Poincaré duality angles and the Dirichlet-to-Neumann operator

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(Dated: February 27, 2013)

On a compact Riemannian manifold with boundary, the absolute and relative cohomology groups appear as certain subspaces of harmonic forms. This paper solves the inverse problem of recovering the relative positions of these subspaces from Cauchy data for differential forms. This solution is also exploited to partially resolve a question of Belishev and Sharafutdinov about whether the Cauchy data for differential forms determines the cup product structure on a manifold with boundary.

1. INTRODUCTION

Consider a closed, smooth, oriented Riemannian manifold M^n. For any p with $0 \leq p \leq n$, the Hodge Decomposition Theorem [7, 10] says that the pth cohomology group $H^p(M;\mathbb{R})$ is isomorphic to the space of closed and co-closed differential p-forms on M. Thus, the space of such forms (called *harmonic fields* by Kodaira) is a concrete realization of the cohomology group $H^p(M;\mathbb{R})$ inside the space $\Omega^p(M)$ of all p-forms on M.

Since M is closed, $\partial M = \emptyset$, so $H^p(M;\mathbb{R}) = H^p(M, \partial M;\mathbb{R})$ and thus the concrete realizations of $H^p(M;\mathbb{R})$ and $H^p(M, \partial M;\mathbb{R})$ coincide. This turns out not to be true for manifolds with non-empty boundary.

When M^n is a compact, smooth, oriented Riemannian manifold with non-empty boundary ∂M, the relevant version of the Hodge Decomposition Theorem was proved by Morrey [13] and Friedrichs [6]. It gives concrete realizations of both $H^p(M;\mathbb{R})$ and $H^p(M, \partial M;\mathbb{R})$ inside the space of harmonic p-fields on M. Not only do these spaces not coincide, they intersect only at 0. Thus, a somewhat idiosyncratic way of distinguishing the closed manifolds from among all compact Riemannian manifolds is as those manifolds for which the concrete realizations of the absolute and relative cohomology groups coincide. It seems plausible that the relative positions of the concrete realizations of $H^p(M;\mathbb{R})$ and $H^p(M, \partial M;\mathbb{R})$ might, in some ill-defined sense, reflect how close the manifold M is to being closed.

The first goal of this paper is to show that these relative positions of cohomology groups can not only be recovered from boundary data, but in fact from boundary data which naturally arises in a well-known and physically significant inverse problem. Specifically, these relative positions are determined by the Cauchy data of harmonic differential forms, which can be represented as the graph of the Dirichlet-to-Neumann operator for differential forms. This operator is a generalization of the classical Dirichlet-to-Neumann operator for functions, which of course is the key operator in the problem of Electrical Impedance Tomography.

More specifically, the relative positions of the concrete realizations of $H^p(M;\mathbb{R})$ and $H^p(M, \partial M;\mathbb{R})$ are encoded by the principal angles between them. These angles were defined in unpublished work of DeTurck and Gluck [4], who called them *Poincaré duality angles*.

DeTurck and Gluck observed that $H^p(M;\mathbb{R})$ and $H^p(M, \partial M;\mathbb{R})$ each have one portion consisting of those cohomology classes coming from the boundary of M and another portion consisting of those classes coming from the "interior" of M. They then showed that these interior and boundary portions manifest themselves as orthogonal subspaces inside the concrete realizations of $H^p(M;\mathbb{R})$ and $H^p(M, \partial M;\mathbb{R})$. This leads to a refinement of the Hodge–Morrey–Friedrichs decomposition which says, in part, that

i. the concrete realizations of $H^p(M;\mathbb{R})$ and $H^p(M,\partial M;\mathbb{R})$ meet only at the origin,
ii. the boundary subspace of each is orthogonal to all of the other, and
iii. the principal angles between the interior subspaces are all acute.

This behavior is depicted in Figure 1.

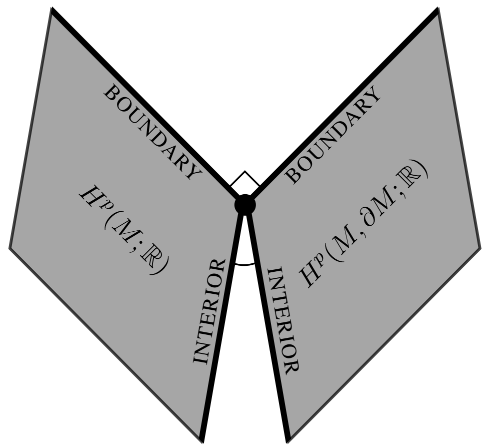

FIG. 1: The concrete realizations of the absolute and relative cohomology groups

The only interesting (i.e., not equal to $\pi/2$) principal angles, then, are the principal angles between the interior subspaces. These angles are invariants of the Riemannian structure on M and are the Poincaré duality angles of the title. They are still poorly understood and the only examples which have been computed explicitly are given in [16], which is otherwise an expanded version of this paper. In all of these examples, the Poincaré duality angles approach zero in sequences of manifolds converging to complete manifolds (it seems not to matter whether the limiting manifold is compact or non-compact), suggesting that they measure the failure of a manifold with boundary to be complete. This remains conjectural, but the behavior of the Poincaré duality angles in examples and the fact that their definition involves a subtle mix of topology and geometry suggests that they encode interesting information about compact Riemannian manifolds with boundary.

The fact that the Poincaré duality angles are determined by the Dirichlet-to-Neumann operator for differential forms is the content of Theorem 1, stated below. This connection will then be exploited to partially reconstruct the cohomology ring structure of M from Cauchy data in Theorem 3.

The Dirichlet-to-Neumann operator for forms was defined by Belishev and Sharafutdinov [1] and generalizes the classical Dirichlet-to-Neumann map for functions which arises in the problem of Electrical Impedance Tomography (EIT). The EIT problem was first posed by Calderón [3] in the context of geoprospecting, but is also of considerable interest in medical imaging.

The Dirichlet-to-Neumann map Λ is a map $\Omega^p(\partial M) \to \Omega^{n-p-1}(\partial M)$; when $n = 3$ this map can be interpreted as an operator on vector fields and is exactly the magnetic Dirichlet-to-Neumann map [2]. The connection to Poincaré duality angles is by way of the Hilbert transform $T = d\Lambda^{-1}$, which is a natural generalization (cf. [1, Section 5]) of the Hilbert transform from complex analysis.

Theorem 1. *If $\theta_1^p, \ldots, \theta_\ell^p$ are the principal angles between the interior subspaces of the concrete realizations of $H^p(M;\mathbb{R})$ and $H^p(M,\partial M;\mathbb{R})$ (i.e. the Poincaré duality angles in dimension p), then the quantities*

$$(-1)^{pn+p+n} \cos^2 \theta_i^p$$

are the non-zero eigenvalues of a suitable restriction of T^2.

In fact, the eigenspaces of the operator T^2 determine a direct-sum decomposition of the traces (i.e. pullbacks to the boundary) of harmonic fields on M, which in turn leads to the following refinement of a theorem of Belishev and Sharafutdinov:

Theorem 2. *Let $\mathcal{E}^p(\partial M)$ be the space of exact p-forms on ∂M. Then the dimension of the quotient $\ker \Lambda / \mathcal{E}^p(\partial M)$ is equal to the dimension of the boundary subspace of $H^p(M;\mathbb{R})$.*

Belishev and Sharafutdinov showed that the cohomology groups of M can be completely determined from the Cauchy data $(\partial M, \Lambda)$ and, in fact, that this data determines the long exact sequence of the pair $(M, \partial M)$. Theorem 1 shows that the data $(\partial M, \Lambda)$ not only determines the interior and boundary subspaces of the cohomology groups, but detects their relative positions as subspaces of differential forms on M.

