THE COMPLETE DIRICHLET-TO-NEUMANN MAP FOR DIFFERENTIAL FORMS

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Abstract. The Dirichlet-to-Neumann map for differential forms on a Riemannian manifold with boundary is a generalization of the classical Dirichlet-to-Neumann map which arises in the problem of Electrical Impedance Tomography. We synthesize the two different approaches to defining this operator by giving an invariant definition of the complete Dirichlet-to-Neumann map for differential forms in terms of two linear operators Φ and Ψ. The pair (Φ, Ψ) is equivalent to Joshi and Lionheart's operator Π and determines Belishev and Sharafutdinov's operator Λ. We show that the Betti numbers of the manifold are determined by Φ and that Ψ determines a chain complex whose homologies are explicitly related to the cohomology groups of the manifold.

1. Introduction

We consider the problem of recovering the topology of a compact, oriented, smooth Riemannian manifold (M, g) with boundary from the Dirichlet-to-Neumann map for differential forms. The classical Dirichlet-to-Neumann map for functions was first defined by Calderón [Cal80], and has been shown to recover surfaces up to conformal equivalence [LU01, Bel03] and real-analytic manifolds of dimension ≥ 3 up to isometry [LTU03].

The classical Dirichlet-to-Neumann map was generalized to an operator on differential forms independently by Joshi and Lionheart [JL05] and Belishev and Sharafutdinov [BS08]. Joshi and Lionheart called their operator Π and showed that the data $(\partial M, \Pi)$ determines the C^∞-jet of the Riemannian metric at the boundary. Krupchyk, Lassas, and Uhlmann have recently extended this result to show that $(\partial M, \Pi)$ determines a real-analytic manifold up to isometry [KLU11].

On the other hand, Belishev and Sharafutdinov called their Dirichlet-to-Neumann map Λ and showed that $(\partial M, \Lambda)$ determines the cohomology groups of the manifold M. Shonkwiler [Sho09] demonstrated a connection between Λ and invariants called Poincaré duality angles and showed that the cup product structure of the manifold M can be partially recovered from

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$(\partial M, \Lambda)$. Tarkhanov [Tar11] demonstrated that the approach can be applied with some modifications to a general elliptic complex.

The operators Π and Λ are similar, but do not appear to be equivalent. One of the advantages of Belishev and Sharafutdinov's Λ, especially for the task of recovering topological data, is that it is defined invariantly. In this paper we provide an invariant definition of Joshi and Lionheart's operator Π, which we give in terms of two auxiliary operators

$$\Phi : \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M) \quad \text{and} \quad \Psi : \Omega^k(\partial M) \to \Omega^{k-1}(\partial M).$$

We can easily show that Λ is determined by Φ and Ψ, so it makes sense to regard Π as the "complete" Dirichlet-to-Neumann operator on differential forms.

Belishev and Sharafutdinov's proof that the Betti numbers of M can be recovered from the data $(\partial M, \Lambda)$ was somewhat circuitous, as it involved determining the dimension of the image of the operator $G = \Lambda \pm d_\partial \Lambda^{-1} d_\partial$. In contrast, it is straightforward to recover the Betti numbers of M from Φ.

Theorem 1. *Let $\beta_k(M) = \dim H^k(M; \mathbb{R})$ be the kth Betti number of M and let Φ_k be the restriction of Φ to $\Omega^k(\partial M)$. Then*

$$\beta_k(M) = \dim \ker \Phi_k.$$

The operator Ψ turns out to be a chain map and the homology of the chain complex $(\Omega^*(\partial M), \Psi)$ is given in terms of a mixture of absolute and relative cohomology groups of M.

Theorem 2. *For any $0 \leq k \leq n-1$,*

$$H_k(\Omega^*(\partial M), \Psi) \simeq H^{k+1}(M, \partial M; \mathbb{R}) \oplus H^k(M; \mathbb{R}).$$

This, in turn, implies that the space of k-forms on ∂M contains an "echo" (detected by Π) of the $(k+1)$st relative cohomology group of M.

Corollary 3. *The space $\Omega^k(\partial M)$ of k-forms on ∂M contains a subspace isomorphic to $H^{k+1}(M, \partial M; \mathbb{R})$ which is distinguished by the Dirichlet-to-Neumann operator Π. Specifically,*

$$(\ker \Psi_k / \operatorname{im} \Psi_{k+1}) / \ker \Phi_k \simeq H^{k+1}(M, \partial M; \mathbb{R}).$$

When $n = 2$ and $k = 0$, Theorem 1 and Corollary 3 imply that all the cohomology groups of a surface are contained in $\Omega^0(\partial M)$.

Corollary 4. *All of the cohomology groups of a surface M with boundary can be realized inside the space of smooth functions on ∂M, where they can be recovered by the Dirichlet-to-Neumann operator Π.*

Since Ψ is a chain map, it is natural to try to define associated cochain maps and compute their cohomologies. In this spirit, we define $\widetilde{\Psi} = \pm \star_\partial \Psi \star_\partial$ and show that it is the adjoint of Ψ. Not surprisingly,

$$H^k(\Omega^*(\partial M), \widetilde{\Psi}) \simeq H_{n-k-1}(\Omega^*(\partial M), \Psi).$$

Finally, we define another cochain map Θ with the same cohomology as $\widetilde{\Psi}$. It turns out that $\Theta = \pm d_\partial \Phi^2$, so the cohomology of $\widetilde{\Psi}$ (and hence the homology of Ψ) is completely determined by the operator Φ. With this in mind, restating Corollary 3 in terms of Φ and specializing to the case $k = 0$ yields the following:

Corollary 5. *A copy of the cohomology group $H^{n-1}(M;\mathbb{R})$ is distinguished by the operator Φ inside $\Omega^0(\partial M)$, the space of smooth functions on ∂M. Specifically,*

$$\ker(d_\partial \Phi^2)/\ker \Phi \simeq H^{n-1}(M;\mathbb{R}).$$

The above results all suggest that the operator Π (and, in particular, Φ) encodes more information about the topology of M than does the operator Λ. Thus far nobody has been able to use Λ to recover the cohomology ring structure on M, but perhaps this will be easier to recover from the operator Π. Another interesting question relates to the linearized inverse problem of recovering the metric: can the results of [Sha09] be strengthened if the data Λ are replaced with the richer data (Φ, Ψ)?

