

# SYMPLECTIC GEOMETRY AND CONNECTIVITY OF SPACES OF FRAMES

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ABSTRACT. The frame homotopy problem, first posed in 2002 and only recently solved by Cahill, Mixon, and Strawn, asks whether the space of finite unit norm tight frames is connected. We generalize that result, showing that spaces of complex frames with arbitrary prescribed norms and frame operators are connected. To do so, we develop a connection between symplectic geometry and frame theory, showing that our frame spaces are closely related to certain flag manifolds and applying a theorem of Atiyah on the connectivity of level sets of momentum maps. These connections to symplectic geometry and flag manifolds may be more broadly useful in the study of frames.

## 1. INTRODUCTION

Speaking loosely, a frame in a Hilbert space  $\mathcal{H}$  is an overcomplete basis for  $\mathcal{H}$ . The overcompleteness of a frame allows for both greater flexibility and greater robustness to data loss, both of which are of substantial importance in a variety of applications.

Frames have a long history in the signal processing community, having been introduced by Duffin and Schaeffer in 1952 [14], though they were relatively neglected until Daubechies, Grossmann, and Meyer’s pioneering work on wavelets in the 1980s [13]. In the 21st century, an interest in *finite frames* (when  $\mathcal{H} = \mathbb{R}^N$  or  $\mathbb{C}^N$ ) led to an explosion of theoretical work and new applications; see [23, 24, 10] for an introduction.

Of particular interest are so-called finite unit-norm tight frames (FUNTFs), which we define below. The interest in FUNTFs is due in part to the fact that they are optimal for signal reconstruction when each measurement has equal power in the presence of additive white Gaussian noise [18]. For example, Rupp and Massey [28] showed that, when all users have the same power, optimal signature sequences in CDMA correspond to FUNTFs.

FUNTFs are known to exist in any dimension [18, 34], but the structure of the spaces of FUNTFs remains fairly mysterious. The simplest possible question about the topology of these spaces – namely, *are they path-connected?* – is known as the *frame homotopy problem*. This problem was apparently posed by Larson in a 2002 REU, though it first appears in the literature in Dykema and Strawn’s 2006 paper [15], in which Conjecture 7.7 states “the space of unit norm, tight complex  $N$  frames in  $\mathbb{C}^k$  is connected for all  $N, k$  with  $N \geq k > 1$ .” The frame homotopy problem was solved for the particular case  $N = 2k$  in [17], but the full conjecture remained open until 2017, when Dykema and Strawn’s conjecture was proved by Cahill, Mixon, and Strawn [5].

The goal of this paper is to extend Cahill, Mixon, and Strawn’s result by generalizing both the “unit norm” and “tight” hypotheses on frames. The key idea is that spaces of complex frames are closely related to nice symplectic manifolds, where some very powerful results from symplectic geometry apply. This application of symplectic machinery to the generalized frame homotopy problem only scratches the surface of a potentially fruitful connection between frame theory and symplectic geometry, so we hope that this paper will inspire others to explore this connection further.

In order to state our result precisely, we recall some definitions. Our results are solely about frames in finite-dimensional complex vector spaces, so we state our definitions in that setting.

A *frame* in  $\mathbb{C}^k$  is a collection  $F = \{f_j\}_{j=1}^N$  of vectors  $f_j \in \mathbb{C}^k$  satisfying

$$a\|v\|^2 \leq \sum_{j=1}^N |\langle v, f_j \rangle|^2 \leq b\|v\|^2 \quad \forall v \in \mathbb{C}^k$$

for some numbers  $0 < a \leq b$  called *frame bounds*. Throughout the paper, we use  $\langle \cdot, \cdot \rangle$  to denote the standard Hermitian product on  $\mathbb{C}^k$  for any  $k$  and  $\|\cdot\|$  its induced norm. A frame is called *tight* if its frame bounds

can be taken to be equal  $a = b$ , in which case we have

$$\sum_{j=1}^N |\langle v, f_j \rangle|^2 = a \|v\|^2$$

Let  $\mathcal{F}^{N,k}$  denote the space of frames of  $N$  vectors in  $\mathbb{C}^k$ . Since the parameters  $N$  and  $k$  will for the most part be fixed throughout the paper, we typically shorten our notation to  $\mathcal{F} = \mathcal{F}^{N,k}$ .

For any frame  $F$ , there is an associated *analysis operator*

$$\begin{aligned} T_F : \mathbb{C}^k &\rightarrow \mathbb{C}^N \\ v &\mapsto (\langle v, f_1 \rangle, \dots, \langle v, f_N \rangle), \end{aligned}$$

a *synthesis operator*

$$\begin{aligned} T_F^* : \mathbb{C}^N &\rightarrow \mathbb{C}^k \\ (z_1, \dots, z_N) &\mapsto \sum_{j=1}^N z_j f_j \end{aligned}$$

and a *frame operator*

$$S_F = T_F^* T_F : \mathbb{C}^k \rightarrow \mathbb{C}^k.$$

Considering the frame  $F = \{f_j\}_{j=1}^N$  as a  $k \times N$  matrix with columns given by the vectors  $f_j$  represented in the standard basis, the above operators are expressed in terms of matrix multiplication as  $T_F(v) = F^*v$  and  $T_F^*(v) = Fv$ , and the frame operator is therefore given by  $S_F = FF^*$ . It is straightforward to show that a frame  $F$  is tight if and only if its frame operator satisfies  $S_F = \frac{1}{a} \text{Id}_k$ , where  $\text{Id}_k$  denotes the identity map on  $\mathbb{C}^k$ .

The space of frames is stratified by subspaces  $\mathcal{F}_S = \mathcal{F}_S^{N,k}$  containing frames with a prescribed frame operator  $S : \mathbb{C}^k \rightarrow \mathbb{C}^k$ . Each of these spaces further decomposes into subspaces  $\mathcal{F}_S(\vec{r})$ , where  $\vec{r} = (r_1, \dots, r_N) \in \mathbb{R}^N$  is a vector of prescribed norms; i.e.,

$$\mathcal{F}_S(\vec{r}) = \{F = \{f_j\}_{j=1}^N \mid T_F^* T_F = S \text{ and } \|f_j\| = r_j \forall j = 1, \dots, N\}.$$

For example, one could consider frames with each vector unit length. If a tight frame satisfies this condition, then it is called a *finite unit-norm tight frame* or *FUNTF*. It is straightforward to show that if a frame is tight and its vectors have unit norm, then its frame bound is forced to be  $a = \frac{k}{N}$ . In our notation, the space of FUNTFs is therefore written as  $\mathcal{F}_{\frac{N}{k} \text{Id}_k}(\vec{1})$ , where  $\vec{1} \in \mathbb{R}^N$  is the vector of all ones.