At the end of their paper, Belishev and Sharafutdinov posed the following question:

> *Can the multiplicative structure of cohomologies be recovered from our data $(\partial M, \Lambda)$? Till now, the authors cannot answer the question.*

A partial answer to Belishev and Sharafutdinov's question can be given in the case of the mixed cup product

$$\cup : H^p(M;\mathbb{R}) \times H^q(M, \partial M;\mathbb{R}) \to H^{p+q}(M, \partial M;\mathbb{R}).$$

Theorem 3. *The boundary data $(\partial M, \Lambda)$ completely determines the mixed cup product when the relative cohomology class is restricted to come from the boundary subspace.*

When the manifold M occurs as a region in Euclidean space, all relative cohomology classes come from the boundary subspace, so Theorem 3 has the following immediate corollary:

Corollary 4. *If M^n is a compact region in $\mathbb{R}^n$, the boundary data $(\partial M, \Lambda)$ completely determines the mixed cup product on M.*

The expression for the reconstruction of the mixed cup product given in the proof of Theorem 3 makes sense even when the relative class does not come from the boundary subspace, suggesting that the data $(\partial M, \Lambda)$ may determine the full mixed cup product. It remains an interesting question whether the Dirichlet-to-Neumann data determines the absolute or relative cup products on M.

2. BACKGROUND

2.1. Poincaré duality angles

Let M^n be a compact, oriented, smooth Riemannian manifold with boundary of dimension n. For each p between 0 and n, let $\Omega^p(M)$ be the space of smooth differential p-forms on M and let $\Omega(M) = \bigoplus_{i=0}^n \Omega^i(M)$ be the algebra of all differential forms on M.

Let $d : \Omega(M) \to \Omega(M)$ be the exterior derivative and let $\mathcal{C}^p(M)$ and $\mathcal{E}^p(M)$ be the space of closed and exact p-forms, respectively. Likewise, let $\delta = d^*$ be the adjoint of d and let $c\mathcal{C}^p(M)$ and $c\mathcal{E}^p(M)$ be the space of co-closed and co-exact p-forms. The Hodge star on M will be denoted $\star$.

The pullbacks to the boundary of various forms will be important, so let $i : \partial M \hookrightarrow M$ be the inclusion. Moreover, denote the exterior derivative, exterior co-derivative, and Hodge star on $\Omega(\partial M)$ by d_∂, δ_∂, and $\star_\partial$, respectively.

The space

$$\mathcal{H}^p(M) := \{\omega \in \Omega^p(M) : d\omega = 0 \text{ and } \delta\omega = 0\}$$

is the space of *harmonic p-fields* on M, and

$$\mathcal{H}^p_N(M) := \{\omega \in \mathcal{H}^p(M) : i^* \star \omega = 0\}$$
$$\mathcal{H}^p_D(M) := \{\omega \in \mathcal{H}^p(M) : i^* \omega = 0\}$$

are the spaces of *Neumann* and *Dirichlet* harmonic fields, respectively. The subscripts N and D likewise indicate Neumann and Dirichlet boundary conditions in the following definitions:

$$c\mathcal{E}^p_N(M) := \{\omega \in \Omega^p(M) : \omega = \delta\xi \text{ for some } \xi \in \Omega^{p+1}(M) \text{ where } i^* \star \xi = 0\}$$
$$\mathcal{E}^p_D(M) := \{\omega \in \Omega^p(M) : \omega = d\eta \text{ for some } \eta \in \Omega^{p-1}(M) \text{ where } i^*\eta = 0\}.$$

Juxtaposition of letters will denote the intersection of the corresponding spaces; for example,

$$c\mathcal{E}\mathcal{H}^p_N(M) := c\mathcal{E}^p(M) \cap \mathcal{H}^p_N(M), \;\; \mathcal{E}\mathcal{H}^p_D(M) := \mathcal{E}^p(M) \cap \mathcal{H}^p_D(M), \;\; \text{etc.}$$

With this notation in place, then, the analogue of the Hodge theorem [8] is the following, which combines the work of Morrey [13] and Friedrichs [6]:

Hodge–Morrey–Friedrichs Decomposition Theorem. *Let M be a compact, oriented, smooth Riemannian manifold with non-empty boundary ∂M. Then the space $\Omega^p(M)$ can be decomposed as*

$$\Omega^p(M) = c\mathcal{E}^p_N(M) \oplus \mathcal{H}^p_N(M) \oplus \mathcal{E}\mathcal{H}^p(M) \oplus \mathcal{E}^p_D(M) \tag{1}$$
$$= c\mathcal{E}^p_N(M) \oplus c\mathcal{E}\mathcal{H}^p(M) \oplus \mathcal{H}^p_D(M) \oplus \mathcal{E}^p_D(M), \tag{2}$$

where the direct sums are L^2-orthogonal. Moreover,

$$H^p(M;\mathbb{R}) \cong \mathcal{H}^p_N(M)$$
$$H^p(M,\partial M;\mathbb{R}) \cong \mathcal{H}^p_D(M)$$

Morrey proved that

$$\Omega^p(M) = c\mathcal{E}^p_N(M) \oplus \mathcal{H}^p(M) \oplus \mathcal{E}^p_D(M) \tag{3}$$

and Friedrichs gave the two decompositions of the harmonic fields:

$$\mathcal{H}^p(M) = \mathcal{H}^p_N(M) \oplus \mathcal{E}\mathcal{H}^p(M) \tag{4}$$
$$= c\mathcal{E}\mathcal{H}^p(M) \oplus \mathcal{H}^p_D(M); \tag{5}$$

both were influenced by the work of Duff and Spencer [5]. The orthogonality of the components in (4) and (5) follows immediately from Green's Formula.

Note that $\mathcal{H}^p_N(M)$ and $\mathcal{H}^p_D(M)$ are not L^2-orthogonal in $\Omega^p(M)$ so cannot both appear in the same orthogonal decomposition of $\Omega^p(M)$. The best that can be done is the following five-term decomposition, which is an immediate consequence of the Hodge–Morrey–Friedrichs decomposition:

Theorem 2.1. *Let M be a compact, oriented, smooth Riemannian manifold with non-empty boundary. Then the space $\Omega^p(M)$ of smooth p-forms on M has the direct-sum decomposition*

$$\Omega^p(M) = c\mathcal{E}^p_N(M) \oplus \mathcal{E}c\mathcal{E}^p(M) \oplus \left(\mathcal{H}^p_N(M) + \mathcal{H}^p_D(M)\right) \oplus \mathcal{E}^p_D(M),$$

where $\mathcal{E}c\mathcal{E}^p(M)$ denotes the space of p-forms which are both exact and co-exact.

In the statement of Theorem 2.1, the symbol $\oplus$ indicates an orthogonal direct sum, whereas the symbol $+$ just indicates a direct sum.

DeTurck and Gluck's key insight, which leads to the definition of Poincaré duality angles, was that the non-orthogonality of $\mathcal{H}^p_N(M)$ and $\mathcal{H}^p_D(M)$ has to do with the fact that some of the cohomology of M comes from the "interior" of M and some comes from the boundary.