2. The operators Φ and Ψ

Throughout this paper, (M, g) will be a smooth, compact, connected, oriented Riemannian manifold of dimension $n \geq 2$ with nonempty boundary. The term "smooth" is used as a synonym for "C^∞-smooth". Let $i : \partial M \hookrightarrow M$ be the identical embedding and let $\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$ be the graded algebra of smooth differential forms on M. We use the standard operators d, δ, Δ, and $\star$ on $\Omega(M)$, as well as their analogues $d_\partial, \delta_\partial, \Delta_\partial$, and $\star_\partial$ on $\Omega(\partial M)$.

Joshi and Lionheart defined their Dirichlet-to-Neumann operator

$$\Pi : \Omega(M)|_{\partial M} \to \Omega(M)|_{\partial M}$$

as

$$\Pi\chi := \left.\frac{\partial\omega}{\partial\nu}\right|_{\partial M},$$

where ν is the unit outward normal vector at the boundary and ω is the solution to the boundary value problem

$$\begin{cases} \Delta\omega = 0 \\ \omega|_{\partial M} = \chi. \end{cases}$$

This boundary value problem has a unique solution for every $\chi \in \Omega(M)|_{\partial M}$ [Sch95, Theorem 3.4.1].

When applied to forms, the meaning of the normal derivative $\partial/\partial\nu$ needs to be specified. Instead, we prefer to give an equivalent definition of Π in invariant terms. To do so, note that the restriction $\omega|_{\partial M}$ is determined by two boundary forms, $i^*\omega$ and $i^* \star \omega$. Likewise, the data $\partial\omega/\partial\nu|_{\partial M}$ are equivalent to the two boundary forms $i^*\star d\omega$ and $i^*\delta\omega$. Hence, we will define the operator

$$\Pi : \Omega^k(\partial M) \times \Omega^{n-k}(\partial M) \to \Omega^{n-k-1}(\partial M) \times \Omega^{k-1}(\partial M)$$

by

$$\Pi \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} i^*\star d\omega \\ i^*\delta\omega \end{pmatrix} \tag{1}$$

where $\omega \in \Omega^k(M)$ is the solution to the boundary value problem

$$\begin{cases} \Delta\omega = 0 \\ i^*\omega = \varphi, \quad i^*\star\omega = \psi. \end{cases} \tag{2}$$

Since Π sends pairs of forms to pairs of forms, it is somewhat cumbersome to work with in practice. Instead of using it directly, we find a pair of operators (Φ, Ψ) which is equivalent to Π. Define the linear operators

$$\Phi : \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M) \quad \text{and} \quad \Psi : \Omega^k(\partial M) \to \Omega^{k-1}(\partial M)$$

by the equalities

$$\Phi\varphi = i^*\star d\omega \quad \text{and} \quad \Psi\varphi = i^*\delta\omega. \tag{3}$$

Here $\omega \in \Omega^k(M)$ is the solution to the boundary value problem

$$\begin{cases} \Delta\omega = 0 \\ i^*\omega = \varphi, \quad i^*\star\omega = 0. \end{cases} \tag{4}$$

Now it is straightforward to express Π in terms of Φ and Ψ. We write Π as the matrix

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix}.$$

Then, comparing (1) and (3),

$$\Pi_{11} = \Phi, \quad \Pi_{21} = \Psi.$$

From (1) and (2), the operators Π_{12} and Π_{22} are given by

$$\Pi_{12}\psi = i^*\star d\varepsilon \quad \text{and} \quad \Pi_{22}\psi = i^*\delta\varepsilon,$$

where ε solves the boundary value problem

$$\begin{cases} \Delta\varepsilon = 0 \\ i^*\varepsilon = 0, \quad i^*\star\varepsilon = \psi. \end{cases}$$

If $\varepsilon \in \Omega^k(M)$ is the solution to this boundary value problem for $\psi \in \Omega^{n-k}(\partial M)$, then the form $\omega = \star\varepsilon$ solves the problem

$$\begin{cases} \Delta\omega = 0 \\ i^*\omega = \psi, \quad i^*\star\omega = 0. \end{cases}$$

Comparing this to (4), we see that

$$\Phi\psi = i^*\star d\omega \quad \text{and} \quad \Psi\psi = i^*\delta\omega. \tag{5}$$

Since

$$i^*\star d\omega = (-1)^{n(n-k)+1} i^*\delta\varepsilon \quad \text{and} \quad i^*\delta\omega = (-1)^{k+1} i^*\star d\varepsilon,$$

(1) and (5) imply that

$$\Pi_{12} = (-1)^{n(n-k)+1}\Psi \quad \text{and} \quad \Pi_{22} = (-1)^{k+1}\Phi \quad \text{on} \quad \Omega^{n-k}(\partial M).$$

Therefore, the operator Π can be expressed in terms of Φ and Ψ as

$$\Pi = \begin{pmatrix} \Phi & (-1)^{n(n-k)+1}\Psi \\ \Psi & (-1)^{k+1}\Phi \end{pmatrix} \quad \text{on} \quad \Omega^k(\partial M) \times \Omega^{n-k}(\partial M). \tag{6}$$

Belishev and Sharafutdinov's version of the Dirichlet-to-Neumann map is the operator

$$\Lambda : \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M)$$

given by

$$\Lambda\varphi = i^*\star d\omega,$$

where $\omega \in \Omega^k(M)$ is a solution to the boundary value problem

$$\begin{cases} \Delta\omega = 0 \\ i^*\omega = \varphi, \quad i^*\delta\omega = 0. \end{cases} \tag{7}$$