As mentioned above, FUNTFs provide optimal reconstructions in the context of measurements of equal power with additive white Gaussian noise. When measurements have unequal power, however, FUNTFs are not optimal. For example, Viswanath and Anantharam showed that optimal CDMA signature sequences when users have different powers correspond to tight frames with squared norms of the frame vectors proportional to user powers [32, 8]. Moreover, quoting Casazza et al. [9], in the presence of colored noise “a tight frame is no longer optimal and the frame operator needs to be matched to the noise covariance matrix” (cf. [3, 33]). Therefore, the spaces  $\mathcal{F}_S(\vec{r})$  of frames with prescribed frame operator and frame vector norms provide optimal reconstructions in the context of inhomogeneous measurement power and/or colored noise.

Our generalization of the frame homotopy conjecture is then the following theorem:

**Main Theorem.** *For any frame operator  $S$  and for any admissible vector of norms  $\vec{r}$ , the space  $\mathcal{F}_S(\vec{r})$  is path-connected.*

The strategy is to represent the frame space  $\mathcal{F}_S(\vec{r})$  as the level set of a momentum map for a torus action on a symplectic manifold. It then follows immediately from a well-known theorem of Atiyah that the space is connected, which easily implies path connectivity due to the well-behaved local topology of the space. This method of proof relies on a pair of classical theorems from symplectic geometry, but we will see below that verifying their applicability to this specific situation boils down to some fairly straightforward linear algebra.

Note that the Main Theorem is only a statement about complex frames, and thus does not address or generalize the frame homotopy problem in real vector spaces, which was the harder part of Cahill, Mixon, and Strawn’s paper [5]. Real frame spaces are not generally symplectic manifolds, so the symplectic machinery does not apply.

We review the relevant ideas from symplectic geometry in Section 2, and then, to introduce these ideas in a more familiar setting, we recover the result of Cahill, Mixon and Strawn for complex FUNTFs in Section 3. We then prove the general theorem in Section 4.

## 2. BASIC CONCEPTS FROM SYMPLECTIC GEOMETRY

A good reference for the definitions and results presented here is [27]. A *symplectic manifold* is a smooth, even-dimensional manifold  $M$  equipped with a closed, nondegenerate 2-form  $\omega$ . Let  $G$  denote a Lie group which acts on  $M$  and let  $\mathfrak{g}$  denote its Lie algebra. Each point  $\xi \in \mathfrak{g}$  determines an *infinitesimal vector field*  $X_\xi$  by the formula

$$X_\xi|_p = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \exp(\epsilon\xi) \cdot p$$

for each  $p \in M$ . In this expression,  $\exp : \mathfrak{g} \rightarrow G$  is the exponential map, so that the quantity being differentiated on the righthand side represents the action of a Lie group element on the point  $p$  for each value of  $\epsilon$ . Using  $\mathfrak{g}^*$  to denote the dual of the Lie algebra  $\mathfrak{g}$ , a *momentum map* for the action is a smooth map  $\mu_G : M \rightarrow \mathfrak{g}^*$  satisfying

$$\omega_p(X, X_\xi|_p) = D_p\mu_G(X)(\xi).$$

for each  $p \in M$ ,  $\xi \in \mathfrak{g}$  and  $X \in T_pM$ . The expression on the righthand side denotes the evaluation of  $D_p\mu_G(X) \in T_{\mu_G(p)}\mathfrak{g}^* \approx \mathfrak{g}^*$  on the vector  $X$ . We note that some authors reverse the arguments of  $\omega$  in this definition, so that our definition will differ by a sign due to skew-symmetry of  $\omega$ . When  $G$  is abelian and  $M$  is the phase space of a mechanical system, the momentum map simply records the conserved quantities guaranteed by Noether's theorem.

If the action of  $G$  admits a momentum map, then the action is called *Hamiltonian*. Hamiltonian actions play a special role in symplectic geometry and one important property is that they induce a quotient operation in the symplectic category. The quotient operation is referred to as a *Marsden–Weinstein reduction* and is defined in the following classical theorem.

**Theorem 2.1** (Marsden–Weinstein Reduction Theorem [26]). *Let  $(M, \omega)$  be a symplectic manifold with a Hamiltonian action of a Lie group  $G$  and let  $\mu_G : M \rightarrow \mathfrak{g}^*$  denote the momentum map for this action. For any regular value  $\xi \in \mathfrak{g}^*$  of  $\mu_G$  which is fixed by the coadjoint action of  $G$ , the space*

$$M //_\xi G := \mu_G^{-1}(\xi)/G$$

has a natural symplectic structure  $\tilde{\omega}$  satisfying

$$(1) \quad \iota^*\omega = \pi^*\tilde{\omega},$$

where  $\iota : \mu_G^{-1}(\xi) \rightarrow M$  and  $\pi : \mu_G^{-1}(\xi) \rightarrow \mu_G^{-1}(\xi)/G$  denote the inclusion and projection maps, respectively.

More generally, let  $\xi \in \mathfrak{g}^*$  be an arbitrary regular value of  $\mu_G$  and let  $\mathcal{O}_\xi$  denote its coadjoint orbit. Then the space

$$M //_\xi G := \mu_G^{-1}(\mathcal{O}_\xi)/G$$

has a natural symplectic structure satisfying the analogue of (1).

Our main technical tool is the following theorem of Atiyah, which is part of the famous Atiyah–Guillemin–Sternberg convexity theorem. The original theorem is [1, Theorem 1] and the statement given here is [12, Theorem 5.21].

**Theorem 2.2** (Atiyah's Connected Level Set Theorem). *Let  $(M, \omega)$  be a compact connected symplectic manifold with a Hamiltonian  $n$ -torus action with momentum map  $\mu : M \rightarrow \mathbb{R}^n$ . Then the nonempty level sets of  $\mu$  are connected.*

Throughout the paper we identify  $(\mathbb{R}^n)^* \approx \mathbb{R}^n$ , specifically choosing the isomorphism to be the one induced by the standard inner product.

## 3. FINITE UNIT NORM TIGHT FRAMES

**3.1. FUNTFs and Grassmannians.** In this section, we consider the space

$$\mathcal{F}_{\frac{N}{k}\text{Id}_k} = \left\{ F \mid FF^* = \frac{N}{k}\text{Id}_k \right\}$$

of tight frames of  $N$  vectors in  $\mathbb{C}^k$  with frame bound  $\frac{k}{N}$ . We are particularly interested in this space of frames with prescribed frame operator because it contains the subset of FUNTFs,  $\mathcal{F}_{\frac{N}{k}\text{Id}_k}(\bar{\mathbb{I}})$ . The results of this subsection can easily be generalized to the space of tight frames with arbitrary frame bound  $a$ .