In absolute cohomology, the interior subspace is very easy to identify. Consider the map $i^*\colon H^p(M;\mathbb{R}) \to H^p(\partial M;\mathbb{R})$ induced by the inclusion. The kernel of i^* certainly deserves to be called the interior portion of $H^p(M;\mathbb{R})$, but it is not clear what the boundary portion should be. Since $H^p(M;\mathbb{R}) \cong \mathcal{H}^p_N(M)$, the interior portion of the absolute cohomology is identifiable as the subspace of the harmonic Neumann fields which pull back to zero in the cohomology of the boundary; i.e.

$$\mathcal{E}_\partial\mathcal{H}^p_N(M) := \{\omega \in \mathcal{H}^p_N(M) : i^*\omega = d_\partial\varphi \text{ for some } \varphi \in \Omega^{p-1}(\partial M)\}.$$

The orthogonal complement of the *interior subspace* $\mathcal{E}_\partial\mathcal{H}^p_N(M)$ inside $\mathcal{H}^p_N(M)$ turns out to be the *boundary subspace* $c\mathcal{E}\mathcal{H}^p_N(M)$.

In turn, the interior and boundary subspaces of $\mathcal{H}^p_D(M)$ are just the images under the Hodge star of the interior and boundary subspaces of $\mathcal{H}^{n-p}_N(M)$, namely

$$c\mathcal{E}_\partial\mathcal{H}^p_D(M) := \{\omega \in \mathcal{H}^p_D(M) : i^*\star\omega = d_\partial\psi \text{ for some } \psi \in \Omega^{n-p-1}(\partial M)\}$$

and $\mathcal{E}\mathcal{H}^p_D(M)$, respectively.

These observations allow the details of Figure 1 to be filled in, as shown in Figure 2.

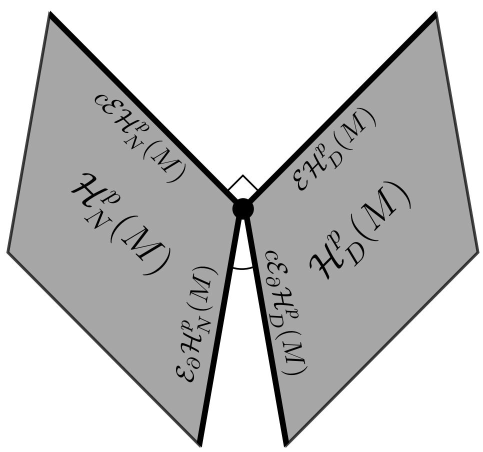

FIG. 2: $\mathcal{H}^p_N(M)$ and $\mathcal{H}^p_D(M)$

With the interior and boundary subspaces given explicitly, the existence of the Poincaré duality angles can now be established:

Theorem 2.2 (DeTurck–Gluck)**.** *Let M^n be a compact, oriented, smooth Riemannian manifold with nonempty boundary ∂M. Then within the space $\Omega^p(M)$ of p-forms on M,*

i. The concrete realizations $\mathcal{H}^p_N(M)$ and $\mathcal{H}^p_D(M)$ of the absolute and relative cohomology groups $H^p(M;\mathbb{R})$ and $H^p(M,\partial M;\mathbb{R})$ meet only at the origin.
ii. The boundary subspace $c\mathcal{E}\mathcal{H}^p_N(M)$ of $\mathcal{H}^p_N(M)$ is orthogonal to all of $\mathcal{H}^p_D(M)$ and the boundary subspace $\mathcal{E}\mathcal{H}^p_D(M)$ of $\mathcal{H}^p_D(M)$ is orthogonal to all of $\mathcal{H}^p_N(M)$.

iii. No larger subspace of $\mathcal{H}^p_N(M)$ is orthogonal to all of $\mathcal{H}^p_D(M)$ and no larger subspace of $\mathcal{H}^p_D(M)$ is orthogonal to all of $\mathcal{H}^p_N(M)$.

iv. The principal angles between the interior subspaces $\mathcal{E}_\partial\mathcal{H}^p_N(M)$ of $\mathcal{H}^p_N(M)$ and $c\mathcal{E}_\partial\mathcal{H}^p_D(M)$ of $\mathcal{H}^p_D(M)$ are all acute.

The fact that $\mathcal{E}_\partial\mathcal{H}^p_N(M)$ and $c\mathcal{E}_\partial\mathcal{H}^p_D(M)$ have the same dimension is a straightforward consequence of Poincaré–Lefschetz duality. The principal angles between these subspaces are the *Poincaré duality angles*.

A complete proof of Theorem 2.2 can be found in [16], which also gives explicit calculations of the Poincaré duality angles of various manifolds defined by removing pieces of complex projective spaces and real Grassmannians.

2.2. The Dirichlet-to-Neumann map

The Dirichlet-to-Neumann map for differential forms is a generalization of the classical Dirichlet-to-Neumann operator for functions. The classical Dirichlet-to-Neumann operator arises in connection with the problem of Electrical Impedance Tomography (EIT), which was originally posed by Calderón [3] in the context of geoprospecting but which is also of particular interest in medical imaging.

The classical Dirichlet-to-Neumann operator $\Lambda_{\text{cl}} : C^\infty(\partial M) \to C^\infty(\partial M)$ is defined by

$$f \mapsto \frac{\partial u}{\partial \nu},$$

where $\Delta u = 0$ on $M^n \subset \mathbb{R}^n$ and $u|_{\partial M} = f$. In dimension ≥ 3, Lee and Uhlmann [12] showed that the problem of EIT is equivalent to determining an associated Riemannian metric g from the Dirichlet-to-Neumann map Λ_{cl}. They also proved that if M is real-analytic, then $(\partial M, \Lambda_{\text{cl}})$ determines the Riemannian metric on M up to isometry.

The classical Dirichlet-to-Neumann map was generalized to differential forms by Belishev and Sharafutdinov [1]. A slightly different generalization was given by Joshi and Lionheart [9] and used by Krupchyk, Lassas, and Uhlmann [11]; these two approaches are reconciled in joint work with Sharafutdinov [15].

If M^n is a compact, oriented, smooth Riemannian manifold with non-empty boundary ∂M, then the Dirichlet-to-Neumann map for p-forms $\Lambda_p : \Omega^p(\partial M) \to \Omega^{n-p-1}(\partial M)$ for any $0 \leq p \leq n-1$ is defined as follows.

If $\varphi \in \Omega^p(\partial M)$ is a smooth p-form on the boundary, then the boundary value problem

$$\Delta\omega = 0, \quad i^*\omega = \varphi \quad \text{and} \quad i^*\delta\omega = 0 \tag{6}$$

can be solved [14, Lemma 3.4.7]. The solution $\omega \in \Omega^p(M)$ is unique up to the addition of an arbitrary harmonic Dirichlet field $\lambda \in \mathcal{H}^p_D(M)$. Define

$$\Lambda_p\varphi := i^* \star d\omega.$$

Then $\Lambda_p\varphi$ is independent of the choice of ω since taking $d\omega$ eliminates the ambiguity in the choice of ω. Define

$$\Lambda := \bigoplus_{i=0}^{n-1} \Lambda_i.$$

When φ is a function (i.e. $\varphi \in \Omega^0(\partial M)$), suppose $u \in \Omega^0(M)$ is a harmonic function which restricts to φ on the boundary. Since $\delta u = 0$, u solves the boundary value problem (6). Hence,

$$\Lambda_0\varphi = i^* \star du = \frac{\partial u}{\partial \nu}\, \mathrm{dvol}_{\partial M} = (\Lambda_{\mathrm{cl}}\varphi)\, \mathrm{dvol}_{\partial M},$$

so Λ is indeed a generalization of the classical Dirichlet-to-Neumann map.

