Looping Animations Using the Modular Flow and Elliptic Functions

Clayton Shonkwiler

Department of Mathematics, Colorado State University; clayton.shonkwiler@colostate.edu

Abstract

This paper describes an approach to generating looping animations using the modular flow and elliptic functions. The modular flow is a flow on lattices with many periodic orbits, and elliptic functions are meromorphic, doubly-periodic functions which can be visualized using domain coloring.

I enjoy making looping animations using interesting mathematics and am always on the lookout for periodic orbits in spaces parametrizing mathematical objects. Every time I teach Complex Analysis I am inspired to try to make animations depicting interesting holomorphic or meromorphic functions, though I have often found it a challenge to come up with looping families of functions which are visually compelling.

The goal here is to describe how to use the modular flow on the space of lattices to get periodic families of elliptic functions. Elliptic functions are meromorphic, doubly-periodic functions, so they are naturally associated with a planar lattice Λ generated by the fundamental periods. In turn, the modular flow is a flow on the space of lattices which has many periodic orbits; these periodic orbits will give us our looping families of elliptic functions, which can then be visualized using domain coloring.

Note All figures in this paper are animations, which should be viewable in Acrobat Reader, Okular, and a few other PDF viewers. Other viewers will only show a single static frame. All animations can also be seen online at https://shonkwiler.org/bridges26. Source code for many of the animations in this paper can be found in the Geometry in Motion repository [10].

The Modular Flow

I was first introduced to this topic by an animation of Etienne Jacob [6]. I will try to give a self-contained summary, but see [2, 5] for nice introductions.

Consider the action of the non-zero real numbers $\mathbb{R}^*$ on the plane $\mathbb{R}^2$ by $s \cdot (x, y) = (sx, s^{-1}y)$. Typical orbits are hyperbolae of the form $xy = c$ for $c \neq 0$, though there are three special orbits, namely the x-axis minus the origin, the y-axis minus the origin, and the origin itself. Extended to an action of $\mathbb{C}^*$ on $\mathbb{CP}^2$, this provides an elementary example motivating the construction of the Geometric Invariant Theory quotient [8].

Since $\mathbb{R}^*$ is disconnected, it is convenient to modify this action by restricting to positive s and think of $s = e^t$: then $t \cdot (x, y) = (e^t x, e^{-t} y)$ defines an action of $\mathbb{R}$ on $\mathbb{R}^2$. Given a lattice $\Lambda \subset \mathbb{R}^2$, we get a new lattice $t \cdot \Lambda$ for each $t \in \mathbb{R}$; this is the *modular flow* on the space of lattices. Since the matrix $\operatorname{diag}(e^t, e^{-t})$ has determinant 1, its action is area-preserving, so the area of the fundamental domain of $t \cdot \Lambda$ is independent of t. In particular, we can interpret the generators of any unit-area Λ with integer coordinates as the columns of some $A \in \operatorname{SL}_2(\mathbb{R})$, the special linear group consisting of 2×2 matrices with determinant 1, and then the generators of $t \cdot \Lambda$ are columns of $\operatorname{diag}(e^t, e^{-t})A$.

This action on $\operatorname{SL}_2(\mathbb{R})$ is very far from periodic, but it turns out that the action on lattices has some periodic orbits: for certain Λ there exists $t_0 > 0$ so that $\Lambda = t_0 \cdot \Lambda$. As written, it's not so clear how to solve for Λ and t_0, but we can extract a matrix equation as follows: if $\Lambda = t_0 \cdot \Lambda$, then for any vertex $\vec{v}$ of Λ we have

that $t_0 \cdot \vec{v}$ is also a vertex of Λ. In turn, since $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the vertices of Λ, we can try to solve the matrix equation

$$AB = \mathrm{diag}(e^{t_0}, e^{-t_0})A \qquad (1)$$

for A and t_0, where $B \in \mathrm{SL}_2(\mathbb{Z})$ is chosen in advance.

For example, with $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ it's straightforward to check that, when $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, $t_0 = \log(1 + \varphi)$ and $A = \begin{bmatrix} \varphi & 1 \\ 1 & -\varphi \end{bmatrix}$ is a solution to (1); see Figure 1a.

(a)   **(b)**

Figure 1: *Two periodic orbits in the space of lattices: (a) the lattice generated by $(\varphi, 1)$ and $(1, -\varphi)$, with period $\log(1 + \varphi) \approx 0.962$; (b) the lattice generated by $\left(2\sqrt{2/3}, -2\sqrt{2/3}\right)$ and $(1, 1)$, with period $\log\left(5 + 2\sqrt{6}\right) \approx 2.292$. Both animations take 2 seconds to show 1 period.*

More generally, taking the transpose of (1) reveals that the rows of A must be eigenvectors of B^T with corresponding eigenvalues e^{t_0} and e^{-t_0}. In other words, if λ_1 and $\lambda_2 = \frac{1}{\lambda_1}$ are the eigenvalues of B, then $t_0 = \pm \log(\lambda_1)$ and the rows of A are the corresponding eigenvectors. This explains golden ratio's appearance above: the characteristic polynomial of $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is $x^3 - 3x + 1$, which has roots $\lambda_1 = \varphi + 1$ and $\lambda_2 = 2 - \varphi$.

