathcal{F}_S^{N,k}$ to indicate the space of frames with prescribed frame operator $S : \mathbb{C}^k \to \mathbb{C}^k$; in particular, $\mathcal{F}_{a\mathrm{Id}_k}$ is the space of a-tight frames. In symplectic terminology (which we will define precisely below), the map $\mu_{\mathrm{U}(k)} : F \mapsto FF^*$ is the momentum map of the (left) Hamiltonian $\mathrm{U}(k)$ action on $\mathbb{C}^{k \times N}$, so each space $\mathcal{F}_S$ – including the a-tight frame space $\mathcal{F}_{a\mathrm{Id}_k}$ – is a level set of this map.

Another interesting class of frames are the *unit-norm frames*, which have all $\|f_j\|^2 = 1$. In a signal reconstruction context these frames produce measurements of equal power [37, 55]. The space of all unit-norm frames also has a natural symplectic description: as we will see, the map $\mu_{\mathrm{U}(1)^N} : \mathbb{C}^{k \times N} \to \mathbb{R}^N$ defined by $\mu_{\mathrm{U}(1)^N}(F) = \left(-\frac{1}{2}\|f_1\|^2, \ldots, -\frac{1}{2}\|f_N\|^2\right)$ is the momentum map of the (right) Hamiltonian action of the torus of diagonal, unitary $N \times N$ matrices on $\mathbb{C}^{k \times N}$. For each $\mathbf{r} = (r_1, \ldots, r_N) \in \mathbb{R}^N$ with $r_j \geq 0$, the space $\mathcal{F}(\mathbf{r})$ of frames with $\|f_j\|^2 = r_j$ is a level set of this momentum map. For example, the space of unit-norm frames $\mathcal{F}(1, \ldots, 1) = \mu_{\mathrm{U}(1)^N}^{-1}\left(-\frac{1}{2}, \ldots, -\frac{1}{2}\right)$.

The frames which are both tight and unit-norm are called *finite unit-norm tight frames* (or FUNTFs). The interest in FUNTFs is due in part to the fact that they are optimal for signal reconstruction when each measurement has equal power in the presence of additive white Gaussian noise and erasures [18, 30, 33]. For example, Rupf and Massey [48] showed that, when all users have the same power, optimal signature sequences in CDMA correspond to FUNTFs.

A FUNTF F must have frame bound $a = \frac{N}{k}$ since $FF^* = a\mathrm{Id}_k$ and $\mathrm{trace}(FF^*) = \mathrm{trace}(F^*F) = \sum_{i=1}^N \|f_i\|^2 = N$, meaning that the space of FUNTFs is exactly $\mathcal{F}_{\frac{N}{k}\mathrm{Id}_k} \cap \mathcal{F}(\mathbf{1})$, where $\mathbf{1}$ is the vector of all 1s. Therefore, FUNTF space is the intersection

$$\mu_{\mathrm{U}(k)}^{-1}\left(\frac{N}{k}\mathrm{Id}_k\right) \cap \mu_{\mathrm{U}(1)^N}^{-1}\left(-\frac{1}{2}, \ldots, -\frac{1}{2}\right),$$

or, equivalently, $\mu^{-1}\left(\frac{N}{k}\mathrm{Id}_k, \left(-\frac{1}{2}, \ldots, -\frac{1}{2}\right)\right)$, where μ is the momentum map of an induced Hamiltonian $\mathrm{U}(k) \times \mathrm{U}(1)^N$ action on $\mathbb{C}^{k \times N}$. More generally, each level set of μ corresponds to a space $\mathcal{F}_S(\mathbf{r})$ of frames with prescribed frame operator S and prescribed squared frame norms $\|f_j\|^2 = r_j > 0$, which is a natural symplectic description of these spaces. If this space is nonempty we say the frame norms are *admissible*; see Theorem 4 below for a complete characterization.

The main message of this paper is that this symplectic perspective provides insight into the structure of these spaces, which are generally poorly understood. Even the FUNTF spaces remain fairly mysterious: they are known to be non-empty [30,59], but the simplest possible question about the topology of FUNTF spaces – namely, *are they path-connected?* – is known as the *frame homotopy conjecture*. This problem was posed by Larson in a 2002 REU and it first appears in the literature in Dykema and Strawn's 2006 paper [24], in which Conjecture 7.7 states "the space of unit norm, tight complex N frames in $\mathbb{C}^k$ is connected for all N, k with $N \geq k > 1$." The frame homotopy conjecture was proved for the particular case $N = 2k$ in [29], but the full conjecture remained open until 2017, when it was proved by Cahill, Mixon, and Strawn [8].

The existing proof of the frame homotopy conjecture is rather technical and, while in principle it might be generalizable to other $\mathcal{F}_S(\mathbf{r})$ spaces, in practice that would be a fairly daunting challenge. However, we will see that the frame homotopy conjecture for complex frames follows rather easily from some classical theorems from symplectic geometry, and moreover that this strategy applies just as well to all other choices of frame operator and squared frame vector norms:

Main Theorem *For any positive-definite frame operator S and for any admissible vector of squared norms $\mathbf{r}$, the space $\mathcal{F}_S(\mathbf{r})$ is path-connected.*

As mentioned above, FUNTFs provide optimal reconstructions in the context of measurements of equal power with additive white Gaussian noise. When measurements have unequal power, however, FUNTFs are not optimal. For example, Viswanath and Anantharam showed that optimal CDMA signature sequences when users have different powers correspond to tight frames with squared norms of the frame vectors proportional to user powers [16,55]. Moreover, quoting Casazza et al. [17], in the presence of colored noise "a tight frame is no longer optimal and the frame operator needs to be matched to the noise covariance matrix" (cf. [7,56]). Therefore, the spaces $\mathcal{F}_S(\mathbf{r})$ of frames with prescribed frame operator and squared frame vector norms provide optimal reconstructions in the context of inhomogeneous measurement power and/or colored noise.