We denote the Stiefel manifold of Hermitian orthonormal  $k$ -tuples of vectors in  $\mathbb{C}^N$  by  $\text{St}_k(\mathbb{C}^N)$ . We denote elements of  $\text{St}_k(\mathbb{C}^N)$  by  $V = \{v_j\}_{j=1}^k$ . Alternatively, we can represent such a  $k$ -tuple by the size  $N \times k$  matrix with column vectors  $v_j$  expressed in the standard basis, hence as a linear map  $V : \mathbb{C}^k \rightarrow \mathbb{C}^N$ . In this representation, the Stiefel manifold is equal to

$$\text{St}_k(\mathbb{C}^N) = \{V \in \text{Lin}(\mathbb{C}^k, \mathbb{C}^N) \mid V^*V = \text{Id}_k\}.$$

We therefore have a natural diffeomorphism  $\mathcal{F}_{\frac{N}{k}\text{Id}_k} \rightarrow \text{St}_k(\mathbb{C}^N)$  given in terms of matrices or linear maps by

$$(2) \quad F \mapsto V = \sqrt{\frac{k}{N}} F^*.$$

The group  $U(k)$  of unitary  $k \times k$  matrices acts naturally on both  $\mathcal{F}_{\frac{N}{k}\text{Id}_k}$  and  $\text{St}_k(\mathbb{C}^N)$ : if  $U \in U(k)$ , then on  $\mathcal{F}_{\frac{N}{k}\text{Id}_k}$  the action is by multiplication on the left by  $U$ , and on  $\text{St}_k(\mathbb{C}^N)$  the action is by multiplication on the right by  $U^*$ . The map (2) is equivariant with respect to these actions. Moreover, this action is free and it is natural to consider the quotient spaces. On the frame side,  $U(k) \backslash \mathcal{F}_{\frac{N}{k}\text{Id}_k}$  is the space of unitary orbits of frames, while on the Stiefel manifold side we obtain the *Grassmannian*

$$\begin{aligned} \text{Gr}_k(\mathbb{C}^N) &= \text{St}_k(\mathbb{C}^N)/U(k) \\ &\approx \{k\text{-dimensional subspaces of } \mathbb{C}^N\} \\ &\approx U(N)/(U(N-k) \times U(k)). \end{aligned}$$

We denote the  $U(k)$ -orbit of a point  $V \in \text{St}_k(\mathbb{C}^N)$  by  $[V] \in \text{Gr}_k(\mathbb{C}^N)$ . The Grassmannian is understood very well from a geometrical and topological perspective, so it serves as an attractive model for studying the geometry and topology of the space of tight frames. In particular, the Grassmannian is a Riemannian symmetric space and a Kähler manifold. In this paper we are primarily interested in the symplectic part of its Kähler structure.

**3.2. Symplectic Structure of the Grassmannian.** In this section we briefly recall how to obtain the canonical symplectic structure on the Grassmannian by realizing  $\text{Gr}_k(\mathbb{C}^N)$  as a Marsden–Weinstein reduction.

The vector space  $\mathbb{C}^{N \times k}$  has a symplectic structure associated to its standard Hermitian inner product, defined on  $X_1, X_2 \in T_V \mathbb{C}^{N \times k} \approx \mathbb{C}^{N \times k}$  by

$$\omega_V(X_1, X_2) = -\text{Im} \langle X_1, X_2 \rangle = -\text{Im} \text{trace}(X_2^* X_1),$$

where the product  $\langle \cdot, \cdot \rangle$  is interpreted as the Hermitian inner product on  $\mathbb{C}^{N \cdot k}$ . The group  $U(k)$  acts on  $\mathbb{C}^{N \times k}$  by right multiplication, and we will show below that this action is Hamiltonian. The Lie algebra  $\mathfrak{u}(k)$  consists of the  $k \times k$  skew-Hermitian matrices, and we identify the dual  $\mathfrak{u}(k)^*$  with the space of Hermitian matrices  $\text{H}(k)$  via the map

$$\begin{aligned} \text{H}(k) &\rightarrow \mathfrak{u}(k)^* \\ A &\mapsto \left( B \mapsto -\frac{i}{2} \text{trace}(AB) \right). \end{aligned}$$

**Proposition 3.1.** *The map*

$$\begin{aligned} \mu_{U(k)} : \mathbb{C}^{N \times k} &\rightarrow \text{H}(k) \approx \mathfrak{u}(k)^* \\ V &\mapsto V^*V. \end{aligned}$$

*is a momentum map for the  $U(k)$ -action on  $\mathbb{C}^{N \times k}$ .*

*Proof.* For  $V \in \mathbb{C}^{N \times k}$ , the derivative of  $\mu_{\mathbf{U}(k)}$  is given by

$$\begin{aligned} D_V \mu_{\mathbf{U}(k)} : \mathbb{C}^{N \times k} &\rightarrow \mathbf{H}(k) \\ X &\mapsto V^* X + X^* V. \end{aligned}$$

The infinitesimal vector field induced by  $B \in \mathfrak{u}(k)$  is given by  $X_B|_V = VB$ , and it follows that for any vector  $X \in T_V \mathbb{C}^{N \times k}$

$$\begin{aligned} (3) \quad \omega_V(X, X_B|_V) &= -\text{Im trace}((VB)^* X) = \frac{i}{2} \text{trace}(B^* V^* X - X^* VB) \\ &= \frac{i}{2} \text{trace}(V^* X B^* - X^* VB) = -\frac{i}{2} \text{trace}(V^* X B + X^* VB) \\ &= -\frac{i}{2} \text{trace}(D_V \mu_{\mathbf{U}(k)}(X) B) = D_V \mu_{\mathbf{U}(k)}(X)(B), \end{aligned}$$

where the equalities in (3) follow from the linearity and cyclic permutation-invariance of the trace operator and the assumption that  $B$  is skew-Hermitian, respectively.  $\square$

It is not hard to see that the identity matrix  $\text{Id}_k$  is a regular value of  $\mu_{\mathbf{U}(k)}$  (we will characterize all regular values of the momentum map in Section 4.1). Since the level set  $\mu_{\mathbf{U}(k)}^{-1}(\text{Id}_k)$  is definitionally the Stiefel manifold  $\text{St}_k(\mathbb{C}^N)$ , it follows from Theorem 2.1 that the Grassmannian inherits a natural symplectic structure, as  $\text{Gr}_k(\mathbb{C}^N) = \mathbb{C}^{N \times k} //_{\text{Id}_k} \mathbf{U}(k)$ . Moreover, the induced symplectic structure is compatible with the canonical Riemannian metric on  $\text{Gr}_k(\mathbb{C}^N)$  inherited from the standard inner product on  $\mathbb{C}^{N \times k}$ . This fact is not crucial for our purposes, but it does allow for the natural characterization of the tangent spaces to  $\text{Gr}_k(\mathbb{C}^N)$  given by the following lemma.

