THE PAULSEN PROBLEM MADE SYMPLECTIC

MARTIN EHLER, MILENA HERING, CHRISTOPHER MANON, TOM NEEDHAM, AND CLAYTON SHONKWILER

1. Introduction

Symplectic geometry [CdS01] provides powerful tools for studying special spaces of complex matrices. We propose that it will be fruitful to use this perspective on Parseval frames, complex fixed unit normal tight frames (FUNTFs), and other spaces of frames. The notion of Hamiltonian reduction of a sympelectic manifold by a compact Lie group is of central importance. This technology has already been used to prove that connectivity properties hold for spaces of FUNTFs (see [NS]), and we argue below that it should have applications to the Paulsen problem.

2. Elements of Hamiltonian Actions on Vector Spaces

Let $V = \mathbb{C}^n$ be a complex vector space equipped with a linear left action by a compact subgroup $K \subset U(n) \subset \mathrm{GL}_n(\mathbb{C})$. Here $U(n)$ is the group of unitary $n \times n$ matrices. The space V comes equipped with a symplectic form $\omega_V = \frac{i}{2}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$. Note that this is a closed differential 2-form on V when it is considered as a real $2n$-dimensional vector space. The action of K on V is said to be symplectic because $k^*(\omega_V) = \omega_V$ for any $k \in K$, where $k^*(\omega_V)$ is the pullback form under the isometry of V defined by k.

For any $v \in V$ we can consider the $n \times n$ matrix $-\frac{i}{2}vv^*$, which can be taken to be an element in the space $\mathcal{H}_n$ of $n \times n$ skew-Hermitian matrices. The space $\mathcal{H}_n$ can be identified with the Lie algebra $\mathfrak{u}_n$ of $U(n)$; and by using the non-degenerate inner product $\langle A, B\rangle = -\frac{i}{2}tr(AB)$, we can view $\mathcal{H}_n$ as the dual space of $\mathfrak{u}_n$. The latter carries a linear group action by $U(n)$ called the coadjoint action, which can be computed on a Hermitian matrix by $Ad_u(A) = uAu^{-1}$ for $u \in U(n)$. There is an inclusion $t_K : \mathfrak{k} \to \mathfrak{u}_n$ and a corresponding surjection on dual spaces $t_K^* : \mathfrak{u}_n^* \to \mathfrak{k}^*$ given by considering the Lie algebras as the tangent spaces to K and $U(n)$ at the identity element. The *moment map* of the action of K on V is a continuous, K-invariant map $\mu_K : V \to \mathfrak{k}^*$ defined by $\mu_K(v) = t_K^*(-\frac{i}{2}vv^*)$.

Let $X \in \mathfrak{k}$ be an element of the Lie algebra, then we can use X to define a vector field on V. At a point $v \in V$ we let $X_v = [\frac{d}{dt}e^{itX}v]_{t=0} \in T_v(V)$, where $T_v(V)$ is the tangent space to V at the point v. It can be verified that the following always holds:

$$d\langle \mu_K(v), X\rangle \mid_v = \omega_V \mid_v (-, X_v). \tag{1}$$

In particular, the differential 1-form associated to the function $\langle \mu_K(-), X\rangle : V \to \mathbb{R}$, where $\langle -, -\rangle : \mathfrak{k} \times \mathfrak{k}^* \to \mathbb{R}$ is the dual pairing, agrees with the 1-form $\omega_V(X_v, -)$ on the tangent space $T_v(V)$. An action by symplectomorphisms with this property is said to be Hamiltonian.

Theorem 2.1. *For any compact subgroup $K \subset U(n) \subset \mathrm{GL}_n(\mathbb{C})$, the induced action of K on $V = \mathbb{C}^n$ is Hamiltonian.*

Hamiltonian group actions are the "correct" group actions to consider in the category of symplectic manifolds; in particular these are the actions for which a good notion of "quotient space" exists. The technical term for such a quotient is "Hamiltonian reduction," and generally speaking when this operation is carried out the result also has the structure of a symplectic space by a result of Marsden and Weinstein [MW74]. Along the way to constructing Hamiltonian reduction of V by K, one considers the so-called level sets of the momentum map μ_K. Let $c \in \mathfrak{k}^*$ be a central element, this means that c is fixed by the coadjoint action of K on $\mathfrak{k}^*$: $Ad_k^*(c) = kck^{-1} = c \; \forall k \in K$. For $c \in \mathfrak{k}^*$ a central element we consider the subspace $\mu_K^{-1}(c) \subset V$. Level sets of the moment map have a number of nice properties, for example they are always connected and K-stable subspaces (see [CdS01], [Sja95]). This property was used by Needham and Shonkwiler [NS] to reprove and generalize a theorem of Cahill, Mixon, and Strawn [CMS17], which states the space of complex fixed unit norm tight frames (FUNTFs) is connected.