We review the relevant ideas from symplectic geometry in Section 2, and then, to introduce these ideas in a more familiar setting, we recover the result of Cahill, Mixon and Strawn for complex FUNTFs in Section 3. We prove the main theorem in Section 4 and suggest some other questions in frame theory which seem accessible to symplectic techniques in Section 5.

2 Basic Concepts from Symplectic Geometry

A good reference for the definitions and results presented here is [45]. Although the objects of study are complex frames, we will work over $\mathbb{R}$ unless explicitly stated otherwise. In particular, when we say "manifold" we mean "real manifold" and when we say "Lie group" we mean "real Lie group".

A *symplectic manifold* is a smooth manifold M equipped with a closed, nondegenerate 2-form ω. The non-degeneracy of ω implies that M must be even-dimensional over $\mathbb{R}$. Let G be a Lie group which acts on M and let $\mathfrak{g}$ denote its Lie algebra. Each point $\xi \in \mathfrak{g}$ determines an *infinitesimal vector field* X_ξ by the formula

$$X_\xi(p) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \exp(\epsilon\xi) \cdot p$$

for each $p \in M$. In this expression, $\exp : \mathfrak{g} \to G$ is the exponential map, so that the quantity being differentiated on the righthand side represents the action of a Lie group element on the point p for each value of ϵ. Using $\mathfrak{g}^*$ to denote the dual of the Lie algebra $\mathfrak{g}$, a *momentum map* for the action is a smooth map $\mu_G : M \to \mathfrak{g}^*$ satisfying

$$\omega_p(X, X_\xi(p)) = D_p\mu_G(X)(\xi)$$

for each $p \in M$, $\xi \in \mathfrak{g}$ and $X \in T_pM$. The map $D_p\mu_G : T_pM \to T_{\mu_G(p)}\mathfrak{g}^*$ is the differential of μ_G at p, so the expression on the right hand side is the evaluation of $D_p\mu_G(X) \in T_{\mu_G(p)}\mathfrak{g}^* \approx \mathfrak{g}^*$ on the vector $\xi \in \mathfrak{g}$. We note that some authors reverse the arguments of ω in this definition, so that our definition will differ by a sign due to skew-symmetry of ω. When G is abelian and M is the phase space of a mechanical system, the momentum map simply records the conserved quantities guaranteed by Noether's theorem.

If the action of G admits a momentum map, then the action is called *Hamiltonian.* Hamiltonian actions play a special role in symplectic geometry and one important property is that they induce a quotient operation in the symplectic category. The quotient operation is referred to as a *symplectic reduction* or *Marsden–Weinstein–Meyer quotient* and is defined in the following classical theorem.

Theorem 1 (Marsden–Weinstein–Meyer Theorem [44,46]) *Let (M,ω) be a symplectic manifold with a Hamiltonian action of a Lie group G and let $\mu_G : M \to \mathfrak{g}^*$ be the momentum map for this action. For any regular value $\xi \in \mathfrak{g}^*$ of μ_G which is fixed by the coadjoint action of G and so that G acts freely on $\mu_G^{-1}(\xi)$, the space*

$$M /\!\!/_\xi G := \mu_G^{-1}(\xi)/G$$

has a natural symplectic structure $\widetilde{\omega}$ satisfying

$$\iota^*\omega = \pi^*\widetilde{\omega}, \tag{3}$$

where $\iota : \mu_G^{-1}(\xi) \to M$ and $\pi : \mu_G^{-1}(\xi) \to \mu_G^{-1}(\xi)/G$ are the inclusion and projection maps, respectively.

More generally, let $\xi \in \mathfrak{g}^*$ be an arbitrary regular value of μ_G and let $\mathcal{O}_\xi$ denote its coadjoint orbit. Then the space

$$M /\!\!/_\xi G := \mu_G^{-1}(\mathcal{O}_\xi)/G$$

has a natural symplectic structure satisfying the analogue of (3).

The assumption that G acts freely on the fiber can be dropped: ξ is a regular value if and only if the action is locally free on the fiber (see, e.g., [11, §23.2]), so the quotient is at worst a symplectic orbifold. In fact, the quotient is a symplectic stratified space even when ξ is not a regular value [51].

Our main technical tool is the following theorem of Atiyah, which is part of the famous Atiyah–Guillemin–Sternberg convexity theorem. The original theorem is [2, Theorem 1] and the statement given here is [12, Theorem 5.21].

Theorem 2 (Atiyah's Connected Level Set Theorem) *Let (M, ω) be a compact connected symplectic manifold with a Hamiltonian n-torus action with momentum map $\mu : M \to \mathbb{R}^n$. Then the nonempty level sets of μ are connected.*

Throughout the paper we identify $(\mathbb{R}^n)^* \approx \mathbb{R}^n$, specifically choosing the isomorphism to be the one induced by the standard inner product.

3 Finite Unit Norm Tight Frames

In this section we will identify the space of FUNTFs as the level set

$$\mu^{-1}\left(\frac{N}{k}\mathrm{Id}_k, \left(-\frac{1}{2}, \dots, -\frac{1}{2}\right)\right)$$

of the momentum map μ of a Hamiltonian action of $\mathrm{U}(k) \times \mathrm{U}(1)^N$ on the vector space $\mathbb{C}^{k\times N}$ of $k \times N$ complex matrices.