**Lemma 3.2.** *The tangent space to  $\text{St}_k(\mathbb{C}^N)$  at the point  $V$  is the vector space*

$$T_V \text{St}_k(\mathbb{C}^N) = \{X \in \mathbb{C}^{N \times k} \mid V^* X + X^* V = 0\}.$$

*Tangents to the orbits of the  $\mathbf{U}(k)$ -action are called vertical tangent vectors. The subspace of vertical tangent vectors is the vertical space*

$$T_V^{\text{vert}} \text{St}_k(\mathbb{C}^N) = \{VB \in \mathbb{C}^{N \times k} \mid B \in \mathfrak{u}(k)\}.$$

*The horizontal space is the subspace of tangent vectors to  $\text{St}_k(\mathbb{C}^N)$  which are orthogonal (with respect to the real part of the standard Hermitian inner product on  $\mathbb{C}^{N \times k}$ ) to all vertical vectors and can be expressed as*

$$T_V^{\text{hor}} \text{St}_k(\mathbb{C}^N) = \{X \in T_V \text{St}_k(\mathbb{C}^N) \mid \text{trace}((X^* V - V^* X) B) = 0 \forall B \in \mathfrak{u}(k)\}.$$

*By the construction of  $\text{Gr}_k(\mathbb{C}^N)$  as the quotient  $\text{Gr}_k(\mathbb{C}^N) = \text{St}_k(\mathbb{C}^N)/\mathbf{U}(k)$ , there is a linear isometry  $T_{[V]} \text{Gr}_k(\mathbb{C}^N) \approx T_V^{\text{hor}} \text{St}_k(\mathbb{C}^N)$ .*

*Proof.* From the implicit function theorem,  $T_V \text{St}_k(\mathbb{C}^N) = \ker D_V \mu_{\mathbf{U}(k)}$ , and the characterization of  $T_V \text{St}_k(\mathbb{C}^N)$  follows. The vertical space description follows from the fact that the infinitesimal vector field  $X_B$  associated to  $B \in \mathfrak{u}(k)$  is given by  $X_B|_V = VB$ . To prove the final claim, note that

$$\text{Re trace}(X^* VB) = \frac{1}{2} \text{trace}(X^* VB + B^* V^* X) = \frac{1}{2} \text{trace}((X^* V - V^* X) B).$$

$\square$

It follows from Theorem 2.1 that the symplectic structure on  $\text{Gr}_k(\mathbb{C}^N)$  can be evaluated by representing tangent vectors to  $[V] \in \text{Gr}_k(\mathbb{C}^N)$  as horizontal elements of  $T_V \text{St}_k(\mathbb{C}^N)$  and using the canonical symplectic form on  $\mathbb{C}^{N \times k}$ . This computational convenience will be used frequently throughout the rest of the paper.

**3.3. Hamiltonian Torus Action.** We now wish to show that  $\text{Gr}_k(\mathbb{C}^N)$  has a Hamiltonian action of the  $(N-1)$ -dimensional torus  $\mathbb{T}^{N-1} = \mathbf{U}(1)^{N-1}$ . The action is induced by the action of  $\mathbb{T}^N$  on  $\text{St}_k(\mathbb{C}^N)$  defined as follows. Fixing the standard basis for  $\mathbb{C}^N$ , we identify the space of linear maps  $\text{Lin}(\mathbb{C}^k, \mathbb{C}^N)$  with the space of  $N \times k$  complex matrices  $\mathbb{C}^{N \times k}$ . We represent elements of  $\text{Lin}(\mathbb{C}^k, \mathbb{C}^N)$  in coordinates using the notation

$$X = (x_1, \dots, x_k) = \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_k^1 \\ x_1^2 & x_2^2 & \cdots & x_k^2 \\ \vdots & \vdots & & \vdots \\ x_1^N & x_2^N & \cdots & x_k^N \end{pmatrix} = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^N \end{pmatrix}$$

in order to easily reference the rows and columns of the matrix. Then an element  $V \in \text{St}_k(\mathbb{C}^N)$  is represented as a matrix  $V = (v_k^j)$  with orthonormal column vectors  $v_k$ . An element  $z$  in the circle  $U(1)$  acts on  $V \in \text{St}_k(\mathbb{C}^N)$  by complex multiplication of the  $j$ th row vector of  $V$  – we thus define an action of  $\mathbb{T}^N$  on  $\text{St}_k(\mathbb{C}^N)$  by these  $N$  independent circle actions.

Since matrix multiplication is associative, the left torus action described above commutes with the right action of  $U(k)$  on  $\text{St}_k(\mathbb{C}^N)$  and therefore descends to a well-defined action on  $\text{Gr}_k(\mathbb{C}^N)$ . However, there is some redundancy in that the diagonal circle in  $\mathbb{T}^N$  acts trivially on the Grassmannian. We therefore take a quotient by the diagonal circle to obtain a free action of  $\mathbb{T}^N/U(1) \approx \mathbb{T}^{N-1}$  on  $\text{Gr}_k(\mathbb{C}^N)$ . An element of the quotient  $\mathbb{T}^N/U(1)$  can be realized concretely as, say, an element of  $\mathbb{T}^N$  with its last entry fixed to be  $1 \in U(1)$ . It is therefore equivalent to realize the  $\mathbb{T}^{N-1}$  action on  $\text{Gr}_k(\mathbb{C}^N)$  as the one induced by  $\mathbb{T}^{N-1}$  acting on the the first  $N - 1$  rows of  $V \in \text{St}_k(\mathbb{C}^N)$ . This torus action on  $\text{Gr}_k(\mathbb{C}^N)$  will be referred to as the *row action*.

For each  $j = 1, \dots, N$ , consider the map  $\tilde{\phi}_j : \text{St}_k(\mathbb{C}^N) \rightarrow \mathbb{R}$  defined on  $V = (v_k^j) \in \text{St}_k(\mathbb{C}^N)$  by

$$\phi_j(V) = \frac{1}{2} \|v^j\|^2.$$

Each of these maps is  $U(k)$ -invariant, so they descend to well-defined maps  $\phi_j : \text{Gr}_k(\mathbb{C}^N) \rightarrow \mathbb{R}$ . We define  $\tilde{\Phi} = (\tilde{\phi}_1, \dots, \tilde{\phi}_{N-1}) : \text{St}_k(\mathbb{C}^N) \rightarrow \mathbb{R}^{N-1}$  and we denote by  $\Phi : \text{Gr}_k(\mathbb{C}^N) \rightarrow \mathbb{R}^{N-1}$  the induced map on the Grassmannian. We remark that if we had considered the full  $\mathbb{T}^N$ -action on  $\text{St}_k(\mathbb{C}^N)$ , the redundancy described above would be reflected in the fact that  $\tilde{\phi}_1 + \dots + \tilde{\phi}_N = \frac{k}{2}$ , so  $\tilde{\phi}_N$  is determined by  $\tilde{\phi}_1, \dots, \tilde{\phi}_{N-1}$ .

