Review of Needham's *Visual Differential Geometry and Forms: A mathematical drama in five acts*

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A reductionist and incomplete summary of *Visual Differential Geometry and Forms*. Tristan Needham's goal is to give a natural and intuitive proof of Gauss's *Theorema Egregium*. The usual proof of this result is infamously inscrutable and depends on apparently miraculous cancellations, making this a noble and ambitious goal, in which Needham largely succeeds. As the title suggests, the book is structured in five Acts, each comprised of a number of (mostly pretty short) Chapters: the first four Acts are devoted to the *Differential Geometry* part of the title, and the fifth to *Forms*.

To give a bit more background, the *Theorema Egregium* says that the Gauss curvature of a surface is an intrinsic quantity; that is, it depends only on the intrinsic geometry of the surface, and not on the particulars of how the surface is placed into 3-dimensional space (or, indeed, if it is placed in space at all). The distinction between intrinsic and extrinsic geometry is subtle and can appear somewhat artificial upon first encounter.

In short, intrinsic geometry is that which is detectable by hypothetical 2-dimensional beings living within the surface and able to measure distances, angles, etc. within it, but with no knowledge of anything which might or might not exist outside the surface itself. Though we are not quite 2-dimensional, this is very close to most of our lived experience with the geometry of the Earth. Think about sailing straight across a stretch of ocean or walking straight across a large salt flat: if you follow a path in which you turn neither left nor right, you're traversing an arc of a great circle, not a straight line in space. This is a consequence of the curvature of the Earth, which is easy to see extrinsically (most obviously from satellite photographs, but also from observation of the sun and stars), but actually is detectable from a purely 2-dimensional perspective. For example, if you staked out all the points at some distance r from a central point and then measured the circumference of the resulting circle, you would get[1] $2\pi R \sin\left(\frac{r}{R}\right)$, where $R \approx 6371$ km is the radius of the Earth. This is *less than* the expected $2\pi r$ you would get in a plane, though the Taylor expansion of sine shows they agree to second order:

$$2\pi R \sin\left(\frac{r}{R}\right) = 2\pi r\left(1 - \frac{r^2}{6R^2} + O(r^4)\right).$$

Alternatively, if you form a triangle on the surface of the Earth with straight (i.e., arcs of great circles) sides, the sum of the interior angles of the triangle will turn out to be $\pi + \frac{A}{R^2}$, where A

[1] Assuming a perfectly spherical Earth.

is the area of the triangle. In both cases, the correction $\frac{1}{R^2}$ that comes up is precisely the Gauss curvature of the sphere. Indeed, after taking the limit as small triangles shrink down to a point, this quantity is essentially Needham's preferred *definition* of Gauss curvature.

When defined in this way, Gauss curvature is manifestly intrinsic. It doesn't matter whether we are actually forming circles and triangles on the western hemisphere of Earth or on the western hemisphere of a gigantic table tennis ball that someone has cut in half and then squeezed along the prime meridian, causing the ball to flex in space: flexing the ball does not change distances *within the ball* (though it certainly does change extrinsic distances *in 3-space*). In one sense, then, the *Theorema Egregium* has become a triviality.

Of course, this really just shifts the burden: Gauss curvature is usually defined *extrinsically*, so the challenge becomes to show that the intrinsic definition is equivalent to the usual one, which is the average curvature of curves formed by slicing the surface in space with planes containing the surface normal.

The standard proof that the intrinsic and extrinsic curvatures are the same is essentially that given by Gauss, which Needham argues was not how Gauss arrived at the result, but rather the argument which he thought would be most incontrovertible, since it is just a long calculation. But "convincing to the leading mathematicians of 1827" and "good for modern learners" are rather different categories,[2] and Needham has done quite a remarkable job of proving Gauss's theorem in a way that provides intuition and geometric insight rather than mere logical rigor. He also manages to more broadly contextualize the result and connect it to other fundamental results and ideas, including the Gauss–Bonnet theorem, index theory, Morse theory, and parallel transport.