Given that, a natural object to consider is the symplectic reduction

$$\begin{aligned}\mu^{-1}\left(\frac{N}{k}\mathrm{Id}_k, \left(-\frac{1}{2}, \dots, -\frac{1}{2}\right)\right) / \left(\mathrm{U}(k) \times \mathrm{U}(1)^N\right) \\ = \mathbb{C}^{k\times N} /\!\!/_{\left(\frac{N}{k}\mathrm{Id}_k, \left(-\frac{1}{2}, \dots, -\frac{1}{2}\right)\right)} \mathrm{U}(k) \times \mathrm{U}(1)^N.\end{aligned}$$

When reducing by a product group it is often helpful to perform the reduction in stages:

$$\begin{aligned}\mathbb{C}^{k\times N} /\!\!/_{\left(\frac{N}{k}\mathrm{Id}_k, \left(-\frac{1}{2}, \dots, -\frac{1}{2}\right)\right)} \mathrm{U}(k) \times \mathrm{U}(1)^N \\ \approx \left(\mathbb{C}^{k\times N} /\!\!/_{\frac{N}{k}\mathrm{Id}_k} \mathrm{U}(k)\right) /\!\!/_{\left(-\frac{1}{2}, \dots, -\frac{1}{2}\right)} \mathrm{U}(1)^N.\end{aligned}$$

We will not actually need to perform the outer reduction: the inner symplectic reduction will turn out to be the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^N)$, which is well-known to be compact and connected, so Atiyah's theorem will imply that the level set of the momentum map for the induced $U(1)^N$ action on the Grassmannian is connected. This level set is simply the quotient $\mathcal{F}_{\frac{N}{k}\mathrm{Id}_k}(\mathbf{1})/\mathrm{U}(k)$ of FUNTF space by the unitary group. Since the unitary group is connected, connectedness of FUNTF space will be a consequence of the following lemma:

Lemma 1 *Let X be a topological space and let G be a connected topological group acting continuously on X. If X/G is connected, then X is connected.*

Proof Since G is connected, the fibers of the quotient map $X \to X/G$ are connected, and the result follows from a standard point-set topology argument (see, e.g., [43, Exercise 5.5]). When X is a compact metric space (as it will be in the cases we care about), this can also be viewed as a special case of the Vietoris mapping theorem [53]. □

3.1 The Symplectic Structure on $\mathbb{C}^{k\times N}$ and the Hamiltonian Actions

The matrix space $\mathbb{C}^{k\times N}$ has a symplectic structure associated to its standard Hermitian inner product, defined on $X_1, X_2 \in T_F\mathbb{C}^{k\times N} \approx \mathbb{C}^{k\times N}$ by

$$\omega_V(X_1, X_2) = -\text{Im}\,\langle X_1, X_2\rangle = -\text{Im trace}(X_1^* X_2),$$

where the product $\langle\cdot,\cdot\rangle$ is interpreted as the Hermitian inner product on $\mathbb{C}^{k\cdot N}$, or equivalently the Frobenius inner product on $k \times N$ matrices.

The group $\text{U}(N)$ of $N \times N$ unitary matrices acts on $\mathbb{C}^{k\times N}$ by right multiplication, and hence so does the subgroup $\text{U}(1)^N$ of diagonal unitary matrices. The effect of this action on $F \in \mathbb{C}^{k\times N}$ is to independently change the phase of each column f_j of F. The Lie algebra $\mathfrak{u}(1)^N \approx \mathbb{R}^N$ is the trivial N-dimensional Lie algebra, and we identify $(\mathfrak{u}(1)^N)^* \approx \mathbb{R}^N$ via the isomorphism induced by the standard inner product.

Proposition 1 *The map*

$$\begin{aligned} &\mu_{\text{U}(1)^N} : \mathbb{C}^{k\times N} \to \mathbb{R}^N \\ &[f_1|f_2|\cdots|f_N] \mapsto \left(-\frac{1}{2}\|f_1\|^2, -\frac{1}{2}\|f_2\|^2, \ldots, -\frac{1}{2}\|f_N\|^2\right) \end{aligned} \tag{4}$$

is a momentum map for the $\text{U}(1)^N$*-action on* $\mathbb{C}^{k\times N}$*.*

Proof Since the circle factors of the torus act independently on each column, it is enough to verify that the $\text{U}(1)$ action on $\mathbb{C}^k$ given by

$$e^{it} \cdot f := f e^{it}$$

has momentum map $\mu_{\text{U}(1)} : f \mapsto -\frac{1}{2}\|f\|^2$. But this is clear: the vector field corresponding to $t \in \mathfrak{u}(1) \approx \mathbb{R}$ is

$$X_t(f) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} f e^{i\epsilon t} = itf,$$

and hence

$$\begin{aligned} \omega_v(X, X_t(f)) &= -\text{Im}\langle X, itf\rangle = -t\,\text{Re}\langle X, f\rangle \\ &= -\frac{1}{2}t\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \text{Re}\langle f + \epsilon X, f + \epsilon X\rangle = D_f\mu_{\text{U}(1)}(X)(t). \end{aligned}$$

□

In turn, the group $\mathrm{U}(k)$ acts on $\mathbb{C}^{k\times N}$ by left multiplication and this action is also Hamiltonian, as we will verify by identifying the momentum map. The Lie algebra $\mathfrak{u}(k)$ consists of the $k\times k$ skew-Hermitian matrices, and we identify the dual $\mathfrak{u}(k)^*$ with the space of Hermitian matrices $\mathcal{H}(k)$ via the map

$$\begin{aligned} \mathcal{H}(k) &\to \mathfrak{u}(k)^* \\ A &\mapsto \left(B \mapsto \frac{i}{2}\mathrm{trace}(AB)\right). \end{aligned} \tag{5}$$