We can now express the operator Λ in terms of Φ and Ψ. Given $\varphi \in \Omega^k(\partial M)$, let $\omega \in \Omega^k(M)$ solve the boundary value problem (7) and set $\psi = i^*\star\omega$. Then ω solves the boundary value problem (2), so we have that

$$\Pi \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} i^*\star d\omega \\ i^*\delta\omega \end{pmatrix} = \begin{pmatrix} \Lambda\varphi \\ 0 \end{pmatrix}.$$

With the help of (6) we can rewrite this equation as the system

$$\Phi\varphi + (-1)^{n(n-k)+1}\Psi\psi = \Lambda\varphi$$
$$\Psi\varphi + (-1)^{k+1}\Phi\psi = 0.$$

Eliminating ψ from the system yields the expression

$$\Lambda = \Phi + (-1)^{n(n-k)+k+1}\Psi\Phi^{-1}\Psi \quad \text{on} \quad \Omega^k(\partial M). \tag{8}$$

The fact that the operator $\Psi\Phi^{-1}\Psi$ is well-defined follows from Corollary 4.3, stated below.

We take this opportunity to record some useful relations involving Φ and Ψ:

Lemma 2.1. *The operators Φ and Ψ satisfy the following relations:*

$$\Phi\Psi = (-1)^k d_\partial\Phi \quad \textit{on} \quad \Omega^k(\partial M), \tag{9}$$

$$\Psi^2 = 0, \tag{10}$$

$$\Psi\Phi = (-1)^{k+1}\Phi d_\partial \quad \textit{on} \quad \Omega^k(\partial M), \tag{11}$$

$$\Phi^2 = (-1)^{kn}(d_\partial\Psi + \Psi d_\partial) \quad \textit{on} \quad \Omega^k(\partial M). \tag{12}$$

Proof. Given $\varphi \in \Omega^k(\partial M)$, let $\omega \in \Omega^k(M)$ solve the boundary value problem (4). Then

$$\Phi\varphi = i^*\star d\omega, \quad \Psi\varphi = i^*\delta\omega. \tag{13}$$

Letting $\xi = \delta\omega$, we certainly have $\Delta\xi = 0$. Pulling ξ and $\star\xi$ back to the boundary yields

$$\begin{aligned} i^*\xi &= i^*\delta\omega = \Psi\varphi \\ i^*\star\xi &= i^*\star\delta\omega = \pm i^*d\star\omega = \pm d_\partial i^*\star\omega = 0. \end{aligned}$$

Therefore, ξ solves the boundary value problem

$$\begin{cases} \Delta\xi = 0 \\ i^*\xi = \Psi\varphi, \quad i^*\star\xi = 0, \end{cases}$$

and so

$$\Phi\Psi\varphi = i^*\star d\xi \quad \text{and} \quad \Psi^2\varphi = i^*\delta\xi. \tag{14}$$

Since $\Delta\omega = 0$, it follows that $d\delta\omega = -\delta d\omega$, which we use to see that

$$\begin{aligned} i^*\star d\xi &= i^*\star d\delta\omega = -i^*\star\delta d\omega = (-1)^k i^*d\star d\omega = (-1)^k d_\partial i^*\star d\omega, \\ i^*\delta\xi &= i^*\delta\delta\omega = 0. \end{aligned}$$

Comparing this with (14), we obtain

$$\Phi\Psi\varphi = (-1)^k d_\partial i^*\star d\omega \quad \text{and} \quad \Psi^2\varphi = 0.$$

With the help of (13), this gives (9) and (10).

Turning to (11), we again let $\omega \in \Omega^k(M)$ solve (4) for a form $\varphi \in \Omega^k(\partial M)$. Let $\varepsilon \in \Omega^{k+1}(M)$ be a solution to the problem

$$\begin{cases} \Delta\varepsilon = 0 \\ i^*\varepsilon = d_\partial\varphi, \quad i^*\star\varepsilon = 0. \end{cases}$$

Then

$$\Phi d_\partial\varphi = i^*\star d\varepsilon, \quad \Psi d_\partial\varphi = i^*\delta\varepsilon. \tag{15}$$

Define $\eta \in \Omega^{n-k-1}(M)$ by

$$\eta = \star d\omega - \star\varepsilon. \tag{16}$$

Clearly, $\Delta\eta = 0$. Moreover,

$$\star\eta = \star\star(d\omega - \varepsilon) = \pm(d\omega - \varepsilon),$$

so

$$i^*\star\eta = \pm i^*(d\omega - \varepsilon) = \pm(d_\partial\varphi - d_\partial\varphi) = 0.$$

Also,

$$i^*\eta = i^*\star d\omega - i^*\star\varepsilon = \Phi\varphi,$$

since $i^*\star\varepsilon = 0$.

Therefore, η solves the boundary value problem

$$\begin{cases} \Delta\eta = 0 \\ i^*\eta = \Phi\varphi, \quad i^*\star\eta = 0. \end{cases}$$

Hence,

$$\Phi^2\varphi = i^*\star d\eta \quad \text{and} \quad \Psi\Phi\varphi = i^*\delta\eta. \tag{17}$$

Using (16) we see that

$$\delta\eta = \delta\star d\omega - \delta\star\varepsilon = \pm\star dd\omega - \delta\star\varepsilon = (-1)^{k+1}\star d\varepsilon.$$

Thus,

$$i^*\delta\eta = (-1)^{k+1} i^*\star d\varepsilon,$$

which, along with (15) and (17), yields

$$\Psi\Phi\varphi = (-1)^{k+1}\Phi d_\partial\varphi,$$

proving (11).