**Proposition 3.3.** *The map  $\Phi : \text{Gr}_k(\mathbb{C}^N) \rightarrow \mathbb{R}^{N-1} \approx (\mathbb{R}^{N-1})^*$  is a momentum map for the row action of the torus  $\mathbb{T}^{N-1}$ .*

The proof of the proposition will use a pair of lemmas. The first is a general fact about momentum maps and the second follows by a straightforward calculation.

**Lemma 3.4.** *Let  $G$  be a Lie group which acts in a Hamiltonian way on a symplectic manifold  $(M, \omega)$ , let  $\mu_G : M \rightarrow \mathfrak{g}^*$  be a momentum map and let  $\xi \in \mathfrak{g}^*$  be a regular value of  $\mu$ . Let  $T_p^{\text{vert}} \mu^{-1}(\xi)$  denote the subspace of vectors which are tangent to the  $G$ -orbit of  $p$ . Then for any  $X \in T_p \mu^{-1}(\xi)$  and any  $Y \in T_p^{\text{vert}} \mu^{-1}(\xi)$ ,  $\omega_p(X, Y) = 0$ .*

*Proof.* Any  $Y \in T_p^{\text{vert}} \mu^{-1}(\xi)$  can be expressed as an infinitesimal vector field  $X_\chi$ , for some  $\chi \in \mathfrak{g}$ . Then

$$\omega_p(X, Y) = \omega_p(X, X_\chi) = D_p \mu_G(X)(\chi) = 0,$$

since  $T_p \mu^{-1}(\xi)$  is the kernel of the map  $D_p \mu_G$ . □

**Lemma 3.5.** *The infinitesimal vector field on  $\text{Gr}_k(\mathbb{C}^N)$  at  $[V]$  associated to  $\xi = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1}$  is represented by the projection of*

$$(4) \quad X_\xi|_V = \begin{pmatrix} i\xi_1 v^1 \\ i\xi_2 v^2 \\ \vdots \\ i\xi_{N-1} v^{N-1} \\ 0 \end{pmatrix}$$

to the horizontal tangent space in  $T_V \text{St}_k(\mathbb{C}^N)$ .

**Example 3.6.** *We remark that the projection step in Lemma 3.5 is necessary, since it is possible for the vector field  $X_\xi|_V$  to have a component along the  $U(k)$ -vertical direction. For example, let*

$$V = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{pmatrix} \in \text{St}_2(\mathbb{C}^3)$$

and let  $\xi = (2, 0) \in \mathbb{R}^2$ . We have

$$X_\xi|_V = \begin{pmatrix} 2i/\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} i/\sqrt{2} & 0 \\ 0 & 0 \\ -i/\sqrt{2} & 0 \end{pmatrix} + \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \\ 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}.$$

The second expression for the vector is decomposed into a horizontal part and a (nontrivial) vertical part. It is also possible that  $X_\xi|_V$  lies completely in the vertical direction; for example, this is the case for  $V = (e_1, e_2) \in \text{St}_2(\mathbb{C}^3)$  (with  $e_j$  denoting standard basis vectors) and any choice of  $\xi$ . This implies that  $[V]$  is a fixed point of the row action on the Grassmannian.

*Proof of Proposition 3.3.* The proof now follows by a calculation. Since  $\text{Gr}_k(\mathbb{C}^N)$  was constructed as a Marsden–Weinstein reduction, the defining condition of a momentum map can be checked in ambient coordinates. We therefore need to check that for a point  $V \in \text{St}_k(\mathbb{C}^N)$ , for a tangent vector  $X \in T_V^{\text{hor}}\text{St}_k(\mathbb{C}^N)$ , and for a vector  $\xi \in \mathbb{R}^{N-1}$ ,

$$(5) \quad D_V \Phi(X)(\xi) = \omega_V(X, X_\xi^{\text{hor}}|_V),$$

where  $X_\xi^{\text{hor}}|_V$  denotes the projection of (4) to  $T_V^{\text{hor}}\text{St}_k(\mathbb{C}^N)$ . Writing  $V = (v_k^j)$  and  $X = (x_k^j)$ , the quantity on the right side of (5) is given by

$$\omega_V(X, X_\xi^{\text{hor}}|_V) = \omega_V(X, X_\xi|_V) = -\text{Im trace}(X_\xi|_V^* X) = \text{Im} \sum_{j,k=1}^{N-1} i\xi_j \overline{v_k^j} x_k^j.$$

The first equality follows by Lemma 3.4 and bilinearity of  $\omega_V$  and the last follows by Lemma 3.5 and an elementary calculation.

To simplify the left hand side of (5), we recall that we have specifically identified  $\mathbb{R}^{N-1}$  with  $(\mathbb{R}^{N-1})^*$  via the standard inner product. Since

$$D_V \Phi(X) = (\text{Re} \langle x^1, v^1 \rangle, \dots, \text{Re} \langle x^{N-1}, v^{N-1} \rangle),$$

our choice of identification yields

$$D_V \Phi(X)(\xi) = \sum_{j=1}^{N-1} \text{Re} \langle x^j, v^j \rangle \xi_j = \text{Im} \sum_{j,k=1}^{N-1} i\xi_j \overline{v_k^j} x_k^j,$$

completing the proof.  $\square$

We can now recover the Cahill, Mixon and Strawn’s result on the connectivity of spaces of complex FUNTFs [5, Theorem 1.1].

**Theorem 3.7.** *The space of complex FUNTFs is path-connected.*

*Proof.* We saw in Proposition 3.3 that  $\Phi$  is the momentum map of a Hamiltonian torus action, and hence Atiyah’s Theorem (Theorem 2.2) implies that each nonempty set  $\Phi^{-1}(\xi)$  is connected. Since  $\text{U}(k)$  is connected and  $\Phi^{-1}(\xi) = \tilde{\Phi}^{-1}(\xi)/\text{U}(k)$ , it follows that  $\tilde{\Phi}^{-1}(\xi)$  is connected as well. Moreover,  $\tilde{\Phi}^{-1}(\xi)$  is a real algebraic set in  $\mathbb{C}^{N \times k} \approx \mathbb{R}^{2 \cdot N \cdot k}$  and is therefore locally path-connected (in fact, it is triangulable by Lojasiewicz’s Triangulation Theorem [25]) so that connectivity implies path connectivity. Fixing  $\xi_0$  to be the constant vector with each entry equal to  $\frac{k}{2N}$ , we have that  $\tilde{\Phi}^{-1}(\xi_0)$  is diffeomorphic via the restriction of the inverse of (2) to the space  $\mathcal{F}_{\frac{N}{k}\text{Id}_k}(\vec{1})$  of FUNTFs, and this completes the proof.  $\square$

#### 4. THE GENERAL CASE

In this section we extend the proof strategy of Theorem 3.7 to treat general spaces of frames with prescribed frame operator and vector of norms and thereby prove the Main Theorem. The extension of the proof is straightforward once we prove some technical preliminary results.