Proposition 2 *The map*

$$\begin{aligned} \mu_{\mathrm{U}(k)} : \mathbb{C}^{k\times N} &\to \mathcal{H}(k) \approx \mathfrak{u}(k)^* \\ F &\mapsto FF^* \end{aligned}$$

is a momentum map for the $\mathrm{U}(k)$*-action on* $\mathbb{C}^{k\times N}$*.*

Proof For $F \in \mathbb{C}^{k\times N}$, the derivative of $\mu_{\mathrm{U}(k)}$ is given by

$$\begin{aligned} D_F\mu_{\mathrm{U}(k)} : \mathbb{C}^{k\times N} &\to \mathcal{H}(k) \\ X &\mapsto FX^* + XF^*. \end{aligned}$$

The infinitesimal vector field induced by $B \in \mathfrak{u}(k)$ is given by $X_B(F) = BF$ at each $F \in \mathbb{C}^{k\times N}$, and it follows that, for any vector $X \in T_F\mathbb{C}^{k\times N} \approx \mathbb{C}^{k\times N}$,

$$\begin{aligned} \omega_F(X, X_B(F)) &= -\mathrm{Im}\ \mathrm{trace}\left(X^*BF\right) = \frac{i}{2}\mathrm{trace}\left(X^*BF - F^*B^*X\right) \\ &= \frac{i}{2}\mathrm{trace}\left(FX^*B - XF^*B^*\right) = \frac{i}{2}\mathrm{trace}\left(FX^*B + XF^*B\right) \\ &= \frac{i}{2}\mathrm{trace}\left(D_F\mu_{\mathrm{U}(k)}(X)B\right) = D_F\mu_{\mathrm{U}(k)}(X)(B), \end{aligned} \tag{6}$$

where the second and third equalities in (6) follow from the linearity and cyclic permutation-invariance of the trace operator, respectively, the fourth equality follows from the assumption that B is skew-Hermitian, and the last is the identification (5) of $\mathcal{H}(k)$ with $\mathfrak{u}(k)^*$. $\square$

The left $\mathrm{U}(k)$ and right $\mathrm{U}(1)^N$ actions commute since matrix multiplication is associative, so we can combine the above actions into a single Hamiltonian $\mathrm{U}(k)\times \mathrm{U}(1)^N$ action on $\mathbb{C}^{k\times N}$ with momentum map

$$\begin{aligned} \mu : \mathbb{C}^{k\times N} &\to \mathcal{H}(k) \times \mathbb{R}^N \\ F &\mapsto \left(FF^*, \left(-\frac{1}{2}\|f_1\|^2, \ldots, -\frac{1}{2}\|f_N\|^2\right)\right). \end{aligned}$$

Observe that the FUNTF space

$$\mathcal{F}_{\frac{N}{k}\mathrm{Id}_k}(\mathbf{1}) = \mu^{-1}\left(\frac{N}{k}\mathrm{Id}_k, \left(-\frac{1}{2}, \ldots, -\frac{1}{2}\right)\right),$$

as desired.

3.2 Reduction in Stages and Grassmannians

As mentioned, we will not actually perform the symplectic reduction by the product group. Instead we focus on reducing by the first factor, which is a reasonable thing to do since – as we will see in Lemma 3 – the frame operator $\frac{N}{k}\mathrm{Id}_k \in \mathcal{H}(k)$ is a regular value of $\mu_{\mathrm{U}(k)}$. The quotient

$$\mathbb{C}^{k\times N} /\!\!/_{\frac{N}{k}\mathrm{Id}_k} \mathrm{U}(k) := \mu^{-1}_{\mathrm{U}(k)}\left(\frac{N}{k}\mathrm{Id}_k\right)/\mathrm{U}(k)$$

is naturally identified with the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^N)$ of k-dimensional linear subspaces of $\mathbb{C}^N$. To see this, observe that

$$\mu^{-1}_{\mathrm{U}(k)}\left(\frac{N}{k}\mathrm{Id}_k\right) = \left\{F \in \mathbb{C}^{k\times N} : FF^* = \frac{N}{k}\mathrm{Id}_k\right\}$$

consists of all $k \times N$ matrices with orthogonal rows, each of norm $\sqrt{\frac{N}{k}}$. This is just a scaled copy of the Stiefel manifold $\mathrm{St}_k(\mathbb{C}^N)$ of k-tuples of Hermitian orthonormal vectors in $\mathbb{C}^N$, and the (free) left $\mathrm{U}(k)$ action corresponds to the standard action on the Stiefel manifold, meaning that the quotient is homeomorphic to $\mathrm{Gr}_k(\mathbb{C}^N) = \mathrm{St}_k(\mathbb{C}^N)/U(k)$. The Grassmannian is understood very well from a geometrical and topological perspective – for example, it is a Riemannian symmetric space and a Kähler manifold – and in particular it is known to be connected.

It follows from Theorem 1 that the Grassmannian inherits a natural symplectic structure from $\mathbb{C}^{k\times N}$ which is compatible with the symplectic structure on $\mathbb{C}^{k\times N}$ in the sense of (3). Moreover, the $\mathrm{U}(1)^N$ action commutes with the $\mathrm{U}(k)$ action and the momentum map (4) is $\mathrm{U}(k)$-equivariant, so both the action and the momentum map descend to $\mathrm{Gr}_k(\mathbb{C}^N)$ [44, Theorem 2]. We call the momentum map $\mathrm{Gr}_k(\mathbb{C}^N) \to \mathbb{R}^N$ of the induced action $\widetilde{\mu}_{\mathrm{U}(1)^N}$.

3.3 The Frame Homotopy Conjecture

We can now give a short proof of the complex frame homotopy conjecture, first proved by Cahill, Mixon and Strawn [8, Theorem 1.1].