The periodic flows starting from different lattices are qualitatively similar, but can already be reasonably visually compelling, as in the animation *Modular Flow* [9]; see Figure 2.

Figure 2: *The animation* Modular Flow *[9].*

Elliptic Functions

While visualizing the modular flow on lattices is already interesting, we can take things a step further by introducing elliptic functions. To motivate this, identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Given a lattice Λ, the quotient $\mathbb{C}/\Lambda$ of the plane by the lattice gives a torus or, in algebraic language, an elliptic curve. That is, each lattice determines a torus.

In turn, we can consider the meromorphic functions on the torus, which simply correspond to doubly-periodic meromorphic functions on $\mathbb{C}$. More precisely, if Λ has fundamental domain with vertices $0, \omega_1, \omega_2$, and $\omega_1 + \omega_2$, then any meromorphic function on $\mathbb{C}/\Lambda$ can be interpreted as a meromorphic function f on $\mathbb{C}$ so that

$$f(z + \omega_1) = f(z) = f(z + \omega_2)$$

for all $z \in \mathbb{C}$. Doubly-periodic meromorphic functions are usually called *elliptic functions*; see [1, Ch. 1] for a nice introduction and [7] for a more comprehensive treatment.

In some sense the most basic elliptic function is the Weierstrass $\wp$-function; to emphasize the dependence on the lattice we will write this as $\wp_\Lambda$. The defining feature of $\wp_\Lambda$ is that its only poles are double poles at each lattice point and that the leading term of its Laurent series at 0 is $\frac{1}{z^2}$; here's the definition:

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).$$

In fact, every elliptic function f with period lattice Λ can be built out of $\wp_\Lambda$ and its derivative $\wp'_\Lambda$. More precisely, every such f can be written as

$$f(z) = R_1(\wp_\Lambda(z)) + \wp'_\Lambda(z) R_2(\wp_\Lambda(z)), \tag{2}$$

where R_1 and R_2 are rational functions.

Now, choose one of the lattices Λ which has a periodic orbit under the modular flow and define an action of $\mathbb{R}$ on the Weierstrass function by $t \cdot \wp_\Lambda := \wp_{t \cdot \Lambda}$. Using (2), we get an induced action on any elliptic function we like which will always have periodic orbits.

With all this machinery in place, we can use domain coloring [3, 4] to visualize the periodic family of elliptic functions. Three examples are shown in Figure 3, though a lot of experimentation is possible here.

(a)   (b)   (c)

Figure 3: *Three examples of periodic families of elliptic functions: (a) the animation* Phase *[11], using the functions $\wp_{t \cdot \Lambda}$; (b) the functions $\wp'_{t \cdot \Lambda}$; (c) a family of Jacobi elliptic functions of type* cn.

In fact, this approach is not strictly limited to elliptic functions: we can use the same idea to get periodic families of quasi-periodic functions like the Weierstrass σ_Λ function, which is implicitly defined by the equation $\frac{d^2}{dz^2} \log(\sigma_\Lambda) = -\wp_\Lambda$. See Figure 4.

Figure 4: *Domain coloring animation of the Weierstrass functions $\sigma_{t\cdot\Lambda}$.*

Acknowledgements

References

[1] T. M. Apostol. *Modular Functions and Dirichlet Series in Number Theory*. ser. Graduate Texts in Mathematics. New York, NY: Springer New York, 1990. vol. 41.

[2] B. Bartlett. The Modular Flow on the Space of Lattices. 2014. https://golem.ph.utexas.edu/category/2014/04/the_modular_flow_on_the_space.html.

[3] F. A. Farris. "Reviews: Visual Complex Analysis." *The American Mathematical Monthly*, vol. 105, no. 6, 1998, pp. 570–576.

[4] F. A. Farris. "Domain Coloring and the Argument Principle." *PRIMUS*, vol. 27, no. 8-9, Oct. 2017, pp. 827–844.

[5] E. Ghys and J. Ley. Lorenz and Modular Flows: A Visual Introduction. 2006. https://www.ams.org/publicoutreach/feature-column/fcarc-lorenz.

[6] E. Jacob. 2022. https://necessary-disorder.tumblr.com/post/704166394298875905/.

[7] D. F. Lawden. *Elliptic Functions and Applications*. ser. Applied Mathematical Sciences. F. John, J. E. Marsden, and L. Sirovich, Eds. New York, NY: Springer, 1989. vol. 80.

[8] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory*. ser. Ergebnisse Der Mathematik Und Ihrer Grenzgebiete. Berlin: Springer-Verlag, 1994. vol. 34.

[9] C. Shonkwiler. Modular Flow. 2022. https://shonkwiler.org/gifs2/modular-flow.

[10] C. Shonkwiler. Geometry in Motion. 2023–2026. https://github.com/shonkwiler/geometry-in-motion.

[11] C. Shonkwiler. Phase. 2025. https://shonkwiler.org/gifs2/phase.