The Hamiltonian reduction of V at level c is defined to be the quotient space $\mu^{-1}(c)/K$, and it can be shown to carry a natural symplectic structure. Part of the utility of this theory is that if c is integral, the reduction coincides with a corresponding "Geometric Invariant Theory" construction in algebraic geometry (see [MFK94]), where the quotient by the corresponding complex group $K^{\mathbb{C}}$ is considered. This is a well-known result due originally to Kempf and Ness [KN79], with important technical extensions to general Kähler manifolds and projective algebraic varieties due to Kirwan [Kir84] and Sjamaar [Sja95]. Let $f : V \to \mathbb{R}$ be $f(v) = -||\mu_K(v) - c||^2$; the flow ϕ_t of the gradient ∇f provides an important ingredient in the proof that these two notions of quotient coincide. We state the relevant result below, let ϕ_∞ be the limit map of the flow ϕ_t (see [Sja95, Proposition 2.4]).

Theorem 2.2. *There is a dense, open set $V^{ss} \subset V$ such that for any $v \in V$,*

$$\phi_\infty(v) \in \mu_K^{-1}(c). \tag{2}$$

3. Parseval Frames and the Paulsen problem

Now we point to several examples from the world of frames the defining conditions of which can be rephrased in the language of moment maps. Let $M_{m\times n}(\mathbb{C})$ $m < n$ be the space of $m \times n$ complex matrices equipped with its left and right actions by $U(m)$ and $U(n)$, respectively. The set of diagonal matrices with modulus 1 entries in $U(n)$ is isomorphic to an n-torus $\mathbb{T}^n = U(1) \times \cdots \times U(1)$. With these identifications, $M_{m\times n}(\mathbb{C})$ has a Hamiltonian action by $U(m) \times \mathbb{T}^n$.

The moment map $\mu_{\mathbb{T}^n}$ is computed by sending $\Phi \in M_{m\times n}(\mathbb{C})$ to $-\frac{1}{2}(\cdots, ||\Phi_i||^2, \cdots)$, where Φ_i is the i-th column of Φ. The dual of the Lie algebra of $\mathbb{T}^n$ can be identified with $\mathbb{R}^n$. Since $\mathbb{T}^n$ is Abelian, every element of $\mathbb{R}^n$ is central. Choosing $c = (c_1, \ldots, c_n)$, we can consider $\mu^{-1}(c) \subset M_{m\times n}(\mathbb{C})$; this is the space of matrices whose i-th column has length $\sqrt{-2c_i}$. In particular, we must have $c_i \in R_{\leq 0}$ for $\mu_{\mathbb{T}^n}(c)$ to be non-empty. The moment map $\mu_{U(m)}$ can be shown to be the *frame operator*: $\mu_{U(m)}(\Phi) = \Phi\Phi^*$. The moment map of a product group is simply the product of the moment maps, so we have:

$$\mu_{U(m)\times\mathbb{T}^n}(\Phi) = \mu_{U(m)}(\Phi) \times \mu_{\mathbb{T}^n}(\Phi) = (\Phi\Phi^*, -\frac{1}{2}||\Phi_1||^2, \cdots, -\frac{1}{2}||\Phi_n||^2). \tag{3}$$

The central elements of the $U(m)$ action on $\mathcal{H}_m$ are precisely the multiples of the identity matrix aI. It follows that we can consider the set of matrices with frame operator a fixed multiple of the identity aI and prescribed column lengths $\sqrt{2c_1}, \ldots, \sqrt{2c_n}$ as the moment level set $\mu^{-1}_{U(m)\times\mathbb{T}^n}(aI, -c_1, \ldots, -c_n)$. Theorem 2.2 then implies that the gradient flow of the *frame potential* $p_{a,\vec{c}}(\Phi) = ||(\Phi\Phi^*, -\frac{1}{2}||\Phi_1||^2, \cdots, -\frac{1}{2}||\Phi_n||^2) - (aI, c_1, \ldots, c_n)||^2$ takes any matrix $\Phi \in M_{m\times n}(\mathbb{C})^{ss}$ into the space $\mu^{-1}_{U(m)\times\mathbb{T}^n}(aI, -c_1, \ldots, -c_n)$.