Finally, (12) is proved along the same lines. From (16) we have

$$\star d\eta = \star d\star(d\omega - \varepsilon) = (-1)^{kn+1}(\delta d\omega - \delta\varepsilon).$$

Again making use of the fact that $\delta d\omega = -d\delta\omega$, this implies that

$$i^*\star d\eta = (-1)^{kn+1}\left(i^*\delta d\omega - i^*\delta\varepsilon\right) = (-1)^{kn}\left(d_\partial i^*\delta\omega + i^*\delta\varepsilon\right).$$

In turn, we can use (13) and (15) to rewrite the above formula as

$$i^*\star d\eta = (-1)^{kn}\left(d_\partial\Psi\varphi + \Psi d_\partial\varphi\right).$$

Comparing with (17), this produces the desired relation (12). □

Remark 2.2. The key properties of the operator Λ are expressed by the equalities

$$\Lambda d_\partial = 0, \quad d_\partial\Lambda = 0, \quad \text{and} \quad \Lambda^2 = 0.$$

It is straightforward to check that these equalities follow from (8) and Lemma 2.1.

3. Recovering the Betti numbers of M from Φ

Belishev and Sharafutdinov showed that the Betti numbers of the manifold M,

$$\beta_k(M) = \dim H^k(M;\mathbb{R}),$$

can be recovered from the data $(\partial M, \Lambda)$. The proof of this fact is somewhat indirect, involving the auxiliary operator

$$G = \Lambda + (-1)^{kn+k+n} d_\partial\Lambda^{-1}d_\partial : \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M). \tag{18}$$

In contrast, it is much more straightforward to recover the Betti numbers of M from the operator Φ.

Theorem 1. *Let $\Phi_k : \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M)$ be the restriction of Φ to $\Omega^k(\partial M)$. Then*

$$\beta_k(M) = \dim\ker\Phi_k.$$

The Hodge–Morrey–Friedrichs decomposition theorem [Sch95, Section 2.4] implies that

$$H^k(M;\mathbb{R}) \simeq \mathcal{H}^k_N(M),$$

where

$$\mathcal{H}^k_N(M) := \{\omega \in \Omega^k(M) : d\omega = 0, \delta\omega = 0, i^*\star\omega = 0\}$$

is the space of harmonic Neumann fields. Since harmonic forms are uniquely determined by their boundary values, $\mathcal{H}^k_N(M) \simeq i^*\mathcal{H}^k_N(M)$, so Theorem 1 is an immediate consequence of the following lemma.

Lemma 3.1. *The kernel of the operator $\Phi_k : \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M)$ consists of the boundary traces of harmonic Neumann fields; i.e.,*

$$\ker \Phi_k = i^*\mathcal{H}^k_N(M).$$

The image of Φ_k coincides with the subspace $(i^*\mathcal{H}^k_N(M))^\perp \subset \Omega^{n-k-1}(\partial M)$ consisting of forms $\psi \in \Omega^{n-k-1}(\partial M)$ satisfying

$$\int_{\partial M} \psi \wedge \xi = 0 \qquad \forall \xi \in i^*\mathcal{H}^k_N(M). \tag{19}$$

In particular, Φ is a Fredholm operator with index zero.

Proof. If $\varphi \in \Omega^k(\partial M)$ such that $\Phi_k\varphi = 0$, then the boundary value problem

$$\begin{cases} \Delta\omega = 0 \\ i^*\omega = \varphi, \quad i^*\star\omega = 0, \quad i^*\star d\omega = 0 \end{cases} \tag{20}$$

is solvable. Using Green's formula,

$$\langle d\omega, d\omega\rangle_{L^2} + \langle \delta\omega, \delta\omega\rangle_{L^2} = \langle \Delta\omega, \omega\rangle_{L^2} + \int_{\partial M} i^*(\omega \wedge \star d\omega - \delta\omega \wedge \star\omega).$$

The right side of this equation equals zero since ω solves the boundary value problem (20). Hence, ω is a harmonic Neumann field since $i^*\star\omega = 0$, and so $\varphi = i^*\omega \in i^*\mathcal{H}^k_N(M)$.

The converse statement is immediate: if $\varphi = i^*\omega$ for $\omega \in \mathcal{H}^k_N(M)$, then ω solves the boundary value problem (20) and hence $\varphi \in \ker\Phi_k$. On the other hand, a form $\psi \in \Omega^{n-k-1}(\partial M)$ is in the image of Φ_k if and only if the boundary value problem

$$\begin{cases} \Delta\omega = 0 \\ i^*\star\omega = 0, \quad i^*\star d\omega = \psi \end{cases}$$

is solvable. The defining condition (19) of $(i^*\mathcal{H}^k_N(M))^\perp$ is precisely the necessary and sufficient condition for the solvability of this boundary value problem [Sch95, Corollary 3.4.8]. □

Corollary 3.2. *The operator $d_\partial\Phi^{-1}$ is well-defined on $\operatorname{im}\Phi_k = (i^*\mathcal{H}^k_N(M))^\perp$; i.e., the equation $\Phi\varphi = \psi$ has a solution φ for every $\psi \in (i^*\mathcal{H}^k_N(M))^\perp$ and $d_\partial\varphi$ is uniquely determined by ψ.*

Proof. A form $\psi \in (i^*\mathcal{H}^k_N(M))^\perp$ belongs to the range of Φ, so the equation $\Phi\varphi = \psi$ is solvable. If $\Phi\varphi_1 = \Phi\varphi_2$, then the form $\varphi_1 - \varphi_2 \in \ker\Phi$ is closed, meaning that $d_\partial\varphi_1 = d_\partial\varphi_2$. □

The apparent similarity between the operator $d_\partial\Phi^{-1}$ and the Hilbert transform $T = d_\partial\Lambda^{-1}$ defined by Belishev and Sharafutdinov is no accident, as the following proposition demonstrates. Thus, the connection to the Poincaré duality angles of M [Sho09, Theorem 4] comes directly from the definition of Φ (and hence Π) without using Λ as an intermediary.