**4.1. Regular Values of  $\mu_{\text{U}(k)}$ .** We return to the map

$$\mu_{\text{U}(k)} : \mathbb{C}^{N \times k} \rightarrow \text{H}(k) \approx \mathfrak{u}(k)^*.$$

In the construction to follow, we will require an understanding of its regular values.

**Lemma 4.1.** *The map  $D_V \mu_{\text{U}(k)} : T_V \mathbb{C}^{N \times k} \approx \mathbb{C}^{N \times k} \rightarrow T_{V^*V} \text{H}(k) \approx \text{H}(k)$  is surjective if and only if  $V$  is full rank.*

*Proof.* If  $V$  is not full rank, then there is a nonzero vector  $v \in \ker(V)$ . Then for any  $X \in \mathbb{C}^{N \times k}$ ,

$$v^* D_V \mu_{U(k)}(X)v = v^* V^* X v + v^* X^* V v = 0.$$

Since  $v^* \text{Id}_k v = v^* v \neq 0$ , we see that this implies  $\text{Id}_k \in \mathfrak{H}(k)$  is not in the image of  $D_V \mu_{U(k)}$ , which is therefore not surjective.

On the other hand, suppose that  $V$  is full rank. To prove that  $D\mu_{U(k)}(V)$  is onto, we choose an arbitrary Hermitian matrix  $W$  and solve the equation

$$(6) \quad V^* X + X^* V = W$$

for  $X$ . Since  $V$  is full rank, it has a left inverse  $V_L^{-1}$ , and we take  $X = \frac{1}{2}(V_L^{-1})^* W$ . Substituting this into the left hand side of (6), we obtain

$$V^* \frac{1}{2}(V_L^{-1})^* W + \left( \frac{1}{2}(V_L^{-1})^* W \right)^* V = \frac{1}{2}(V_L^{-1} V)^* W + \frac{1}{2} W^* V_L^{-1} V = \frac{1}{2}(W + W^*) = W,$$

where the last equality follows because  $W$  is Hermitian.  $\square$

**Lemma 4.2.** *The regular values of  $\mu_{U(k)}$  are exactly the invertible, positive-definite Hermitian matrices.*

*Proof.* We first note that any such matrix  $S$  has a square root; i.e. a size  $k \times k$  matrix  $W$  with  $W^* W = S$ . We then construct a size  $N \times k$  block matrix

$$V = \begin{pmatrix} W \\ 0 \end{pmatrix},$$

which satisfies  $\mu_{U(k)}(V) = V^* V = W^* W = S$ . Thus  $\mu_{U(k)}$  surjects onto the space of invertible, positive-definite Hermitian matrices.

Now note that  $V^* V = S$  is invertible if and only if  $V$  is full rank. Indeed, it is clear that invertibility of  $S$  implies that  $V$  has trivial kernel. On the other hand, suppose that  $V^* V$  has a nonzero vector  $w$  in its kernel. Then

$$0 = \langle V^* V w, w \rangle = \langle V w, V w \rangle$$

implies that  $w \in \ker(V)$ , and  $V$  is not full rank. Together with Lemma 4.1, this completes the proof.  $\square$

**4.2. The  $S$ -Grassmannian.** For each invertible, positive-definite Hermitian linear map  $S : \mathbb{C}^k \rightarrow \mathbb{C}^k$ , we define the  $S$ -Grassmannian to be the space

$$\text{Gr}_S(\mathbb{C}^N) = \{[V] \mid V \in \text{Lin}(\mathbb{C}^k, \mathbb{C}^N), V^* V = S\},$$

where  $[V]$  denotes the orbit of the map  $V$  under the action of  $U(k)$  by precomposition.

We wish to show that  $\text{Gr}_S(\mathbb{C}^N)$  is realized as a (generalized) Marsden–Weinstein reduction. For any positive-definite, invertible Hermitian matrix  $S \in \mathfrak{H}(k) \approx \mathfrak{u}(k)^*$ , Lemma 4.2 implies that the space

$$\text{St}_S(\mathbb{C}^N) = \{V \in \mathbb{C}^{N \times k} \mid V^* V = S\}$$

is a smooth submanifold of  $\mathbb{C}^{N \times k}$ . However, such a matrix  $S$  is fixed by the coadjoint action if and only if it is a multiple of the identity matrix, so  $\text{St}_S(\mathbb{C}^N)$  is generally not preserved by the action of  $U(k)$ . We must therefore consider the more general version of Marsden–Weinstein reduction involving coadjoint orbits, and we define

$$\tilde{\text{St}}_S(\mathbb{C}^N) = \{V \in \mathbb{C}^{N \times k} \mid V^* V = U^* S U \text{ for some } U \in U(k)\} = \mu_{U(k)}^{-1}(\mathcal{O}_S).$$

We then realize  $\text{Gr}_S(\mathbb{C}^N)$  as

$$\text{Gr}_S(\mathbb{C}^N) = \mathbb{C}^{N \times k} //_{\mathcal{O}_S} U(k) = \tilde{\text{St}}_S(\mathbb{C}^N) / U(k).$$

This representation is valid, as any  $U(k)$ -equivalence class  $[V]$  has a representative which maps to exactly  $S$  under  $\mu_{U(k)}$ . Indeed, if  $V^* V = U^* S U$ , then  $\tilde{V} = V U^*$  satisfies  $\tilde{V}^* \tilde{V} = S$ , and  $[\tilde{V}] = [V]$ .

The dimension of  $\text{Gr}_S(\mathbb{C}^N)$  depends on  $S$ . For an invertible, positive-definite Hermitian matrix  $S \in \mathfrak{H}(k)$  with  $\ell$  distinct (necessarily real) eigenvalues  $\lambda_1 > \lambda_2 > \dots > \lambda_\ell > 0$  with multiplicities  $k_1, k_2, \dots, k_\ell$  adding to  $k$ , there is a diffeomorphism

$$\mathcal{O}_S \approx U(k) / U(k_1) \times U(k_2) \times \dots \times U(k_\ell)$$



(see [2, Section II.1.d]). In other words, if  $d_1 = k_1, d_2 = k_1 + k_2, d_3 = k_1 + k_2 + k_3, \dots, d_\ell = k_1 + k_2 + \dots + k_\ell = k$ , then  $\mathcal{O}_S$  is the *flag manifold*

$$\text{Fl}_k(d_1, \dots, d_\ell) = \{(P_1, \dots, P_\ell) \mid P_j \text{ is a } d_j\text{-dimensional subspace of } \mathbb{C}^k \text{ of such that } P_j \subset P_{j+1}\}.$$

It follows that

$$\dim \mathcal{O}_S = k^2 - k_1^2 - k_2^2 - \dots - k_\ell^2.$$

For a regular value  $S$  of the momentum map  $\mu_{U(k)}$ , the  $S$ -Grassmannian  $\text{Gr}_S(\mathbb{C}^N)$  is therefore a smooth manifold of dimension  $2k(N - k) + \dim \mathcal{O}_S$ .