Two key lemmas, both in Belishev and Sharafutdinov's argument and for Section 3, are the following:

Lemma 2.3 (Belishev–Sharafutdinov)**.** *If $\varphi \in \Omega^p(\partial M)$ and $\omega \in \Omega^p(M)$ solves the boundary value problem (6), then $d\omega \in \mathcal{H}^{p+1}(M)$ and $\delta\omega = 0$. Hence, (6) is equivalent to the boundary value problem*

$$\Delta\omega = 0, \quad i^*\omega = \varphi \quad \textit{and} \quad \delta\omega = 0. \tag{7}$$

Lemma 2.4 (Belishev–Sharafutdinov)**.** *For any $0 \le p \le n-1$, the kernel of Λ_p coincides with the image of Λ_{n-p-1}. Moreover, a form $\varphi \in \Omega^p$ belongs to $\ker \Lambda_p = \operatorname{im} \Lambda_{n-p-1}$ if and only if $\varphi = i^*\omega$ for some harmonic field $\omega \in \mathcal{H}^p(M)$. In other words,*

$$i^*\mathcal{H}^p(M) = \ker \Lambda_p = \operatorname{im} \Lambda_{n-p-1}.$$

Knowledge of the kernel of the Dirichlet-to-Neumann map yields lower bounds on the Betti numbers $b_p(M)$ of M and $b_p(\partial M)$ of ∂M:

Theorem 2.5 (Belishev–Sharafutdinov)**.** *The kernel $\ker \Lambda_p$ of the Dirichlet-to-Neumann map Λ_p contains the space $\mathcal{E}^p(\partial M)$ of exact p-forms on ∂M and*

$$\dim\left[\ker \Lambda_p / \mathcal{E}^p(\partial M)\right] \le \min\left\{b_p(M), b_p(\partial M)\right\}.$$

The fact that the above is only an inequality and not an equality is a little unsatisfying. Theorem 2 resolves this defect, showing that, in fact, $\ker \Lambda_p/\mathcal{E}^p(\partial M)$ recovers precisely the cohomology of M which comes from the boundary.

Two other operators come to attention in Belishev and Sharafutdinov's story. The first is the *Hilbert transform* T, defined as $T := d_\partial \Lambda^{-1}$. The Hilbert transform is obviously not well-defined on all forms on ∂M, but it is well-defined on $i^*\mathcal{H}^p(M) = \operatorname{im} \Lambda_{n-p-1}$ for any $0 \le p \le n-1$ because, although Λ_{n-p-1} has a large kernel and hence Λ_{n-p-1}^{-1} is not well-defined on its own, composing with the exterior derivative d_∂ eliminates this ambiguity since $\ker \Lambda_{n-p-1}$ consists of closed forms by Lemma 2.4. The analogy between the map T and the classical Hilbert transform from complex analysis is explained in Belishev and Sharafutdinov's Section 5.

The other interesting operator $G_p : \Omega^p(\partial M) \to \Omega^{n-p-1}(\partial M)$ is defined as

$$G_p := \Lambda_p + (-1)^{pn+p+n} d_\partial \Lambda_{n-p-2}^{-1} d_\partial.$$

Letting $G = \bigoplus_{i=0}^{n-1} G_i$, note that $G = \Lambda \pm T d_\partial$.

Belishev and Sharafutdinov's main theorem shows that knowledge of Λ (and thus of G) yields knowledge of the cohomology of M:

Theorem 2.6 (Belishev–Sharafutdinov)**.** *For any $0 \le p \le n-1$,*

$$\operatorname{im} G_{n-p-1} = i^*\mathcal{H}_N^p(M).$$

Since harmonic Neumann fields are uniquely determined by their pullbacks to the boundary, this means that $\operatorname{im} G_{n-p-1} \cong \mathcal{H}_N^p(M) \cong H^p(M;\mathbb{R})$. In other words, the boundary data $(\partial M, \Lambda)$ completely determines the absolute cohomology groups of M.

By Poincaré–Lefschetz duality, $H^p(M;\mathbb{R}) \cong H^{n-p}(M,\partial M;\mathbb{R})$, so the above theorem immediately implies that $(\partial M,\Lambda)$ also determines the relative cohomology groups of M.

A key feature of Theorem 2.6 is that the cohomology groups $H^p(M;\mathbb{R})$ and $H^p(M,\partial M;\mathbb{R})$ are not just determined abstractly by $(\partial M,\Lambda)$, but can be realized as particular subspaces of differential forms on ∂M. The content of Theorem 1 is that these shadows of $H^p(M;\mathbb{R})$ and $H^p(M,\partial M;\mathbb{R})$ on the boundary determine the relative positions of the spaces $\mathcal{H}^p_N(M)$ and $\mathcal{H}^p_D(M)$ and hence recover whatever geometric information is encoded by those positions.

3. THE DIRICHLET-TO-NEUMANN MAP AND POINCARÉ DUALITY ANGLES

Suppose the manifold M has Poincaré duality angles $\theta_1^p,\dots,\theta_k^p$ in dimension p; i.e. $\theta_1^p,\dots,\theta_k^p$ are the principal angles between the interior subspaces $\mathcal{E}_\partial\mathcal{H}^p_N(M)$ and $c\mathcal{E}_\partial\mathcal{H}^p_D(M)$.

If $\mathsf{proj}_D : \mathcal{H}^p_N(M) \to \mathcal{H}^p_D(M)$ is the orthogonal projection, then the images of the boundary and interior subspaces are $\mathsf{proj}_D c\mathcal{E}\mathcal{H}^p_N(M) = 0$ and $\mathsf{proj}_D\mathcal{E}_\partial\mathcal{H}^p_N(M) = c\mathcal{E}_\partial\mathcal{H}^p_D(M)$. Since the cosines of the principal angles between two k-planes are the singular values of the orthogonal projection from one to the other, the $\cos\theta_i^p$ are the non-zero singular values of proj_D. Likewise, if $\mathsf{proj}_N : \mathcal{H}^p_D(M) \to \mathcal{H}^p_N(M)$ is the orthogonal projection, the $\cos\theta_i^p$ are also the non-zero singular values of proj_N.

Thus, for $1 \le i \le k$, the quantities $\cos^2\theta_i^p$ are the non-zero eigenvalues of the compositions

$$\mathsf{proj}_N \circ \mathsf{proj}_D \quad \text{and} \quad \mathsf{proj}_D \circ \mathsf{proj}_N.$$

It is this interpretation of the Poincaré duality angles which yields the connection to the Dirichlet-to-Neumann map.

Specifically, the Hilbert transform $T = d\Lambda^{-1}$ is closely related to the projections proj_D and proj_N, as illustrated by the following propositions:

Proposition 3.1. *If $\omega \in \mathcal{H}^p_N(M)$ and $\mathsf{proj}_D\,\omega = \eta \in \mathcal{H}^p_D(M)$ is the orthogonal projection of ω onto $\mathcal{H}^p_D(M)$, then*

$$Ti^*\omega = (-1)^{np+1}\, i^*\star\eta.$$

Proposition 3.2. *If $\lambda \in \mathcal{H}^p_D(M)$ and $\mathsf{proj}_N\,\lambda = \sigma \in \mathcal{H}^p_N(M)$ is the orthogonal projection of λ onto $\mathcal{H}^p_N(M)$, then*

$$Ti^*\star\lambda = (-1)^{n+p+1}\, i^*\sigma.$$

Proposition 3.2 is proved by applying the Hodge star and then invoking Proposition 3.1.