Proposition 3.3. $d_\partial\Lambda^{-1} = d_\partial\Phi^{-1}$, *where the term on the right-hand side is understood to be the restriction of* $d_\partial\Phi^{-1}$ *to* $\operatorname{im}\Lambda = i^*\mathcal{H}^k(M)$*, the space of boundary traces of harmonic fields.*

Proof. Suppose $\varphi \in \operatorname{im}\Lambda = i^*\mathcal{H}^k(M)$. Then $\varphi = i^*\omega$ for some $\omega \in \mathcal{H}^k(M)$. The Friedrichs decomposition says that

$$\mathcal{H}^k(M) = c\mathcal{E}\mathcal{H}^k(M) \oplus \mathcal{H}^k_D(M),$$

where

$$\begin{aligned} c\mathcal{E}\mathcal{H}^k(M) &= \{\delta\xi \in \Omega^k(M) : d\delta\xi = 0\} \\ \mathcal{H}^k_D(M) &= \{\eta \in \Omega^k(M) : d\eta = 0, \delta\eta = 0, i^*\eta = 0\}. \end{aligned}$$

Hence,

$$\omega = \delta\xi + \eta \in c\mathcal{E}\mathcal{H}^k(M) \oplus \mathcal{H}^k_D(M).$$

The form $\xi \in \Omega^{k+1}(M)$ can be chosen such that ξ is closed, $\Delta\xi = 0$, and $i^*\xi = 0$ [Sch95, p. 87, Remark 2]. Therefore,

$$\begin{cases} \Delta \star \xi = 0, \\ i^*\star(\star\xi) = 0, \\ i^*\delta\star\xi = \pm i^*\star d\star\star\xi = \pm i^*\star d\xi = 0. \end{cases}$$

This implies that $\star\xi$ solves the boundary value problems associated to both Λ and Φ, so

$$\Lambda i^*\star\xi = i^*\star d\star\xi = (-1)^{nk+1} i^*\delta\xi = (-1)^{nk+1} i^*\omega = (-1)^{nk+1}\varphi$$

and

$$\Phi i^*\star\xi = i^*\star d\star\xi = (-1)^{nk+1} i^*\delta\xi = (-1)^{nk+1} i^*\omega = (-1)^{nk+1}\varphi.$$

Hence,

$$d_\partial\Lambda^{-1}\varphi = (-1)^{nk+1} d_\partial i^*\star\xi = d_\partial\Phi^{-1}\varphi,$$

so we conclude that, indeed, $d_\partial\Lambda^{-1} = d_\partial\Phi^{-1}$. □

4. The homology of the chain complex $(\Omega^*(\partial M), \Psi)$

We saw in Lemma 2.1 that $\Psi^2 = 0$, so it is natural to ask: what is the homology of the chain complex $(\Omega^*(\partial M), \Psi)$?

Theorem 2. *For any $0 \leq k \leq n-1$, if $\Psi_k : \Omega^k(\partial M) \to \Omega^{k-1}(\partial M)$ is the restriction of Ψ to the space of k-forms on ∂M, then*

$$H_k(\Omega^*(\partial M), \Psi) = \frac{\ker \Psi_k}{\operatorname{im} \Psi_{k+1}} \simeq H^{k+1}(M, \partial M; \mathbb{R}) \oplus H^k(M; \mathbb{R}).$$

In other words, the homology groups of $(\Omega^*(\partial M), \Psi)$ contain the absolute cohomology groups of M in the same dimension and echoes of the relative cohomology groups of M in one higher dimension. This behavior is similar to that exhibited by the cohomology of harmonic forms studied by Cappell, DeTurck, Gluck, and Miller [CDGM06].

Since $H^k(M; \mathbb{R}) \simeq \ker \Phi_k$ (by Theorem 1) and since it will turn out that $\operatorname{im} \Psi_{k+1}$ completely misses $\ker \Phi_k$, we can isolate the echo of the $(k+1)$st relative cohomology group of M inside the space of k-forms on ∂M.

Corollary 3. *The space $\Omega^k(\partial M)$ of k-forms on ∂M contains a space isomorphic to $H^{k+1}(M, \partial M; \mathbb{R})$ which is distinguished by the Dirichlet-to-Neumann operator Π. Specifically,*

$$(\ker \Psi_k / \operatorname{im} \Psi_{k+1}) / \ker \Phi_k \simeq H^{k+1}(M, \partial M; \mathbb{R}).$$

When $n = 2$ and $k = 0$, Theorem 1 and Corollary 3 imply that $H^0(M; \mathbb{R})$ and $H^1(M, \partial M; \mathbb{R})$ can be distinguished inside the space of functions on ∂M. Moreover, by Poincaré–Lefschetz duality, $H^0(M; \mathbb{R}) \simeq H^2(M, \partial M; \mathbb{R})$ and $H^1(M, \partial M; \mathbb{R}) \simeq H^1(M; \mathbb{R})$. Since $H^0(M, \partial M; \mathbb{R})$ and $H^2(M; \mathbb{R})$ are both trivial, we have the following corollary.

Corollary 4. *All of the cohomology groups of a surface M with boundary can be realized inside the space of smooth functions on ∂M, where they can be recovered by the Dirichlet-to-Neumann operator Π.*

Theorem 2 will follow from Lemmas 4.1 and 4.2, which describe the kernel and image of Ψ.

Lemma 4.1. *If $\Psi_k : \Omega^k(\partial M) \to \Omega^{k-1}(\partial M)$ is the restriction of Ψ to the space of k-forms on ∂M, then $\ker \Psi_k$ is a direct sum of three spaces:*

(i) *The pullbacks of harmonic Neumann fields*

$$i^* \mathcal{H}_N^k(M) = \ker \Phi_k.$$

(ii) *The space*

$$\ker G_k \cap i^* \left((\mathcal{C}^k(M))^\perp \right),$$

which consists of the pullbacks of k-forms with conjugates on M which are perpendicular to the space of closed forms.

(iii) *A space isomorphic to $H^{k+1}(M, \partial M; \mathbb{R})$.*

The operator G_k is the restriction to $\Omega^k(\partial M)$ of the operator G defined in (18).