Theorem 3 *The space $\mathcal{F}_{\frac{N}{k}\mathrm{Id}_k}(\mathbf{1})$ of complex FUNTFs is path-connected.*

Proof Since the Grassmannian is compact and connected, Atiyah's Theorem (Theorem 2) tells us that $\widetilde{\mu}^{-1}_{\mathrm{U}(1)^N}\left(-\frac{1}{2},\dots,-\frac{1}{2}\right) \subset \mathrm{Gr}_k(\mathbb{C}^N)$ is connected. But this level set is the quotient

$$\mu^{-1}\left(\frac{N}{k}\mathrm{Id}_k, \left(-\frac{1}{2},\dots,-\frac{1}{2}\right)\right)/\,\mathrm{U}(k) = \mathcal{F}_{\frac{N}{k}\mathrm{Id}_k}(\mathbf{1})/\mathrm{U}(k),$$

so Lemma 1 implies that $\mathcal{F}_{\frac{N}{k}\mathrm{Id}_k}(\mathbf{1})$ is connected as well. Moreover, $\mathcal{F}_{\frac{N}{k}\mathrm{Id}_k}(\mathbf{1})$ is a real algebraic set in $\mathbb{C}^{k\times N} \approx \mathbb{R}^{2\cdot k\cdot N}$ and is therefore locally path-connected (in fact, it is triangulable by Łojasiewicz's Triangulation Theorem [41]) so that connectivity implies path connectivity. □

4 The General Case

In this section we extend the proof strategy of Theorem 3 to treat general spaces of frames with prescribed frame operator and vector of squared norms and thereby prove the Main Theorem. The setup is essentially the same: given a Hermitian $k \times k$ matrix S and a vector $\mathbf{r} = (r_1, \ldots, r_N)$ with $r_i \geq 0$, the space $\mathcal{F}_S(\mathbf{r})$ of frames F with frame operator $FF^* = S$ and squared frame norms $\|f_i\|^2 = r_i$ arises as a fiber of the momentum map μ for the $\mathrm{U}(k) \times \mathrm{U}(1)^N$ action on $\mathbb{C}^{k \times N}$:

$$\mathcal{F}_S(\mathbf{r}) = \mu^{-1}\left(S, -\frac{1}{2}\mathbf{r}\right).$$

The challenge is that $\left(S, -\frac{1}{2}\mathbf{r}\right) \in \mathcal{H}(k) \times \mathbb{R}^N$ is not fixed by the coadjoint action of $\mathrm{U}(k) \times \mathrm{U}(1)^N$. The basic issue is that, for $U \in \mathrm{U}(k)$ and $F \in \mathcal{F}_S(\mathbf{r})$,

$$\mu(UF) = \left(UFF^*U^*, \left(-\frac{1}{2}\|Uf_1\|^2, \ldots, -\frac{1}{2}\|Uf_{N-1}\|^2\right)\right) = \left(USU^*, -\frac{1}{2}\mathbf{r}\right)$$

and $USU^* \neq S$ in general. This means that $\mathrm{U}(k) \times \mathrm{U}(1)^N$ does not preserve $\mathcal{F}_S(\mathbf{r})$ and hence there is no quotient.

However, there is still a natural symplectic reduction at hand: the more general reduction over a coadjoint orbit

$$\mathbb{C}^{k \times N} /\!\!/_{\mathcal{O}_{\left(S, -\frac{1}{2}\mathbf{r}\right)}} \mathrm{U}(k) \times \mathrm{U}(1)^N := \mu^{-1}\left(\mathcal{O}_{\left(S, -\frac{1}{2}\mathbf{r}\right)}\right) / \left(\mathrm{U}(k) \times \mathrm{U}(1)^N\right),$$

where $\mathcal{O}_{\left(S, -\frac{1}{2}\mathbf{r}\right)}$ is the coadjoint orbit of $\left(S, -\frac{1}{2}\mathbf{r}\right) \in \mathcal{H}(k) \times \mathbb{R}^N$. Since the coadjoint action of the torus $\mathrm{U}(1)^N$ on $\mathbb{R}^N$ is trivial, this is just

$$\mathcal{O}_{\left(S, -\frac{1}{2}\mathbf{r}\right)} = \mathcal{O}_S \times \left\{-\frac{1}{2}\mathbf{r}\right\},$$

where $\mathcal{O}_S = \{USU^* : U \in \mathrm{U}(k)\}$ is the orbit of $S \in \mathcal{H}(k)$ under the conjugation action of $\mathrm{U}(k)$ on $\mathcal{H}(k)$. Hence,

$$\begin{aligned}\widetilde{\mathcal{F}}_S(\mathbf{r}) &:= \mu^{-1}\left(\mathcal{O}_{\left(S, -\frac{1}{2}\mathbf{r}\right)}\right) \\ &= \{F \in \mathbb{C}^{k \times N} : FF^* = USU^* \text{ for some } U \in \mathrm{U}(k) \text{ and } \|f_i\|^2 = r_i \text{ for all } i\}\end{aligned}$$

is the set of all frames with frame operator conjugate to S and squared frame norms given by the r_i.

Paralleling the FUNTF case, we will only perform the "inner" reduction by $\mathrm{U}(k)$. As we will see in Lemma 3, S is a regular value of $\mu_{\mathrm{U}(k)}$ when it is invertible and positive definite. Since this is always true of frame operators, we can perform the reduction

$$\widetilde{\mathcal{F}}_S(\mathbf{r})/\mathrm{U}(k) = \mathbb{C}^{k \times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k),$$

which will turn out to be a generalization of a Grassmannian called a flag manifold [26,28,47,57,58], which is compact and connected, so we can again apply Atiyah's connectedness theorem.