Proof of Proposition 3.1. Using the first Friedrichs decomposition (4),

$$\omega = \delta\xi + \eta \in c\mathcal{E}\mathcal{H}^p(M) \oplus \mathcal{H}^p_D(M). \tag{8}$$

Since ω satisfies the Neumann boundary condition,

$$0 = i^*\star\omega = i^*\star(\delta\xi + \eta) = i^*\star\delta\xi + i^*\star\eta,$$

so

$$i^*\star\eta = -i^*\star\delta\xi. \tag{9}$$

On the other hand, since η satisfies the Dirichlet boundary condition,

$$i^*\omega = i^*(\delta\xi + \eta) = i^*\delta\xi + i^*\eta = i^*\delta\xi = (-1)^{np+n+2}\, i^*\star d \star \xi.$$

The form ξ can be chosen such that

$$\Delta\xi = 0 \quad \text{and} \quad d\xi = 0$$

(see (4.11) from Chapter 2 of Schwarz [14] or Section 2 of Belishev and Sharafutdinov [1]), which means that $\star\,\xi$ solves the boundary value problem

$$\Delta\varepsilon = 0, \quad i^*\varepsilon = i^*\star\,\xi, \quad \text{and} \quad i^*\delta\varepsilon = 0.$$

Therefore,

$$(-1)^{np+n+2}\,\Lambda i^*\star\,\xi = (-1)^{np+n+2}\, i^*\star\, d\star\,\xi = i^*\omega.$$

Then, using the definition of the Hilbert transform $T = d_\partial\Lambda^{-1}$ and (9),

$$\begin{aligned} Ti^*\omega = (-1)^{np+n+2}\, d_\partial\Lambda^{-1}\Lambda i^*\star\,\xi &= (-1)^{np+n}\, d_\partial i^*\star\,\xi \\ &= (-1)^{np+2}\, i^*\star\,\delta\xi \\ &= (-1)^{np+1}\, i^*\star\,\eta, \end{aligned}$$

as desired. □

Consider the restriction $\widetilde{T}_p$ of the Hilbert transform T to $i^*\mathcal{H}^p_N(M) \subset \Omega^p(\partial M)$. If $\eta \in \mathcal{H}^p_D(M)$ then $\star\,\eta \in \mathcal{H}^{n-p}_N(M)$, so Proposition 3.1 implies that the image of $\widetilde{T}_p$ is contained in the domain of $\widetilde{T}_{n-p}$ and hence that the square $\widetilde{T}^2 = \widetilde{T}_{n-p} \circ \widetilde{T}_p$ is well-defined. Since $\widetilde{T}_p$ and $\widetilde{T}_{n-p}$ are closely related to the orthogonal projections proj_D and proj_N, it should come as no surprise that $\widetilde{T}^2$ is closely related to the composition $\mathsf{proj}_N \circ \mathsf{proj}_D$, the eigenvalues of which are the $\cos^2\theta^p_i$. Indeed, the Poincaré duality angles are recovered from the Dirichlet-to-Neumann map by way of $\widetilde{T}^2$:

Theorem 1. *If $\theta^p_1, \ldots, \theta^p_k$ are the principal angles between $\mathcal{E}_\partial\mathcal{H}^p_N(M)$ and $c\mathcal{E}_\partial\mathcal{H}^p_D(M)$ (i.e. the Poincaré duality angles in dimension p), then the quantities*

$$(-1)^{pn+p+n}\cos^2\theta^p_i$$

are the non-zero eigenvalues of $\widetilde{T}^2 = \widetilde{T}_{n-p} \circ \widetilde{T}_p$.

Proof. If $\delta\alpha \in c\mathcal{E}\mathcal{H}^p_N(M)$, the boundary subspace of $\mathcal{H}^p_N(M)$, then, by Theorem 2.2, the form $\delta\alpha$ is orthogonal to $\mathcal{H}^p_D(M)$. By Proposition 3.1, then, $\widetilde{T}_p i^*\delta\alpha = 0$, so $i^*c\mathcal{E}\mathcal{H}^p_N(M)$ is contained in the kernel of $\widetilde{T}^2$.

Combining Propositions 3.1 and 3.2, the eigenforms of $\widetilde{T}^2$ are precisely the eigenforms of $\mathsf{proj}_N \circ \mathsf{proj}_D$. Moreover, if $\omega_i \in \mathcal{E}_\partial\mathcal{H}^p_N(M)$ is an eigenform of $\mathsf{proj}_N \circ \mathsf{proj}_D$ corresponding to a non-zero eigenvalue, then ω_i is a Neumann field realizing the Poincaré duality angle θ^p_i. Hence

$$\mathsf{proj}_N \circ \mathsf{proj}_D(\omega_i) = \cos^2\theta^p_i\,\omega_i.$$

Therefore,

$$\widetilde{T}^2 i^*\omega_i = T^2 i^*\omega_i = (-1)^{pn+p+n}\cos^2\theta^p_i\, i^*\omega_i,$$

so the $(-1)^{pn+p+n}\cos^2\theta^p_i$ are the non-zero eigenvalues of $\widetilde{T}^2$. □

Note that the domain of $\widetilde{T}^2$ is $i^*\mathcal{H}^p_N(M) = \operatorname{im} G_{n-p-1}$, which is determined by the Dirichlet-to-Neumann map. Thus, Theorem 1 implies that the Dirichlet-to-Neumann operator not only determines the cohomology groups of M, as shown by Belishev and Sharafutdinov, but determines the interior and boundary cohomology.

Corollary 3.3. *The boundary data* $(\partial M, \Lambda)$ *distinguishes the interior and boundary cohomology of* M.

Proof. By Theorem 1, the pullback $i^*c\mathcal{E}\mathcal{H}^p_N(M)$ of the boundary subspace is precisely the kernel of the operator $\widetilde{T}^2$, while the pullback $i^*\mathcal{E}_\partial\mathcal{H}^p_N(M)$ of the interior subspace is the image of $\widetilde{T}^2$. Since harmonic Neumann fields are uniquely determined by their pullbacks to the boundary,

$$i^*c\mathcal{E}\mathcal{H}^p_N(M) \cong c\mathcal{E}\mathcal{H}^p_N(M) \quad \text{and} \quad i^*\mathcal{E}_\partial\mathcal{H}^p_N(M) \cong \mathcal{E}_\partial\mathcal{H}^p_N(M),$$

so the interior and boundary absolute cohomology groups are determined by the data $(\partial M, \Lambda)$. Since, for each p, $\star\, c\mathcal{E}\mathcal{H}^p_N(M) = \mathcal{E}\mathcal{H}^{n-p}_D(M)$ and $\star\,\mathcal{E}_\partial\mathcal{H}^p_N(M) = c\mathcal{E}_\partial\mathcal{H}^{n-p}_D(M)$, the interior and boundary relative cohomology groups are also determined by the data $(\partial M, \Lambda)$. □

4. A DECOMPOSITION OF THE TRACES OF HARMONIC FIELDS

Aside from the obvious connection that it gives between the Dirichlet-to-Neumann map and the Poincaré duality angles, Theorem 1 also implies that the traces of harmonic Neumann fields have the following direct-sum decomposition:

$$i^*\mathcal{H}^p_N(M) = i^*c\mathcal{E}\mathcal{H}^p_N(M) + i^*\mathcal{E}_\partial\mathcal{H}^p_N(M). \tag{10}$$

Since $\mathcal{H}^p_N(M) = c\mathcal{E}\mathcal{H}^p_N(M) \oplus \mathcal{E}_\partial\mathcal{H}^p_N(M)$, the space $i^*\mathcal{H}^p_N(M)$ is certainly the sum of the subspaces $i^*c\mathcal{E}\mathcal{H}^p_N(M)$ and $i^*\mathcal{E}_\partial\mathcal{H}^p_N(M)$. Theorem 1 implies that $i^*c\mathcal{E}\mathcal{H}^p_N(M)$ is contained in the kernel of $\widetilde{T}^2$ and that $\widetilde{T}^2$ is injective on $i^*\mathcal{E}_\partial\mathcal{H}^p_N(M)$. Therefore, the spaces $i^*c\mathcal{E}\mathcal{H}^p_N(M)$ and $i^*\mathcal{E}_\partial\mathcal{H}^p_N(M)$ cannot overlap, so the sum in (10) is indeed direct.