Lemma 4.2. *The image of the operator $\Psi_{k+1} : \Omega^{k+1}(\partial M) \to \Omega^k(\partial M)$ is precisely the space*

$$\ker G_k \cap i^*\Big((\mathcal{C}^k(M))^\perp \Big).$$

Proof of Lemma 4.1. Suppose $\varphi \in \Omega^k(\partial M)$ such that $\Psi\varphi = 0$. Then, if $\omega \in \Omega^k(M)$ solves the boundary value problem (4), we have that

$$0 = \Psi\varphi = i^*\delta\omega. \tag{21}$$

Using the Hodge-Morrey decomposition of $\Omega^k(M)$ [Sch95, Theorem 2.4.2],

$$\omega = \delta\xi + \kappa + d\zeta \in c\mathcal{E}^k_N(M) \oplus \mathcal{H}^k(M) \oplus \mathcal{E}^k_D(M), \tag{22}$$

where

$$c\mathcal{E}^k_N(M) = \{\omega \in \Omega^k(M) : \omega = \delta\xi \text{ for some } \xi \in \Omega^{k+1}(M) \text{ with } i^* \star \xi = 0\}$$
$$\mathcal{H}^k(M) = \{\omega \in \Omega^k(M) : d\omega = 0, \delta\omega = 0\}$$
$$\mathcal{E}^k_D(M) = \{\omega \in \Omega^k(M) : \omega = d\zeta \text{ for some } \zeta \in \Omega^{k-1}(M) \text{ with } i^*\zeta = 0\}.$$

Equations (21) and (22) imply that

$$0 = i^*\delta\omega = i^*\delta(\delta\xi + \kappa + d\zeta) = i^*\delta d\zeta. \tag{23}$$

Since $\delta d\zeta$ is co-exact and since the space of co-exact k-forms is precisely the orthogonal complement of the space of k-forms satisfying the Dirichlet boundary condition, (23) implies that $\delta d\zeta = 0$. Hence, $d\zeta$ is co-closed—but $\mathcal{E}^k_D(M)$ is precisely the orthogonal complement of the space of co-closed k-forms, so it follows that $d\zeta = 0$.

Therefore,

$$\omega = \delta\xi + \kappa$$

is co-closed. Since both ω and $\delta\xi \in c\mathcal{E}^k_N(M)$ satisfy the Neumann boundary condition, κ must be a harmonic Neumann field. Moreover, since both ω and κ are harmonic, it follows that $\delta\xi$ is harmonic. Hence,

$$\omega = \delta\xi + \kappa \in (c\mathcal{E}^k_N(M) \cap \ker\Delta) \oplus \mathcal{H}^k_N(M)$$

and so

$$\varphi = i^*\omega \in i^*(c\mathcal{E}^k_N(M) \cap \ker\Delta) + i^*\mathcal{H}^k_N(M). \tag{24}$$

Conversely, forms in this space are clearly in the kernel of Ψ.

In (24) the sum of spaces is not, *a priori*, direct, but directness of the sum follows immediately from the fact that harmonic forms are uniquely determined by their boundary values [Sch95, Theorem 3.4.10].

The term $i^*\mathcal{H}^k_N(M) = \ker\Phi_k$ in (24) is exactly the space described in (i), so the lemma will follow from showing that $i^*(c\mathcal{E}^k_N(M) \cap \ker\Delta)$ is the direct sum of the spaces described in (ii) and (iii).

Suppose, then, that $\varphi \in i^*(c\mathcal{E}^k_N(M) \cap \ker \Delta)$; i.e., that $\omega = \delta\xi$. Since $0 = \Delta\omega = \Delta\delta\xi$, we know that

$$0 = (d\delta + \delta d)\delta\xi = \delta d\delta\xi,$$

so $d\delta\xi$ is co-closed, meaning that $d\delta\xi \in \mathcal{H}^{k+1}(M)$; specifically, $d\delta\xi \in \mathcal{E}\mathcal{H}^{k+1}(M)$. On the other hand, for any $d\gamma \in \mathcal{E}\mathcal{H}^{k+1}(M)$, there is a unique choice of primitive γ that is in $c\mathcal{E}^k_N(M) \cap \ker \Delta$. Hence,

$$c\mathcal{E}^k_N(M) \cap \ker \Delta \simeq \mathcal{E}\mathcal{H}^{k+1}(M).$$

In turn, since forms in $c\mathcal{E}^k_N(M) \cap \ker \Delta$ are uniquely determined by their pullbacks to the boundary, this implies that

$$i^*(c\mathcal{E}^k_N(M) \cap \ker \Delta) \simeq \mathcal{E}\mathcal{H}^{k+1}(M).$$

Applying the Hodge star to the space $c\mathcal{E}^k_N(M) \cap \ker \Delta$ yields Cappell, DeTurck, Gluck, and Miller's space EHarm^{n-k}. Thinking in those terms, $\delta\xi \in c\mathcal{E}^k_N(M)$ is a harmonic, co-exact form, but the primitive ξ is not necessarily harmonic. There are two possibilities:

Case 1: If ξ is harmonic, then

$$0 = \Delta\xi = (d\delta + \delta d)\xi = d\delta\xi + \delta d\xi,$$

meaning that $d\delta\xi = -\delta d\xi$ is both exact and co-exact. Since $\Delta\delta\xi = 0$, this means that $\delta\xi$ has a conjugate form (in the sense of [BS08, Section 5]). This implies that $i^*\delta\xi \in \ker G_k$ [BS08, Theorem 5.1]. Since $\delta\xi$ is orthogonal to the space of closed k-forms on M, we have

$$\varphi = i^*\delta\xi \in \ker G_k \cap i^*\left((\mathcal{C}^k(M))^\perp\right),$$

which is the space in (ii).