Let $a = 1$ and $c_i = \frac{m}{2n}$, then matrices $\Phi \in \mu^{-1}_{U(m)\times\mathbb{T}^n}(I, -\frac{m}{2n}, \ldots, -\frac{m}{2n})$ are known as Parseval frames. A matrix $\Psi \in M_{m\times n}(\mathbb{C})$ is said to be an ϵ-nearly equal norm Parseval frame if $(1-\epsilon)I \preceq \Psi\Psi^* \preceq (1+\epsilon)I$, $(1-\epsilon)\frac{m}{n} \leq ||\Psi e_j||^2 \leq (1+\epsilon)\frac{m}{n}$, where e_j is the j-th canonical basis member of $\mathbb{C}^n$. Recently Hamilton and Moitra [HM] have shown that for any ϵ-nearly equal norm Parseval frame there is an actual Parseval frame Φ such that $||\Phi - \Psi||^2 \leq 40\epsilon d^2$, where $||-||$ denotes the Frobenius norm. Finding bounds for $||\Phi - \Psi||^2$ is known as the Paulsen problem.

We conjecture an inequality of the form $||\Phi - \Psi||^2 \leq C\epsilon d^{2-\alpha}$ with $0 \leq \alpha < 1$ using the gradient flow of the frame potential. Following observations of Lerman [Ler05], we intend to bound the distance between a matrix Ψ and the end point of its flow $\phi_\infty(\Phi)$ using a Łojasiewicz estimate. Similar “constrained” gradient flow methods have been used in [BC10] and [CFM12], which translate roughly to utilizing the gradient flow for the moment maps $\mu_{\mathbb{T}^n}$ or $\mu_{U(m)}$ alone, rather than in concert.

An alternate approach to the Paulsen problem can be seen in [Ehl15] and [HM], where one normalizes the columns of $\Gamma\Psi$, where $\Gamma \in \mathrm{GL}_m(\mathbb{C})$ is a matrix depending on Ψ. We observe that the resulting matrix is in the same $\mathrm{GL}_m(\mathbb{C}) \times (\mathbb{C}^*)^n$ orbit as Ψ. Another consequence of the fact that symplectic reduction and GIT quotient coincide implies that the matrix constructed in [Ehl15] must be in the same $U(m) \times \mathbb{T}^n$ orbit as the matrix we construct [Sja95, Proposition 1.15]. The same holds for the approaches in [HM], [BC10] and [CFM12]. It follows that the procedure consisting of first using the gradient flow, and then optimizing over the $U(m) \times \mathbb{T}^n$ orbit must produce the state-of-the art answer to the Paulsen problem.

We expect there are other applications of symplectic geometry to other spaces of frames. In particular, spaces of finite complex fusion frames should be be amenable to symplectic techniques.

References

[BC10] Bernhard G. Bodmann and Peter G. Casazza. The road to equal-norm Parseval frames. *J. Funct. Anal.*, 258(2):397–420, 2010.

[CdS01] Ana Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[CFM12] Peter G. Casazza, Matthew Fickus, and Dustin G. Mixon. Auto-tuning unit norm frames. *Appl. Comput. Harmon. Anal.*, 32(1):1–15, 2012.

[CMS17] Jameson Cahill, Dustin G. Mixon, and Nate Strawn. Connectivity and irreducibility of algebraic varieties of finite unit norm tight frames. *SIAM J. Appl. Algebra Geom.*, 1(1):38–72, 2017.

[Ehl15] Martin Ehler. Preconditioning filter bank decomposition using structured normalized tight frames. *J. Appl. Math.*, pages Art. ID 861563, 12, 2015.

[HM] L. Hamilton and A. Moitra. The paulsen problem made simple. arXiv:1809.04726 [math.FA].

[Kir84] Frances Clare Kirwan. *Cohomology of quotients in symplectic and algebraic geometry*, volume 31 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1984.

[KN79] George Kempf and Linda Ness. The length of vectors in representation spaces. In *Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978)*, volume 732 of *Lecture Notes in Math.*, pages 233–243. Springer, Berlin, 1979.

[Ler05] Eugene Lerman. Gradient flow of the norm squared of a moment map. *Enseign. Math. (2)*, 51(1-2):117–127, 2005.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[MW74] Jerrold Marsden and Alan Weinstein. Reduction of symplectic manifolds with symmetry. *Rep. Mathematical Phys.*, 5(1):121–130, 1974.

[NS] T. Needham and C. Shonkwiler. Symplectic geometry and connectivity of spaces of frames. arXiv:1804.05899 [math.FA].

[Sja95] Reyer Sjamaar. Holomorphic slices, symplectic reduction and multiplicities of representations. *Ann. of Math. (2)*, 141(1):87–129, 1995.

University of Vienna
E-mail address: martin.ehler@univie.ac.at

University of Edinburgh
E-mail address: m.hering@ed.ac.uk

University of Kentucky
E-mail address: Christopher.Manon@uky.edu

The Ohio State University
E-mail address: needham.71@osu.edu

Colorado State University
E-mail address: clayton.shonkwiler@colostate.edu