At this point let me stop and be explicit about what I think is particularly valuable about this book: the main story is about a very classical subject (surfaces in space) and pitched to an undergraduate audience, but **presented in a way that broadly reflects how modern differential geometers think about these ideas**. Of course, one can make elementary material completely inscrutable using sufficiently advanced machinery,[3] so equally important is Needham's remarkable ability to refine these ideas to their simple, intuitive cores and to visualize anything and everything, be it with his painstakingly clear diagrams or his demonstrations with bananas, summer squash, durians, and many other fruits and vegetables.

An aside on the fruit and vegetable constructions. Even apart from their clear utility in giving the reader hands-on experience with the material, well beyond the inflatable sphere and plastic donut demonstrations that many of us use when teaching differential geometry classes, they are often quite brilliant in their own right. I would point in particular to §22.3, entitled "Potato-Peeler Transport," which demonstrates how to parallel transport a vector along a curve on a surface. See Figure 1 for a demonstration; in words, draw the curve on your fruit or vegetable (Needham uses a pomelo, I'm using a grapefruit) and tape a toothpick to the starting point to indicate the initial vector, then use a potato peeler to remove a narrow strip of rind along the curve (including the toothpick), lay it flat on the table (where it won't be straight unless the initial curve was a geodesic), tape toothpicks along the curve that are parallel (in the plane) to the original one, and then lay the strip back on the surface: the toothpicks exactly indicate the parallel transport

[2] As I write these words, I am looking at a copy of do Carmo's *Differential Geometry of Curves and Surfaces* [4] with a broken spine—the result of my throwing it across the room almost two decades ago in sheer frustration at the section in which the *Theorema Egregium* is proved.

[3] I will gesture here in the direction of Linderholm's *Mathematics Made Difficult* [12], though with the warning that some of the fictional vignettes are regrettable.

of the vector along the curve! While Needham found an analogous explanation in Levi-Civita's treatise [11, p. 102], the demonstration with fruit, potato peeler, and toothpicks is rather more effective than Levi-Civita's un-illustrated "We can now take for our definition of surface parallelism on σ along T the parallelism which we have associated with the developable σ_T."

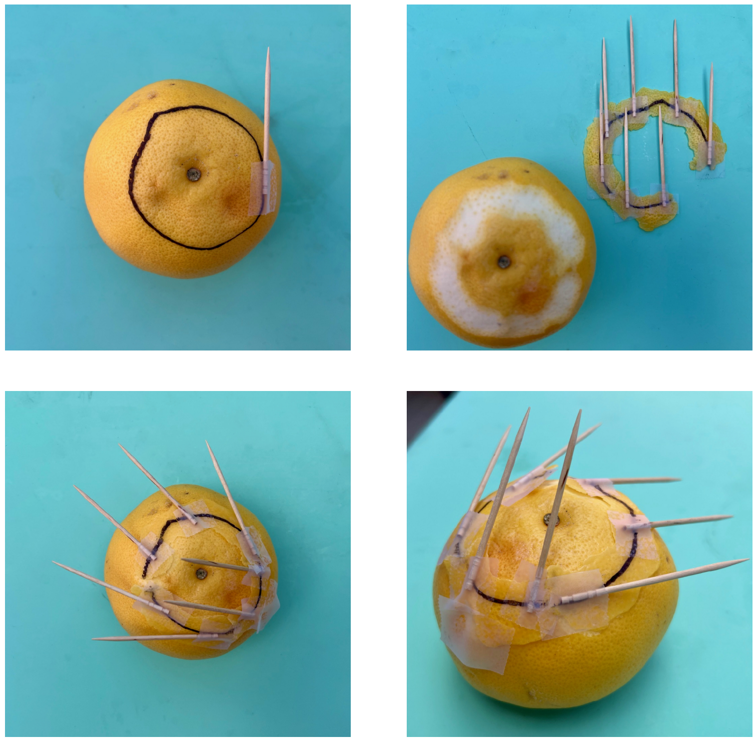

Figure 1: Potato-peeler parallel transport.