Let  $S$  be a Hermitian matrix with eigenvalue multiplicities as above. Then  $S$  is diagonalizable by unitary matrices, and it follows easily that  $\text{Gr}_S(\mathbb{C}^N)$  only depends on these multiplicities. That is, if  $S'$  is a Hermitian matrix with the same multiplicities, then  $\text{Gr}_{S'}(\mathbb{C}^N) = \text{Gr}_S(\mathbb{C}^N)$ . This observation suggests that  $\text{Gr}_S(\mathbb{C}^N)$  is itself a flag manifold, which we now prove.

**Proposition 4.3.** *With  $S, k_i$ , and  $d_i$  as above,  $\text{Gr}_S(\mathbb{C}^N) \approx \text{Fl}_N(d_1, \dots, d_\ell, N)$ .*

*Proof.* Let  $W$  be a square root of  $S$ , as in the proof of Lemma 4.2, and let

$$V_0 = \begin{pmatrix} W \\ 0 \end{pmatrix} \in \mathbb{C}^{N \times k},$$

so that  $V_0^* V_0 = S$ . Define  $R = V_0 V_0^* \in \mathbb{H}(N)$  and let  $\tilde{\mathcal{O}}_R$  be its coadjoint orbit.

The matrix  $R$  is rank  $k$  and has the same nonzero eigenvalues as  $S$ . Since  $S$  is invertible and hence does not have zero as an eigenvalue, the eigenvalue multiplicities of  $R$  are  $k_1, \dots, k_\ell, N - k$ , meaning that  $\tilde{\mathcal{O}}_R$  is a copy of the flag manifold  $\text{Fl}_N(d_1, \dots, d_\ell, N)$ .

Now, for any  $[V] \in \text{Gr}_S(\mathbb{C}^N)$ , the Gramian  $VV^*$  has the same spectrum as  $R$ , and so lies in  $\tilde{\mathcal{O}}_R$ . Moreover, if  $V_1, V_2$  represent the same class in  $\text{Gr}_S(\mathbb{C}^N)$ , then  $V_2 = V_1 U$  for some  $U \in U(k)$ , and hence

$$V_2 V_2^* = (V_1 U)(V_1 U)^* = V_1 U U^* V_1^* = V_1 V_1^*.$$

Therefore,  $[V] \mapsto VV^*$  defines a smooth map  $G : \text{Gr}_S(\mathbb{C}^N) \rightarrow \tilde{\mathcal{O}}_R \approx \text{Fl}_N(d_1, \dots, d_\ell, N)$ , which we claim is a diffeomorphism.

To see that this map is injective, suppose  $G([V_1]) = G([V_2])$  for  $[V_1], [V_2] \in \text{Gr}_S(\mathbb{C}^N)$ ; i.e.,  $V_1 V_1^* = V_2 V_2^*$ . This implies  $V_1$  and  $V_2$  have the same left singular vectors as well as the same singular values. Also, since  $V_1^* V_1$  and  $V_2^* V_2$  are both conjugate to  $S$ , and hence to each other, by unitary matrices, the right singular vectors of  $V_1$  and  $V_2$  are related by a unitary transformation. But then simply writing out the singular value decompositions of  $V_1$  and  $V_2$  shows that  $V_2 = V_1 U$  for some  $U \in U(k)$ , and hence  $[V_1] = [V_2]$ .

On the other hand, to see that  $G$  is surjective, suppose  $P \in \tilde{\mathcal{O}}_R$ . Then

$$P = U^* R U = U^* V_0 V_0^* U = (U^* V_0)(U^* V_0)^*$$

for some  $U \in U(n)$ . But then  $[U^* V_0] \in \text{Gr}_S(\mathbb{C}^N)$  since

$$(U^* V_0)^*(U^* V_0) = V_0^* U U^* V_0 = V_0^* V_0 = S,$$

so  $P = G([U^* V_0])$ .

We've now shown that  $G$  is bijective, and the inverse map  $P \mapsto [U^* V_0]$  is clearly smooth, so this completes the proof.  $\square$

**4.3. Spaces of Frames with Prescribed Frame Operator and Norms.** Recall that  $\mathcal{F}_S$  denotes the space of frames with prescribed frame operator  $S$ , which is necessarily positive-definite, invertible and Hermitian. Also recall that for an admissible vector of norms  $\vec{r}$ ,  $\mathcal{F}_S(\vec{r})$  denotes the space of frames with prescribed frame operator  $S$  and vector norms prescribed by  $\vec{r}$ . By *admissible*, we simply mean that  $\mathcal{F}_S(\vec{r})$  is nonempty. The admissible vectors of norms are completely characterized by the following theorem.

**Theorem 4.4** (Casazza–Leon [11]). *A vector  $\vec{r} = (r_1, \dots, r_N)$  is an admissible vector of norms for  $\mathcal{F}_S$ , where  $S$  is a prescribed frame operator with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ , if and only if*

$$\sum_{j=1}^N r_j^2 = \sum_{j=1}^k \lambda_j \quad \text{and} \quad \sum_{j=1}^{\ell} r_j^2 \leq \sum_{j=1}^{\ell} \lambda_j \quad \forall 1 \leq \ell \leq k.$$

We remark that this theorem follows immediately from the classical Schur-Horn theorem [20, 29]. A modern exposition of the Schur-Horn theorem using ideas from symplectic geometry appears in [22].

The map (2) is easily adapted to give a diffeomorphism  $\mathcal{F}_S \approx \text{St}_S(\mathbb{C}^N)$  for any invertible, positive-definite Hermitian matrix  $S$ ; specifically, the adapted map is simply

$$F \mapsto V = F^*.$$

Our strategy for showing that general spaces of frames  $\mathcal{F}_S(\vec{r})$  are connected follows the strategy that we used in the case of FUNTFs. That is, we show that  $\mathcal{F}_S(\vec{r})$  is closely related to a level set of a momentum map of a torus action.