In fact, slightly more is true. Removing the restriction on the domain, the operator T^2 is a map $i^*\mathcal{H}^p(M) \to i^*\mathcal{H}^p(M)$. To see this, recall that $T = d_\partial\Lambda^{-1}$, so the domain of T^2 is certainly

$$\operatorname{im} \Lambda_{n-p-1} = i^*\mathcal{H}^p(M).$$

Moreover (again using the fact that $T = d_\partial\Lambda^{-1}$), the image of T^2 is contained in $\mathcal{E}^p(\partial M)$ which, by Theorem 2.5, is contained in $\ker \Lambda_p = i^*\mathcal{H}^p(M)$.

By Theorem 2.1, the space $\mathcal{H}^p(M)$ of harmonic p-fields on M admits the decomposition

$$\mathcal{H}^p(M) = \mathcal{E}c\mathcal{E}^p(M) \oplus \left(\mathcal{H}^p_N(M) + \mathcal{H}^p_D(M)\right). \tag{11}$$

Since $\mathcal{H}^p_N(M)$ can be further decomposed as

$$\mathcal{H}^p_N(M) = c\mathcal{E}\mathcal{H}^p_N(M) \oplus \mathcal{E}_\partial\mathcal{H}^p_N(M)$$

and since $i^*\mathcal{H}^p_D(M) = \{0\}$, this means that

$$i^*\mathcal{H}^p(M) = i^*\mathcal{E}c\mathcal{E}^p(M) + i^*c\mathcal{E}\mathcal{H}^p_N(M) + i^*\mathcal{E}_\partial\mathcal{H}^p_N(M), \tag{12}$$

where the right hand side is just a sum of spaces and not, *a priori*, a direct sum. However, two harmonic fields cannot pull back to the same form on the boundary unless they differ by a Dirichlet field. Since (11) is a direct sum, elements of $\mathcal{E}c\mathcal{E}^p(M) \oplus c\mathcal{E}\mathcal{H}^p_N(M) \oplus \mathcal{E}_\partial\mathcal{H}^p_N(M)$ cannot differ by a Dirichlet field, so the right hand side of (12) is a direct sum. A characterization of the summands follows from Theorem 1 and a description of how T^2 acts on $i^*\mathcal{E}c\mathcal{E}^p(M)$.

Lemma 4.1. *The restriction of T^2 to the subspace $i^*\mathcal{E}c\mathcal{E}^p(M)$ is $(-1)^{np+p}$ times the identity map.*

Proof. Suppose $\varphi \in i^*\mathcal{E}c\mathcal{E}^p(M)$. Then

$$\varphi = i^* d\gamma = i^*\delta\xi$$

for some form $d\gamma = \delta\xi \in \mathcal{E}c\mathcal{E}^p(M)$. The form $\xi \in \Omega^{p+1}(M)$ can be chosen such that

$$\Delta\xi = 0, \quad d\xi = 0.$$

Therefore,

$$\Delta \star \xi = \star \Delta\xi = 0, \quad i^*\delta \star \xi = (-1)^p \, i^* \star d\xi = 0,$$

so

$$\varphi = i^*\delta\xi = (-1)^{np+1} \, i^* \star d \star \xi = (-1)^{np+1} \, \Lambda i^* \star \xi.$$

Since $\gamma \in \Omega^{p-1}(M)$ can be chosen such that

$$\Delta\gamma = 0, \quad \delta\gamma = 0,$$

this means that

$$\Lambda i^*\gamma = i^* \star d\gamma = i^* \star \delta\xi = (-1)^{p+1} \, i^* d \star \xi = (-1)^{p+1} d_\partial i^* \star \xi = (-1)^{np+p} d_\partial \Lambda^{-1}\varphi.$$

Hence, applying T^2 to φ yields

$$T^2\varphi = T d_\partial \Lambda^{-1}\varphi = (-1)^{np+p} \, T\Lambda i^*\gamma = (-1)^{np+p} d_\partial \Lambda^{-1}\Lambda i^*\gamma.$$

Simplifying further, this implies that

$$T^2\varphi = (-1)^{np+p} d_\partial i^*\gamma = (-1)^{np+p} \, i^* d\gamma = (-1)^{np+p} \, \varphi.$$

Since the choice of $\varphi \in i^*\mathcal{E}c\mathcal{E}^p(M)$ was arbitrary, this implies that T^2 is $(-1)^{np+p}$ times the identity map on $i^*\mathcal{E}c\mathcal{E}^p(M)$, completing the proof of the lemma. □

Lemma 4.1 has the following immediate consequence:

Proposition 4.2. *The data $(\partial M, \Lambda)$ determines the direct-sum decomposition*

$$\ker \Lambda_p = i^*\mathcal{H}^p(M) = i^*\mathcal{E}c\mathcal{E}^p(M) + i^*c\mathcal{E}\mathcal{H}^p_N(M) + i^*\mathcal{E}_\partial\mathcal{H}^p_N(M), \tag{13}$$

Proof. Since the sum in (13) was already seen to be direct, Lemma 4.1 implies that $i^*\mathcal{E}c\mathcal{E}^p(M)$ is the $(-1)^{np+p}$-eigenspace of T^2. Likewise, Theorem 1 says that $i^*c\mathcal{E}\mathcal{H}^p_N(M)$ is the kernel of T^2 and $i^*\mathcal{E}_\partial\mathcal{H}^p_N(M)$ is the sum of the eigenspaces corresponding to the eigenvalues $\cos^2\theta^p_i$. Since T^2 is certainly determined by $(\partial M, \Lambda)$, this completes the proof of the proposition. □

The decomposition (13) turns out not, in general, to be orthogonal; equivalently, the operator T^2 is not self-adjoint. In particular, as will be shown in Theorem 2,

$$i^*\mathcal{E}c\mathcal{E}^p(M) + i^*\mathcal{E}_\partial\mathcal{H}^p_N(M) = \mathcal{E}^p(\partial M),$$

but $i^*c\mathcal{E}\mathcal{H}^p_N(M)$ is not usually contained in $\mathcal{H}^p(\partial M)$. This is somewhat surprising since elements of $c\mathcal{E}\mathcal{H}^p_N(M)$ are harmonic fields and their pullbacks to ∂M must be non-trivial in cohomology.

However, the decomposition (13) does yield a refinement of Theorem 2.5:

Theorem 2. *Let* $\Lambda_p : \Omega^p(\partial M) \to \Omega^{n-p-1}(\partial M)$ *be the Dirichlet-to-Neumann operator. Then* $\ker \Lambda_p$ *has the direct-sum decomposition*

$$\ker \Lambda_p = i^*\mathcal{H}^p(M) = i^* c\mathcal{E}\mathcal{H}^p_N(M) + \mathcal{E}^p(\partial M). \tag{14}$$

Therefore,

$$\ker \Lambda_p / \mathcal{E}^p(\partial M) \cong c\mathcal{E}\mathcal{H}^p_N(M),$$

so the dimension of this space is equal to the dimension of the boundary subspace of $H^p(M;\mathbb{R})$.