So what is the big deal about intrinsic versus extrinsic geometry? Broadly speaking, there are many spaces of interest, both mathematically and as configuration spaces for physical systems, in which we are interested in geometric features like curvature but do not have access to extrinsic measurements. Two examples which Needham explores in some detail are hyperbolic geometry and general relativity.

The hyperbolic plane was the first generally accepted model of a non-Euclidean geometry, meaning it satisfies the first four of Euclid's axioms but not the parallel postulate. Since Euclidean geometry is complete and self-contained, any proposed alternative also needs to be, and so cannot depend on its relationship to some larger space that contains it.[4] Chapter 5 ("The Pseudosphere and the Hyperbolic Plane") is wonderful, building up from hand-built models of the pseudosphere to Beltrami's construction of the upper half plane model (including, rather delightfully, using Snell's

[4]Indeed, spherical geometry is now recognized as a classic non-Euclidean geometry, but seems not to have been seriously considered as such prior to the 19th century because the sphere's geometry was apparently induced by the Euclidean 3-space in which it sat. It was only the *Theorema Egregium* and related results which showed that spherical geometry can be self-contained.

Law to determine its geodesics) to the disk model and Escher's *Circle Limit I*.

General relativity explains gravity as curvature of spacetime. Because we live inside the universe, we can't understand or quantify any extrinsic geometry it might have. Therefore, in order to have any predictive value this curvature must be something we can measure intrinsically. As a student of Penrose and a disciple of Misner, Thorne, and Wheeler's *Gravitation* [13], relativity understandably looms large for Needham, and indeed the statement and explanation of Einstein's field equation serves as the climax of Act IV of the book. To get there requires forays into general n-dimensional manifolds, the Riemann curvature tensor, and much else that would typically appear nowhere in an undergraduate differential geometry book, but certainly builds upon the foundations established in studying the geometry of surfaces. While this inevitably involves some compromises, every step along the way is grounded in physical intuition[5] and visual examples, so the broad narrative arc is compelling and convincing (though certainly quite challenging!).

There are many possible directions one could go in pursuing more advanced topics in differential geometry, but I applaud Needham's choice to focus on what he's passionate about rather than trying for an impossible completeness. And his insistence on actually trying to explain relativity, rather than merely hand-waving in its direction as a motivation for studying curvature, parallel transport, and all the rest, seems inextricably tied to his boundless enthusiasm and expositional confidence, both of which are evident throughout. His enthusiasm, in particular, is quite infectious, and I would hope it would be effective in drawing in new learners.

That being said, I do think Needham's passion and confidence led him a bit astray in Act V, which develops the theory of differential forms, including Maxwell's equations, Cartan's moving frames, and curvature forms, concluding with the curvature of the Schwarzschild black hole. While ending with black holes connects this Act with what precedes it, overall it feels somewhat disconnected, especially since it is much less visual than the first four Acts. The move away from visual explanations reflects the fact that the expository demands of this material are somewhat different than in the rest of the book. One example where the seams show: Needham's intuitive and basically implicit definition of tangent vectors works for the discussion of the *Theorema Egregium*, general relativity, etc., but struggles to support the cognitive load of differential forms. Differential forms are quite reasonably defined as (special) functions with vector inputs, but with no explicitly-stated intrinsic definition of the tangent space I think new learners would be hard-pressed to articulate what, exactly, the domains of these functions actually are.

Is this book suitable for use in the classroom? Not having taught from it, I can't say for sure, and I am mindful of Needham's words from the Prologue:

> I have made no attempt to write this book as a classroom textbook. While I hope that some brave souls may nevertheless choose to use it for that purpose—as some previously did with [*Visual Complex Analysis*]—my primary goal has been to communicate a majestic and powerful subject to the reader as honestly and as lucidly as I am able, regardless of whether that reader is a tender neophyte, or a hardened expert.