**4.4. Proof of the Main Theorem.** To prove our connectivity result in the general case, we follow the procedure that was used in the FUNTF setting. Going back through the proofs in Section 3.3, notice that the symplectic calculations were actually performed in the ambient space  $\mathbb{C}^{N \times k}$ . The ability to do so relied on the construction of  $\text{Gr}_k(\mathbb{C}^N)$  as a Marsden–Weinstein reduction and on the fact that the torus action on  $\text{Gr}_k(\mathbb{C}^N)$  is induced by an action on  $\mathbb{C}^{N \times k}$ . Since we have also realized  $\text{Gr}_S(\mathbb{C}^N)$  as a Marsden–Weinstein reduction, the definitions and proofs generalize immediately. In particular the row action of  $\mathbb{T}^{N-1}$  is still well-defined on  $\text{Gr}_S(\mathbb{C}^N)$ , as is the map  $\Phi : \text{Gr}_S(\mathbb{C}^N) \rightarrow \mathbb{R}^{N-1}$ . Moreover, we have the following generalized proposition.

**Proposition 4.5.** *The map  $\Phi$  is a momentum map for the row action of  $\mathbb{T}^{N-1}$  on  $\text{Gr}_S(\mathbb{C}^N)$ .*

An alternative way to arrive at this fact is via the identification of  $\text{Gr}_S(\mathbb{C}^N)$  with  $\text{Fl}_N(d_1, \dots, d_\ell, N)$ , since the torus action is just the standard maximal torus action on the flag manifold and  $\Phi$  is its corresponding momentum map.

Finally, let  $\tilde{\Phi} : \tilde{\text{St}}_S(\mathbb{C}^N) \rightarrow \mathbb{R}^{N-1}$  and  $\bar{\Phi} : \text{St}_S(\mathbb{C}^N) \rightarrow \mathbb{R}^{N-1}$  be the corresponding maps on  $\tilde{\text{St}}_S(\mathbb{C}^N)$  and  $\text{St}_S(\mathbb{C}^N)$ . If  $V \in \text{St}_S(\mathbb{C}^N)$  and  $\xi = (\xi_1, \dots, \xi_N)$  where  $\xi_j = \tilde{\phi}_j(V) = \frac{1}{2} \|v^j\|^2$ , then

$$\xi_1 + \dots + \xi_N = \frac{1}{2} \text{tr}(VV^*) = \frac{1}{2} \text{tr}(V^*V) = \frac{1}{2} \text{tr}(S).$$

Hence  $\xi$  is determined by its first  $N - 1$  entries. For a vector of frame norms  $\vec{r} \in \mathbb{R}^N$ , let

$$\xi(\vec{r}) = \frac{1}{2} (r_1^2, \dots, r_{N-1}^2) \in \mathbb{R}^{N-1}.$$

Then  $\mathcal{F}_S(\vec{r})$  can be identified with  $\bar{\Phi}^{-1}(\xi(\vec{r}))$  and we are ready to prove our main theorem.

**Main Theorem.** *For any frame operator  $S$  and for any admissible vector of norms  $\vec{r}$ , the space  $\mathcal{F}_S(\vec{r})$  is path-connected.*

*Proof.* Fix a frame operator  $S$ , an admissible vector of norms  $\vec{r}$  and let  $\xi = \xi(\vec{r})$ . Proposition 4.5 and Atiyah’s theorem combine to imply that  $\Phi^{-1}(\xi)$  is connected, and hence that  $\tilde{\Phi}^{-1}(\xi) \subset \tilde{\text{St}}_S(\mathbb{C}^N)$  is connected. By the same algebraic set argument used in the FUNTF case,  $\tilde{\Phi}^{-1}(\xi)$  is also path-connected. For any points  $V_0, V_1 \in \tilde{\Phi}^{-1}(\xi) \subset \tilde{\text{St}}_S(\mathbb{C}^N)$ , there is a continuous path  $\tilde{V}_t : [0, 1] \rightarrow \tilde{\Phi}^{-1}(\xi)$  joining them. There exists a continuous path  $U_t : [0, 1] \rightarrow \text{U}(k)$  such that  $\tilde{V}_t^* V_t = U_t^* S U_t$ , and we amend our original path to obtain  $V_t : [0, 1] \rightarrow \text{St}_S(\mathbb{C}^N)$  via the formula  $V_t = \tilde{V}_t U_t^*$ . Since  $U_t$  is unitary, this alteration of the path fixes the row norms for all  $t$ , so that  $V_t$  is a path in  $\bar{\Phi}^{-1}(\xi) \subset \text{St}_S(\mathbb{C}^N)$ . The theorem then follows from the identification  $\bar{\Phi}^{-1}(\xi) \approx \mathcal{F}_S(\vec{r})$ .  $\square$

## 5. DISCUSSION

The symplectic techniques we have used in this paper should have other interesting applications to frame theory. For example, (an open, dense subset of) the biquotient  $\text{U}(k) \backslash \mathcal{F}_{\frac{N}{k} \text{Id}_k}(\vec{1}) / \mathbb{T}^{N-1} \approx \text{Gr}_k(\mathbb{C}^N) //_{\xi(\vec{1})} \mathbb{T}^{N-1}$  of the space of FUNTFs in  $\mathbb{C}^k$  is toric [16], and the corresponding moment polytope should be the eigenstep polytope (cf. [4]). This means that, from a measure-theoretic perspective, FUNTF spaces are equivalent to the product of Lebesgue measure on the eigenstep polytope, the standard product measure on a torus, and Haar measure on  $\text{U}(k)$  [7]. Building on work with equilateral polygons in  $\mathbb{R}^3$  [6], there is an explicit algorithm for sampling random FUNTFs in  $\mathbb{C}^2$  [30], but a similar algorithm should exist in all dimensions.

Either experimentally or theoretically, it would be interesting to get estimates or bounds on the probability that FUNTFs have various nice properties.

Also, symplectic geometry can work well in infinite dimensions. The Marsden–Weinstein reduction operation was already applied to infinite-dimensional symplectic manifolds in Marsden and Weinstein’s initial paper on the subject (see Examples 6 and 7 in [26, Section 4]). Moreover, analogues of the Atiyah–Guillemin–Sternberg connectivity and convexity theorems have been extended to several infinite-dimensional settings [19, 31]. This suggests that the symplectic ideas used here may also be relevant to the study of frames in infinite-dimensional Hilbert spaces.

The versions of our frame spaces  $\mathcal{F}_S(\vec{r})$  for frames in  $\mathbb{R}^k$  are generally not symplectic, though they should appear as Lagrangian submanifolds of the complex frame spaces. Hence, it is not obvious how to extend our main theorem to real frame spaces, and in fact the direct translation of the statement of our theorem cannot be true. For example, it follows from work of Kapovich and Millson [21] that the space of tight frames in  $\mathbb{R}^2$  with frame vector lengths  $(2, 2, 2, 1, 1, 1)$  is not connected, so the correct statement of the real version of our theorem must necessarily be more complicated.

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