Conversely, if $\varphi \in \ker G_k \cap i^*\big((\mathcal{C}^k(M))^\perp\big)$, then $\varphi = i^*\delta\xi$ for some $\delta\xi \in c\mathcal{E}^k_N(M)$ which has a conjugate form. This implies that $d\delta\xi$ is both exact and co-exact, and it is straightforward to check that ξ can be chosen to be harmonic.

Case 2: If ξ is not harmonic, then it belongs to the space

$$\mathcal{N}^k := \{\delta\xi \in c\mathcal{E}^k_N(M) \cap \ker \Delta : \Delta\xi \neq 0\}.$$

This space is isomorphic to $H^{k+1}(M, \partial M; \mathbb{R})$ [CDGM06, Lemma 3], and so $i^*\mathcal{N}^k$ is the space given in (iii).

The directness of the sum

$$\left(\ker G_k \cap i^*\left((\mathcal{C}^k(M))^\perp\right)\right) + i^*\mathcal{N}^k$$

again follows from the fact that harmonic forms are uniquely determined by their boundary values. □

We can now determine the image of Ψ_{k+1}.

Proof of Lemma 4.2. Suppose $\vartheta \in \Omega^k(\partial M)$ such that $\vartheta = \Psi\varphi$ for some $\varphi \in \Omega^{k+1}(\partial M)$. If $\omega \in \Omega^{k+1}(M)$ solves the boundary value problem (4), then $\vartheta = \Psi\varphi = i^*\delta\omega$.

Since ω satisfies the Neumann boundary condition,

$$\delta\omega \in c\mathcal{E}_N^k(M).$$

Moreover, since Δ commutes with the co-differential,

$$\Delta\delta\omega = \delta\Delta\omega = 0,$$

and so

$$\delta\omega \in c\mathcal{E}_N^k(M) \cap \ker \Delta.$$

Since ω is itself harmonic, this is precisely the situation described in Case 1 of the proof of Lemma 4.1, so

$$\vartheta = i^*\delta\omega \in \ker G_k \cap\, i^*\Big((\mathcal{C}^k(M))^\perp \Big).$$

Conversely, if $\vartheta = i^*\delta\zeta$ for $\delta\zeta \in c\mathcal{E}_N^k(M) \cap \ker\Delta$ with ζ harmonic, then

$$\Delta\zeta = 0 \quad \text{and} \quad i^* \star \zeta = 0,$$

so $\vartheta = i^*\delta\zeta = \Psi i^*\zeta$ is in the image of Ψ. ☐

Corollary 4.3.

$$\ker\Phi_k \subset \ker\Psi_k \quad \textit{and} \quad \operatorname{im}\Psi_k \subset \operatorname{im}\Phi_{n-k}.$$

Proof. The fact that $\ker\Phi_k \subset \ker\Psi_k$ is an immediate consequence of Lemma 4.1.

Now, suppose $\varphi \in \operatorname{im}\Psi_k$. Then, by Lemma 4.2, $\varphi \in \ker G_{k-1}$, meaning $\varphi = i^*\omega$ for $\omega \in \Omega^{k-1}(M)$ satisfying

$$\Delta\omega = 0, \quad \delta\omega = 0, \quad \text{and} \quad d\omega = \star\, d\eta$$

for some $\eta \in \Omega^{n-k-1}(M)$ with $\Delta\eta = 0$ and $\delta\eta = 0$ [BS08, Theorem 5.1]. Therefore, for any $\lambda_N \in \mathcal{H}_N^{n-k}(M)$,

(25)
$$\int_{\partial M} \varphi \wedge i^*\lambda_N = \pm \int_{\partial M} i^*\omega \wedge i^*(\star\star\lambda_N) = \pm \left[\langle d\omega, \star\lambda_N \rangle_{L^2(M)} - \langle \omega, \delta \star \lambda_N \rangle_{L^2(M)} \right]$$

by Green's formula. The second term on the right hand side vanishes since λ_N is closed, while the first is equal to

$$\langle \star\, d\eta, \star\, \lambda_N \rangle_{L^2(M)} = \langle d\eta, \lambda_N \rangle_{L^2(M)} = 0. \tag{26}$$

The first equality above is due to the fact that $\star$ is an isometry and the second follows because $\mathcal{H}_N^{n-k}(M)$ is orthogonal to the space of exact forms on M.

Putting (25) and (26) together shows that

$$\int_{\partial M} \varphi \wedge i^*\lambda_N = 0$$

for any $\lambda_N \in \mathcal{H}_N^{n-k}(M)$, so Lemma 3.1 implies that $\varphi \in \operatorname{im}\Phi_{n-k}$, as desired. ☐

5. Cochain maps and the adjoint of Ψ

Since Ψ is a chain map whose homologies are interesting, it seems natural to try to find associated cochain maps and compute their cohomologies. In fact, there are two such maps,

$$\widetilde{\Psi} := (-1)^{k(n-1)} \star_\partial \Psi \star_\partial \quad \text{and} \quad \Theta := (-1)^{(k+1)(n-1)} \Phi\Psi\Phi.$$

By definition both are maps $\Omega^k(\partial M) \to \Omega^{k+1}(\partial M)$.

5.1. The operator $\widetilde{\Psi}$.

The fact that $\widetilde{\Psi}^2 = 0$ is immediate:

$$\widetilde{\Psi}^2 = \pm \star_\partial \Psi \star_\partial \star_\partial \Psi \star_\partial = \pm \star_\partial \Psi^2 \star_\partial = 0,$$

since $\Psi^2 = 0$.

Let $\widetilde{\Psi}^k$ be the restriction of $\widetilde{\Psi}$ to $\Omega^k(\partial M)$. Since $\star_\partial$ is an isomorphism,

$$\ker \widetilde{\Psi}^k \simeq \ker \Psi_{n-k-1} \quad \text{and} \quad \operatorname{im} \widetilde{\Psi}^{k-1} \simeq \operatorname{im} \Psi_{n-k},$$

and so

$$H^k(\Omega^*(\partial M), \widetilde{\Psi}) \simeq H_{n-k-1}(\Omega^*(\partial M), \Psi). \tag{27}$$

Thus, we can use Theorem 2 to determine the cohomology groups of $\widetilde{\Psi}$.