Nonetheless, I think this book could form a good foundation for either a course on non-Euclidean geometry or an undergraduate differential geometry course. For non-Euclidean geometry, I would focus mostly on Acts I & II, maybe also with some selections (or at least inspiration) from Chapters 14 (polyhedral Gauss–Bonnet) and 22 (parallel transport). For undergraduate differential geometry,

[5] I never expected to see an explanation of neap tides in a differential geometry book, but now I have.

I think it would be just about possible to cover Chapters 1–17, 21–25, and 27 in a (busy) semester. In both cases, there is much else for students to explore (including alternative proofs of the Gauss–Bonnet theorem, relativity, and differential forms) either in final projects or on their own. The fact that there is so much more to explore in the book is, to my taste, a virtue. As Needham so eloquently puts it, "Great mathematical ideas not only put past mysteries to rest, they also reveal *new* ones: tunnels at the end of the light!" [p. 65].

In case it wasn't clear from the title, this book is very definitely a sibling to Needham's previous book, *Visual Complex Analysis* [14] (VCA). As in VCA, many arguments are based on a "Newtonian form of geometric reasoning" [p. 33] making liberal use of *ultimate equality*: two quantities A and B depending on a parameter ϵ are said to be ultimately equal, denoted $A \asymp B$, if $\lim_{\epsilon \searrow 0} \frac{A}{B} = 1$. For those familiar with VCA, this is presumably either appealing or off-putting. I found this style to be quite effective, developing much better intuition than other approaches to this material, but the arguments do depend on the reader having a very good handle on Euclidean and spherical geometry, which may pose a teaching challenge for those looking to use this book in the classroom.

I recently re-read Frank Farris' review of VCA [5], and many of his comments apply equally well here.[6] For example, he lauds VCA as a book in which "we have not only the clarity of the facts but the helpful voice of the author. [...] This is a book in which the author has been willing to make himself available as our teacher. His own voice enters in a rather charming way." Farris concludes with the hope that VCA will "inspire other books in which the voice of the author is vividly present to teach and explain."

The present book obviously falls into this tradition, but I think we have actually seen a flourishing of mathematical writing with a strong authorial voice in the 25 years since VCA was published. The near-infinite capacity of web servers and Amazon warehouses, coupled with the ease of self-publishing and publishers' perpetual pursuit of the long tail of demand, probably has a lot to do with that, but VCA deserves credit as well. Standout examples at the undergraduate level include: Nathan Carter's *Visual Group Theory* [3], which is primarily focused on visualization and is much more interested in helping the reader to get to know groups very concretely than in developing an abstract, formal theory; Benedict Gross, Joe Harris, and Emily Riehl's *Fat Chance: Probability from 0 to 1* [9], framed more as an introduction to a foreign language than a traditional math book and filled with asides that expose the artifices inherent in mathematical writing; and Robert Ghrist's calculus books and videos [7,8], which are absolutely packed with distinctive, detailed illustrations (including lots of visual gags) and exude a practitioner's enthusiasm for material sometimes considered standard and boring. I would also briefly point to graduate texts by Bradley, Bryson, and Terilla [2] (a categorical approach to point-set topology!), Gallier and Quaintance [6] (a computational perspective on differential and Riemannian geometry!), and Vakil [15] (and its companion picture book(!) on spectral sequences [16]). The Ph.D. dissertations of Tai-Danae Bradley [1] and Piper H [10] also deserve to be mentioned here for telling compelling and engaging (but also mathematically rigorous!) stories.

As previously mentioned, what seems particularly special to me about Needham's book is not just the author's unique voice or the visual presentation of the material, but the fact that it conveys a faithful sense of how (many) research mathematicians actually think about differential geometry. So, while I echo Farris' call for more books with a "vividly present" (rather than omniscient and disembodied) authorial voice, I also hope that this book will inspire others at all levels that embrace modern practitioners' modes of thinking about their subject.

[6] This review is also notable because it seems to be where domain coloring of complex functions was first introduced.

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