Proof. Using (13), the decomposition (14) will follow from the fact that

$$\mathcal{E}^p(\partial M) = i^*\mathcal{E}c\mathcal{E}^p(M) + i^*\mathcal{E}_\partial\mathcal{H}^p_N(M). \tag{15}$$

The right hand side is certainly contained in the space on the left. To see the other containment, suppose $d_\partial\varphi \in \mathcal{E}^p(\partial M)$. Then let $\varepsilon \in \Omega^{p-1}(M)$ solve the boundary value problem

$$\Delta\varepsilon = 0, \quad i^*\varepsilon = \varphi, \quad i^*\delta\varepsilon = 0.$$

Then $d\varepsilon \in \mathcal{H}^p(M)$ and $i^*d\varepsilon = d_\partial i^*\varepsilon = d_\partial\varphi$. Now, using Theorem 2.1,

$$d\varepsilon = d\gamma + \lambda_N + \lambda_D \in \mathcal{E}c\mathcal{E}^p(M) \oplus \left(\mathcal{H}^p_N(M) + \mathcal{H}^p_D(M)\right).$$

Since λ_D is a Dirichlet field,

$$d_\partial\varphi = i^*d\varepsilon = i^*d\gamma + i^*\lambda_N.$$

Therefore,

$$i^*\lambda_N = d_\partial\varphi - d_\partial i^*\gamma = d_\partial(\varphi - i^*\gamma)$$

is an exact form, so $\lambda_N \in \mathcal{E}_\partial\mathcal{H}^p_N(M)$. Hence,

$$d_\partial\varphi = i^*d\gamma + i^*\lambda_N \in i^*\mathcal{E}c\mathcal{E}^p(M) + i^*\mathcal{E}_\partial\mathcal{H}^p_N(M),$$

so $\mathcal{E}^p(\partial M) \subset i^*\mathcal{E}c\mathcal{E}^p(M) + i^*\mathcal{E}_\partial\mathcal{H}^p_N(M)$ and (15) follows.

The decompositions (13) and (15) together imply that

$$\ker \Lambda_p = i^*\mathcal{H}^p(M) = i^* c\mathcal{E}\mathcal{H}^p_N(M) + \mathcal{E}^p(\partial M),$$

meaning that

$$\ker \Lambda_p / \mathcal{E}^p(\partial M) \cong i^* c\mathcal{E}\mathcal{H}^p_N(M).$$

However, since the elements of $c\mathcal{E}\mathcal{H}^p_N(M)$ are harmonic Neumann fields and harmonic Neumann fields are uniquely determined by their pullbacks to the boundary, $i^*c\mathcal{E}\mathcal{H}^p_N(M) \cong c\mathcal{E}\mathcal{H}^p_N(M)$, so

$$\ker \Lambda_p / \mathcal{E}^p(\partial M) \cong c\mathcal{E}\mathcal{H}^p_N(M),$$

as desired. □

5. PARTIAL RECONSTRUCTION OF THE MIXED CUP PRODUCT

The goal of this section is to state and prove Theorem 3, which says that the mixed cup product

$$\cup : H^p(M;\mathbb{R}) \times H^q(M,\partial M;\mathbb{R}) \to H^{p+q}(M,\partial M;\mathbb{R})$$

can be at least partially recovered from the boundary data $(\partial M, \Lambda)$ for any p and q.

Since $H^p(M;\mathbb{R}) \cong \text{im } G_{n-p-1}$ and

$$H^q(M,\partial M;\mathbb{R}) \cong H^{n-q}(M;\mathbb{R}) \cong \text{im } G_{q-1},$$

an absolute cohomology class $[\alpha] \in H^p(M;\mathbb{R})$ and a relative cohomology class $[\beta] \in H^q(M,\partial M;\mathbb{R})$ correspond, respectively, to forms on the boundary

$$\varphi \in \text{im } G_{n-p-1} \subset \Omega^p(\partial M) \quad \text{and} \quad \psi \in \text{im } G_{q-1} \subset \Omega^{n-q}(\partial M).$$

In turn, the relative cohomology class

$$[\alpha] \cup [\beta] \in H^{p+q}(M,\partial M;\mathbb{R}) \cong H^{n-p-q}(M;\mathbb{R})$$

corresponds to a form

$$\theta \in \text{im } G_{p+q-1} \subset \Omega^{n-p-q}(\partial M).$$

More concretely, the class $[\alpha]$ is represented by the Neumann harmonic field $\alpha \in \mathcal{H}^p_N(M)$, the class $[\beta]$ is represented by the Dirichlet harmonic field $\beta \in \mathcal{H}^q_D(M)$ and

$$\varphi = i^*\alpha, \qquad \psi = i^* \star \beta.$$

The form $\alpha \wedge \beta \in \Omega^{p+q}(M)$ is closed since

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p\, \alpha \wedge d\beta = 0.$$

Therefore, using the second Friedrichs decomposition (5),

$$\alpha \wedge \beta = \delta\xi + \eta + d\zeta \in c\mathcal{E}\mathcal{H}^{p+q}(M) \oplus \mathcal{H}^{p+q}_D(M) \oplus \mathcal{E}^{p+q}_D(M).$$

The form $\alpha \wedge \beta$ satisfies the Dirichlet boundary condition since β does, so $\delta\xi = 0$ and the above decomposition simplifies as

$$\alpha \wedge \beta = \eta + d\zeta. \tag{16}$$

This means that the relative cohomology class $[\alpha] \cup [\beta] = [\alpha \wedge \beta]$ is represented by the form $\eta \in \mathcal{H}^{p+q}_D(M)$ and that

$$\theta = i^* \star \eta.$$

Reconstructing the mixed cup product is equivalent to determining the form $i^*\star\eta$ from the forms $\varphi = i^*\alpha$ and $\psi = i^* \star \beta$. This can be done in the case that β comes from the boundary subspace of $\mathcal{H}^q_D(M)$:

Theorem 3. *The boundary data $(\partial M, \Lambda)$ completely determines the mixed cup product when the relative cohomology class comes from the boundary subspace. Specifically, with notation as above, if $\beta \in \mathcal{E}\mathcal{H}^q_D(M)$, then*

$$i^* \star \eta = (-1)^p\, \Lambda(\varphi \wedge \Lambda^{-1}\psi). \tag{17}$$

When M can be embedded in the Euclidean space $\mathbb{R}^n$ of the same dimension, all of the cohomology of M must be carried by the boundary. Thus, the following, which may be relevant to Electrical Impedance Tomography, is an immediate corollary of Theorem 3:

Corollary 4. *If M^n is a compact region in $\mathbb{R}^n$, the boundary data $(\partial M, \Lambda)$ completely determines the mixed cup product on M.*

The obvious conjecture is that the result of Theorem 3 holds without the hypothesis that β comes from the boundary subspace:

Conjecture 5.1. *The boundary data $(\partial M, \Lambda)$ determines the mixed cup product on M. In particular, with notation as above,*

$$i^* \star \eta = (-1)^p \, \Lambda(\varphi \wedge \Lambda^{-1}\psi).$$

Before proving Theorem 3, it is necessary to verify that the right hand side of (17) is even well-defined.

Lemma 5.2. *The expression $\Lambda(\varphi \wedge \Lambda^{-1}\psi)$ is well-defined.*

Proof. Since $\star \beta$ is a harmonic field,

$$\psi = i^* \star \beta \in i^* \mathcal{H}^{n-q}(M) = \text{im } \Lambda_{q-1}$$

by Lemma 2.4. Therefore, $\psi = \Lambda\mu$ for some $\mu \in \Omega^{q-1}(\partial M)$, so the expression $\Lambda^{-1}\psi = \mu$ seems to make sense.

Of course, Λ has a large kernel, so the expression $\Lambda^{-1}\psi$ is not well-defined: for any $(q-1)$-form $\sigma \in \ker \Lambda$, the form $\mu + \sigma$ is another valid choice for $\Lambda^{-1}\psi$. To see that this ambiguity does not matter, it suffices to show that

$$\Lambda(\varphi \wedge (\mu + \sigma)) = \Lambda(\varphi \wedge \mu)$$

for any such $\sigma \in \ker \Lambda$. Certainly,

$$\Lambda(\varphi \wedge (\mu + \sigma)) = \Lambda(\varphi \wedge \mu) + \Lambda(\varphi \wedge \sigma), \tag{18}$$

so the goal is to show that $\Lambda(\varphi \wedge \sigma) = 0$.