The goals of this section, then, are: (i) to identify the regular values of $\mu_{\mathrm{U}(k)}$; (ii) to identify $\mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k)$ as a flag manifold and to observe that it is compact and connected; and (iii) to see that connectedness of $\widetilde{\mathcal{F}}_S(\mathbf{r})$ implies $\mathcal{F}_S(\mathbf{r})$ is connected.

4.1 Regular Values of $\mu_{\mathrm{U}(k)}$

In order to identify the regular values of $\mu_{\mathrm{U}(k)} : \mathbb{C}^{k\times N} \to \mathcal{H}(k)$, we need to know when its differential is surjective.

Lemma 2 *For $F \in \mathbb{C}^{k\times N}$, the differential*

$$D_F\mu_{\mathrm{U}(k)} : T_F\mathbb{C}^{k\times N} \approx \mathbb{C}^{k\times N} \to T_{FF^*}\mathcal{H}(k) \approx \mathcal{H}(k)$$

is surjective if and only if F is full rank; equivalently, if and only if F is a frame.

Proof If F is not full rank, then there is a nonzero vector $v \in \ker(F^*)$. Then for any $X \in \mathbb{C}^{k\times N}$,

$$v^* D_F\mu_{\mathrm{U}(k)}(X)v = v^*FX^*v + v^*XF^*v = 0.$$

Since $v^*\mathrm{Id}_k v = v^*v \neq 0$, we see that this implies $\mathrm{Id}_k \in \mathcal{H}(k)$ is not in the image of $D_F\mu_{\mathrm{U}(k)}$, which is therefore not surjective.

On the other hand, suppose that F is full rank. To prove that $D_F\mu_{\mathrm{U}(k)}$ is onto, we choose an arbitrary $W \in \mathcal{H}(k)$ and need to show that the equation

$$D_F\mu_{\mathrm{U}(k)}(X) = FX^* + XF^* = W \tag{7}$$

has a solution $X \in \mathbb{C}^{k\times N}$. Since F is full rank, it has a right inverse F_R^{-1}, and we take $X = \frac{1}{2}W\left(F_R^{-1}\right)^*$. Substituting this into the left hand side of (7), we obtain

$$\begin{aligned} F\left(\frac{1}{2}W\left(F_R^{-1}\right)^*\right)^* + \frac{1}{2}W\left(F_R^{-1}\right)^* F^* &= \frac{1}{2}FF_R^{-1}W^* + \frac{1}{2}W\left(FF_R^{-1}\right)^* \\ &= \frac{1}{2}\left(W^* + W\right) = W, \end{aligned}$$

where the last equality follows because W is Hermitian. ◻

Lemma 3 *The regular values of $\mu_{\mathrm{U}(k)}$ are exactly the invertible, positive-definite Hermitian matrices.*

Proof We first note that any such matrix S has a square root; i.e. a size $k \times k$ matrix W with $WW^* = S$. We then construct a size $k \times N$ block matrix

$$F = \begin{pmatrix} W & 0 \end{pmatrix},$$

which satisfies $\mu_{\mathrm{U}(k)}(F) = FF^* = WW^* = S$. Thus, $\mu_{\mathrm{U}(k)}$ surjects onto the space of invertible, positive-definite Hermitian matrices.

Now note that $FF^* = S$ is invertible if and only if F is full rank. Indeed, it is clear that invertibility of S implies that F^* has trivial kernel. On the other hand, suppose that FF^* has a nonzero vector w in its kernel. Then

$$0 = \langle FF^* w, w\rangle = \langle F^* w, F^* w\rangle$$

implies that $w \in \ker(F^*)$, and F is not full rank. Together with Lemma 2, this completes the proof. □

4.2 A Flag Manifold

For invertible, positive-definite $S \in \mathcal{H}(k)$, Lemma 3 and Theorem 1 imply that

$$\mu^{-1}_{\mathrm{U}(k)}\left(\mathcal{O}_S\right)/\mathrm{U}(k) = \mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k)$$

is a symplectic manifold.

This space plays the same role as the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^N)$ did in the previous section. The key fact about the Grassmannian was that it was compact and connected, and so our goal now is to identify this space and see that it is also compact and connected.

For an invertible, positive-definite Hermitian matrix $S \in \mathcal{H}(k)$ with ℓ distinct (necessarily real) eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0$ with multiplicities $k_1, k_2, \ldots, k_\ell$ adding to k, there is a diffeomorphism

$$\mathcal{O}_S \approx \mathrm{U}(k)/\left(\mathrm{U}(k_1) \times \mathrm{U}(k_2) \times \cdots \times \mathrm{U}(k_\ell)\right)$$

(see [3, Section II.1.d]). In other words, if $d_1 = k_1$, $d_2 = k_1 + k_2$, $d_3 = k_1 + k_2 + k_3, \ldots, d_\ell = k_1 + k_2 + \cdots + k_\ell = k$, then $\mathcal{O}_S$ is the *flag manifold* $\mathrm{Fl}_k(d_1, \ldots, d_\ell)$ consisting of ℓ-tuples $(P_1, \ldots, P_\ell)$ of d_j-dimensional subspaces $P_j \subset \mathbb{C}^k$ satisfying $P_j \subset P_{j+1}$. It follows that

$$\dim \mathcal{O}_S = k^2 - k_1^2 - k_2^2 - \cdots - k_\ell^2.$$

For a regular value S of the momentum map $\mu_{\mathrm{U}(k)}$, then, $\mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k)$ is a smooth manifold of dimension $2k(N-k) + \dim \mathcal{O}_S$.

Let S be a Hermitian matrix with eigenvalue multiplicities as above. Then S is diagonalizable by unitary matrices, and it follows easily that $\mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k)$ only depends on these multiplicities. That is, if S' is a Hermitian matrix with the same multiplicities, then $\mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k) \approx \mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_{S'}} \mathrm{U}(k)$. This observation suggests that $\mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k)$ is itself a flag manifold, which we now prove.

Proposition 3 *With S, k_i, and d_i as above,* $\mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k) \approx \mathrm{Fl}_N(d_1, \ldots, d_\ell, N)$.

Proof Let W be a square root of S, as in the proof of Lemma 3, and let

$$F_0 = \begin{pmatrix} W & 0 \end{pmatrix} \in \mathbb{C}^{k\times N},$$

so that $F_0 F_0^* = S$. Define $R = F_0^* F_0 \in \mathcal{H}(N)$ and let $\widetilde{\mathcal{O}}_R$ be its orbit under the coadjoint (conjugation) action of $\mathrm{U}(N)$ on $\mathcal{H}(N)$.

The matrix R is rank k and has the same nonzero eigenvalues as S. Since S is invertible and hence does not have zero as an eigenvalue, the eigenvalue multiplicities of R are $k_1, \ldots, k_\ell, N-k$, meaning that $\widetilde{\mathcal{O}}_R$ is a copy of the flag manifold $\mathrm{Fl}_N(d_1, \ldots, d_\ell, N)$.

Now, for any $[F] \in \mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k)$, which we think of as the frames in $\widetilde{\mathcal{F}}_S$ which are unitarily equivalent to F, the Gramian F^*F has the same spectrum as R, and so lies in $\widetilde{\mathcal{O}}_R$. Moreover, if F_1, F_2 represent the same class in $\mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k)$, then $F_2 = UF_1$ for some $U \in \mathrm{U}(k)$, and hence

$$F_2^* F_2 = (UF_1)^*(UF_1) = F_1^* U^* U F_1 = F_1^* F_1.$$

Therefore, $[F] \mapsto F^*F$ defines a smooth map $G : \mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k) \to \widetilde{\mathcal{O}}_R \approx \mathrm{Fl}_N(d_1, \ldots, d_\ell, N)$, which we claim is a diffeomorphism.

To see that this map is injective, suppose $G([F_1]) = G([F_2])$ for $[F_1], [F_2] \in \mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k)$; i.e., $F_1^* F_1 = F_2^* F_2$. This implies F_1 and F_2 have the same right singular vectors as well as the same singular values. Also, since $F_1 F_1^*$ and $F_2 F_2^*$ are both conjugate to S, and hence to each other, by unitary matrices, the left singular vectors of V_1 and V_2 are related by a unitary transformation. But then simply writing out the singular value decompositions of F_1 and F_2 shows that $F_2 = UF_1$ for some $U \in U(k)$, and hence $[F_1] = [F_2]$.

On the other hand, to see that G is surjective, suppose $P \in \widetilde{\mathcal{O}}_R$. Then

$$P = U^* R U = U^* F_0^* F_0 U = (F_0 U)^*(F_0 U)$$

for some $U \in \mathrm{U}(n)$. But then $[F_0 U] \in \mathbb{C}^{k\times N} /\!\!/_{\mathcal{O}_S} \mathrm{U}(k)$ since

$$(F_0 U)(F_0 U)^* = F_0 U U^* F_0^* = F_0 F_0^* = S,$$

so $P = G([F_0 U])$.

We've now shown that G is bijective, and the inverse map $P \mapsto [F_0 U]$ is clearly smooth, so this completes the proof. □

Notice, in particular, that the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^N) = \mathrm{Fl}_N(k, N)$, so this generalizes the construction in the FUNTF case.

Moreover, just like the Grassmannian, the flag manifold inherits a natural symplectic structure from $\mathbb{C}^{k\times N}$ and the $\mathrm{U}(1)^N$ action on $\mathbb{C}^{k\times N}$ descends to the flag manifold with corresponding momentum map $\widetilde{\mu}_{\mathrm{U}(1)^N}$.

The flag manifold

$$\mathrm{Fl}_N(d_1, \ldots, d_\ell, N) = \mathrm{U}(N) / \left(\mathrm{U}(k_1) \times \mathrm{U}(k_2) \times \cdots \times \mathrm{U}(k_\ell) \times \mathrm{U}(N-k)\right)$$

is clearly connected, since both Lie groups involved in its definition are, so Atiyah's theorem applies in this setting. Since

$$\widetilde{\mu}^{-1}_{\mathrm{U}(1)^N}\left(-\frac{1}{2}\mathbf{r}\right) = \widetilde{\mathcal{F}}_S(\mathbf{r})/\mathrm{U}(k),$$

Lemma 1 tells us that connectedness of the level set implies connectedness of $\widetilde{\mathcal{F}}_S(\mathbf{r})$.

Of course, connectedness only makes sense when $\widetilde{\mathcal{F}}_S(\mathbf{r})$ is nonempty. For invertible, positive-definite $S \in \mathcal{H}(k)$, we say that a vector $\mathbf{r} = (r_1, \ldots, r_N)$ with $r_i > 0$ is *admissible* if $\widetilde{\mathcal{F}}_S(\mathbf{r})$ is nonempty. The admissible vectors of squared norms are completely characterized by the following theorem.

Theorem 4 (Casazza and Leon [20]) *A vector* $\mathbf{r} = (r_1, \ldots, r_N)$ *is an admissible vector of squared norms for a frame operator* S *with eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ *if and only if*

$$\sum_{j=1}^{N} r_j = \sum_{j=1}^{k} \lambda_j \quad \textit{and} \quad \sum_{j=1}^{\ell} r_j \leq \sum_{j=1}^{\ell} \lambda_j \;\forall\, 1 \leq \ell \leq k.$$

This theorem follows immediately from the classical Schur–Horn theorem [34,49]. A modern exposition of the Schur–Horn theorem using ideas from symplectic geometry appears in [36].