Since the kernel of Λ consists of pullbacks of harmonic fields, there is some $\tau \in \mathcal{H}^{q-1}(M)$ such that $\sigma = i^*\tau$. Then

$$\varphi \wedge \sigma = i^*\alpha \wedge i^*\tau = i^*(\alpha \wedge \tau).$$

Both α and τ are harmonic fields, so the form $\alpha \wedge \tau$ is closed, meaning that

$$\alpha \wedge \tau = \chi + d\varepsilon \in \mathcal{H}^{p+q-1}(M) \oplus \mathcal{E}_D^{p+q-1}(M)$$

by the Morrey decomposition (3). Since $i^*d\varepsilon = 0$,

$$\varphi \wedge \sigma = i^*(\alpha \wedge \tau) = i^*\chi,$$

so χ solves the boundary value problem

$$\Delta\chi = 0, \quad i^*\chi = \varphi \wedge \sigma, \quad i^*\delta\chi = 0.$$

Thus, by definition of the Dirichlet-to-Neumann map,

$$\Lambda(\varphi \wedge \sigma) = i^{*}\star d\chi = 0$$

since χ is closed. Applying this to (18) gives that

$$\Lambda(\varphi \wedge (\mu + \sigma)) = \Lambda(\varphi \wedge \mu),$$

so the expression $\Lambda(\varphi \wedge \Lambda^{-1}\psi)$ is indeed well-defined. □

The above argument only depended on the fact that $\beta \in \mathcal{H}^q_D(M)$, so the expression $\Lambda(\varphi \wedge \Lambda^{-1}\psi)$ is well-defined regardless of whether or not β lives in the boundary subspace. Thus, Conjecture 5.1 is at least plausible.

Proof of Theorem 3. Suppose β comes from the boundary subspace of $\mathcal{H}^q_D(M)$; i.e.,

$$\beta = d\beta_1 \in \mathcal{E}\mathcal{H}^q_D(M).$$

Since $\star\, d\beta_1$ is a harmonic field, Lemma 2.4 implies that $i^{*}\star\, d\beta_1$ is in the image of Λ. In fact, since β_1 can be chosen to be harmonic and co-closed, β_1 solves the boundary value problem

$$\Delta\varepsilon = 0, \quad i^{*}\varepsilon = i^{*}\beta_1, \quad i^{*}\delta\varepsilon = 0.$$

Hence,

$$\psi = i^{*}\star\, d\beta_1 = \Lambda i^{*}\beta_1.$$

Therefore, $\Lambda^{-1}\psi = i^{*}\beta_1$ (up to the ambiguity mentioned in the proof of Lemma 5.2), so

$$\Lambda(\varphi \wedge \Lambda^{-1}\psi) = \Lambda(\varphi \wedge i^{*}\beta_1).$$

On the other hand,

$$\alpha \wedge d\beta_1 = (-1)^p\, d(\alpha \wedge \beta_1)$$

is exact, so η is also exact:

$$\eta = \alpha \wedge d\beta_1 - d\zeta = d\left[(-1)^p\, \alpha \wedge \beta_1 - \zeta\right].$$

Letting $\eta' := (-1)^p\, \alpha \wedge \beta_1 - \zeta$, the form $\eta = d\eta' \in \mathcal{E}\mathcal{H}^{p+q}_D(M)$ belongs to the boundary subspace of $\mathcal{H}^{p+q}_D(M)$.

Substituting into the decomposition (16) gives that

$$\alpha \wedge d\beta_1 = d\eta' + d\zeta,$$

so the goal is to show that $(-1)^p\, \Lambda(\varphi \wedge \Lambda^{-1}\psi) = i^{*}\star\, d\eta'$. Using the definition of φ and the fact that $\Lambda^{-1}\psi = i^{*}\beta_1$,

$$(-1)^p\, \varphi \wedge \Lambda^{-1}\psi = (-1)^p\, i^{*}\alpha \wedge i^{*}\beta_1 = (-1)^p\, i^{*}(\alpha \wedge \beta_1). \tag{19}$$

Moreover, since α is closed,

$$d\left((-1)^p\, \alpha \wedge \beta_1\right) = \alpha \wedge d\beta_1 = d\eta' + d\zeta,$$

so the conclusion that

$$(-1)^p\, \Lambda(\varphi \wedge \Lambda^{-1}\psi) = i^{*}\star\, d\eta'$$

will follow from:

Proposition 5.3. *Let m be an integer such that $1 \leq m \leq n$. Given an exact Dirichlet form*

$$d\rho + d\varepsilon \in \mathcal{EH}_D^m(M) \oplus \mathcal{E}_D^m(M),$$

suppose $\gamma \in \Omega^{m-1}(M)$ is any primitive of $d\rho + d\varepsilon$; i.e. $d\gamma = d\rho + d\varepsilon$. Then

$$\Lambda i^*\gamma = i^* \star d\rho.$$

To see that Theorem 3 follows, note that $(-1)^p \alpha \wedge \beta_1$ is a primitive for $\alpha \wedge d\beta_1 = d\eta' + d\zeta$, so Proposition 5.3 and (19) imply that

$$i^* \star d\eta' = \Lambda i^*((-1)^p \alpha \wedge \beta_1) = (-1)^p \Lambda(\varphi \wedge \Lambda^{-1}\psi),$$

completing the proof of Theorem 3 modulo proving Proposition 5.3. □

Proof of Proposition 5.3. First, note that a primitive ρ for $d\rho$ can be chosen such that $\Delta\rho = 0$ and $\delta\rho = 0$. Also, by definition of the space $\mathcal{E}_D^m(M)$, a primitive ε for $d\varepsilon$ can be chosen such that $i^*\varepsilon = 0$. Then $\rho + \varepsilon$ is a primitive for $d\rho + d\varepsilon$ and

$$i^*(\rho + \varepsilon) = i^*\rho.$$

Since ρ is harmonic and co-closed,

$$\Lambda\, i^*(\rho + \varepsilon) = \Lambda\, i^*\rho = i^* \star d\rho. \tag{20}$$

Now, suppose γ is another primitive of $d\rho + d\varepsilon$. Then the form $\gamma - \rho - \varepsilon$ is closed, and so can be decomposed as

$$\gamma - \rho - \varepsilon = \kappa_1 + d\kappa_2 \in \mathcal{H}^{m-1}(M) \oplus \mathcal{E}_D^{m-1}(M).$$

Then

$$i^*(\gamma - \rho - \varepsilon) = i^*(\kappa_1 + d\kappa_2) = i^*\kappa_1$$

since $d\kappa_2$ is a Dirichlet form. Using Lemma 2.4, this means that

$$\Lambda i^*(\gamma - \rho - \varepsilon) = \Lambda i^*\kappa_1 = 0$$

since $i^*\kappa_1 \in i^*\mathcal{H}^{m-1}(M) = \ker \Lambda$.

Combining this with (20) gives that

$$\Lambda i^*\gamma = \Lambda i^*(\gamma - \rho - \varepsilon + \rho + \varepsilon) = \Lambda i^*(\gamma - \rho - \varepsilon) + \Lambda i^*(\rho + \varepsilon) = i^* \star d\rho,$$

as desired. □

6. ACKNOWLEDGEMENTS

I am very grateful for the discussions about Poincaré duality angles and the Dirichlet-to-Neumann operator that I have had with Dennis DeTurck, Herman Gluck, Rafal Komendarczyk, Vladimir Sharafutdinov, and David Shea Vela-Vick.

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