4.3 Proof of the Main Theorem

We are now ready to prove our main theorem.

Main Theorem *For any positive-definite frame operator* S *and for any admissible vector of squared norms* $\mathbf{r}$*, the space* $\mathcal{F}_S(\mathbf{r})$ *is path-connected.*

Proof Fix a frame operator S and an admissible vector of squared norms $\mathbf{r}$. The flag manifold $\mathrm{Fl}_N(d_1, \ldots, d_\ell, N)$ is compact and connected, so Atiyah's Theorem (Theorem 2) implies $\widetilde{\mu}^{-1}_{\mathrm{U}(1)^N}\left(-\frac{1}{2}\mathbf{r}\right)$ is connected. In turn, we have just seen this implies $\widetilde{\mathcal{F}}_S(\mathbf{r})$ is connected. By the same algebraic set argument used in the FUNTF case, $\widetilde{\mathcal{F}}_S(\mathbf{r})$ is also path-connected.

For any points $F_0, F_1 \in \mathcal{F}_S(\mathbf{r}) \subset \widetilde{\mathcal{F}}_S(\mathbf{r})$, there is a continuous path $\widetilde{F}_t : [0,1] \to \widetilde{\mathcal{F}}_S(\mathbf{r})$ joining them. There exists a continuous path $U_t : [0,1] \to \mathrm{U}(k)$ with $\widetilde{F}_t \widetilde{F}_t^* = U_t S U_t^*$ and $U_0 = \mathrm{Id}_k = U_1$, so we can amend our original path to obtain $F_t : [0,1] \to \mathcal{F}_S(\mathbf{r})$ via the formula $F_t = U_t^* \widetilde{F}_t$, which ensures $F_t F_t^* = S$ for all t. Since U_t is unitary, this alteration of the path fixes the column norms for all t, so that F_t is a path in $\mathcal{F}_S(\mathbf{r}) \subset \mathcal{F}_S$ connecting F_0 and F_1, and the theorem follows. □

5 Discussion

The symplectic approach to thinking about frames that we have introduced in this paper should be much more broadly applicable. There are several promising directions for further applications of symplectic ideas to important problems in frame theory. For example, since we have seen that the space of FUNTFs appears as a level set of the momentum map corresponding to the Hamiltonian $U(k) \times U(1)^N$ action on $\mathbb{C}^{k\times N}$, flowing along the negative gradient directions of the squared norm of the momentum map [40] becomes a viable means of "fixing up" a frame which is nearly a FUNTF. This gives a new approach to attacking the Paulsen problem [6,15,31,39] which we intend to take up in a future paper; see [25] for a brief introduction to the key ideas.

Symplectic geometry should also be relevant to the phase retrieval problem [4,5,10,21,54], which can be cast in the following way: an unknown signal vector $v \in \mathbb{C}^k$ is mapped to $\mathbb{C}^N$ by the analysis operator from (2) and then the result is fed into the momentum map (4) of the Hamiltonian $U(1)^N$ action on $\mathbb{C}^N \approx \mathbb{C}^{1\times N}$, and the problem is to invert this composite map up to a global phase ambiguity.

A key feature of compact symplectic manifolds is that they have a natural probability measure (the *Liouville measure*) induced by the symplectic volume form. In the case of Kähler manifolds, which all of the spaces under discussion are, the symplectic volume form and Riemannian volume form agree, so Liouville measure is the Riemannian measure. However, this measure is often much more accessible to symplectic than to Riemannian techniques. For example, (an open, dense subset of) the quotient $\mathcal{F}_{\frac{N}{k}\mathrm{Id}_k}(\mathbf{1})/(\mathrm{U}(k)\times \mathrm{U}(1)^N) \approx \mathrm{Gr}_k(\mathbb{C}^N) /\!\!/_{\left(-\frac{1}{2},\ldots,-\frac{1}{2}\right)} \mathrm{U}(1)^N$ of the space $\mathcal{F}_{\frac{N}{k}\mathrm{Id}_k}(\mathbf{1})$ of FUNTFs in $\mathbb{C}^k$ is toric [27], which means the Liouville measure has a simple structure [14]. Building on work with equilateral polygons in $\mathbb{R}^3$ [13], this leads to an explicit algorithm for sampling random FUNTFs in $\mathbb{C}^2$ [50], but a similar algorithm should exist in all dimensions and for all $\mathcal{F}_S(\mathbf{r})$ spaces. Either experimentally or theoretically, it would be interesting to get estimates or bounds on the probability that frames in these spaces have various nice properties. For example, it should be possible to use this machinery to prove that the spark-deficient frames [1,42] have measure zero in all $\mathcal{F}_S(\mathbf{r})$ spaces.

Also, symplectic geometry can work well in infinite dimensions. The symplectic reduction operation was already applied to infinite-dimensional symplectic manifolds in Marsden and Weinstein's initial paper on the subject (see Examples 6 and 7 in [44, Section 4]). Moreover, analogues of the Atiyah–Guillemin–Sternberg connectivity and convexity theorems have been extended to several infinite-dimensional settings [32,52]. This suggests that the symplectic ideas developed here may also be relevant to the study of frames in infinite-dimensional Hilbert spaces.

Finally, we caution that spaces of frames in $\mathbb{R}^k$ are generally not symplectic, though they should appear as Lagrangian submanifolds of the corresponding complex frame spaces. Hence, it is not obvious how to extend our main theorem or other symplectic arguments to real frame spaces. In fact, the direct translation of the statement of our main theorem cannot be true: it follows from work of Kapovich and Millson [35] that the space of tight frames in $\mathbb{R}^2$ with squared frame vector norms $(4, 4, 4, 1, 1, 1)$ is not connected, so the correct statement of the real version of our theorem must necessarily be more complicated.

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