Proposition 5.1. *The cohomology groups of the cochain complex* $(\Omega^*(\partial M), \widetilde{\Psi})$ *are*

$$H^k(\Omega^*(\partial M), \widetilde{\Psi}) \simeq H^{n-k}(M; \mathbb{R}) \oplus H^{n-k-1}(M, \partial M; \mathbb{R})$$

The obvious guess, suggested by experience with Λ and by the duality given in (27), is that $\widetilde{\Psi}$ is the adjoint of Ψ.

Proposition 5.2. $\widetilde{\Psi}$ *is the adjoint of* Ψ*.*

Proof. The proof follows along similar lines to the proof that $\Lambda^* = \star_\partial \Lambda \star_\partial$ [BS08, p. 132].

Let $\varphi \in \Omega^k(\partial M)$ and $\psi \in \Omega^{n-k}(\partial M)$. Suppose $\omega \in \Omega^k(M)$ solves the boundary value problem (4) and that $\eta \in \Omega^{n-k}(M)$ solves the equivalent boundary value problem for ψ.

The key step is to show that

$$(-1)^{k+1} \int_{\partial M} \varphi \wedge \Psi\psi = (-1)^{kn+n+1} \int_{\partial M} \psi \wedge \Psi\varphi. \tag{28}$$

Provided this is true, we can re-write the above equation as

$$(-1)^{kn+k+1} \langle \varphi, \star_\partial \Psi\psi \rangle_{L^2(\partial M)} = -\langle \psi, \star_\partial \Psi\varphi \rangle_{L^2(\partial M)}$$

or, equivalently,

$$\langle \varphi, \star_\partial \Psi\psi \rangle_{L^2(\partial M)} = (-1)^{k(n-1)} \langle \psi, \star_\partial \Psi\varphi \rangle_{L^2(M)}.$$

Letting $\psi = \star_\partial \psi'$, this becomes

$$\langle \varphi, \star_\partial \Psi \star_\partial \psi' \rangle_{L^2(\partial M)} = (-1)^{k(n-1)} \langle \star_\partial \psi', \star_\partial \Psi\varphi \rangle_{L^2(\partial M)} = (-1)^{k(n-1)} \langle \psi', \Psi\varphi \rangle_{L^2(\partial M)},$$

since $\star_\partial$ is an isometry. Therefore,

$$\Psi^* = (-1)^{k(n-1)} \star_\partial \Psi \star_\partial = \widetilde{\Psi},$$

as desired.

To prove (28) we note that, by Green's formula,

$$\int_{\partial M} \varphi \wedge \Psi\psi = \int_{\partial M} i^*\omega \wedge i^*\delta\eta = (-1)^{n(k+1)+n+1} \int_{\partial M} i^*\omega \wedge i^*(\star\, d \star \eta)$$
$$= (-1)^{kn+1} \left(\langle d\omega, d \star \eta \rangle_{L^2(M)} - \langle \omega, \delta d \star \eta \rangle_{L^2(M)} \right). \tag{29}$$

Notice that

$$-\langle \omega, \delta d \star \eta \rangle_{L^2(M)} = \langle \omega, d\delta \star \eta \rangle_{L^2(M)}$$

since $0 = \star\, \Delta\eta = \Delta \star \eta = d\delta \star \eta + \delta d \star \eta$. In turn,

$$\langle \delta\omega, \delta \star \eta \rangle_{L^2(M)} = \langle \omega, d\delta \star \eta \rangle_{L^2(M)} - \int_{\partial M} i^*\delta \star \eta \wedge i^* \star \omega.$$

Since $i^*\star\omega = 0$, the second term on the right hand side vanishes. Therefore, we can re-write (29) as

$$\int_{\partial M} \varphi \wedge \Psi\psi = (-1)^{kn+1} \left(\langle d\omega, d \star \eta \rangle_{L^2(M)} + \langle \delta\omega, \delta \star \eta \rangle_{L^2(M)} \right). \tag{30}$$

Completely analogous reasoning yields the expression

$$\int_{\partial M} \psi \wedge \Psi\varphi = (-1)^{kn+n+1} \left(\langle d\eta, d \star \omega \rangle_{L^2(M)} + \langle \delta\eta, \delta \star \omega \rangle_{L^2(M)} \right) \tag{31}$$

Therefore, (28) follows from (30) and (31) because

$$\langle d\omega, d \star \eta \rangle_{L^2(M)} = \langle \star\, d\omega, \star\, d \star \eta \rangle_{L^2(M)} = (-1)^{k(n+1)} \langle \delta \star \omega, \delta\eta \rangle_{L^2(M)}$$
$$\langle \delta\omega, \delta \star \eta \rangle_{L^2(M)} = \langle \star\, \delta\omega, \star\, \delta \star \eta \rangle_{L^2(M)} = (-1)^{k(n+1)} \langle d \star \omega, d\eta \rangle_{L^2(M)}$$

(the first equality in each line is due to the fact that $\star$ is an isometry). □

5.2. **The operator Θ.** There are several different equivalent ways of expressing the operator $\Theta = (-1)^{(k+1)(n+1)} \Phi\Psi\Phi$. Using (9),

$$\Theta = (-1)^{(k+1)(n+1)} \Phi\Psi\Phi = (-1)^{kn} d_\partial \Phi^2. \tag{32}$$

On the other hand, using (11),

$$\Theta = (-1)^{(k+1)(n+1)} \Phi\Psi\Phi = (-1)^{n(k+1)} \Phi^2 d_\partial. \tag{33}$$

Finally, combining (12) with (33) yields

$$\Theta = (-1)^{n(k+1)} \Phi^2 d_\partial = (d_\partial \Psi + \Psi d_\partial) d_\partial = d_\partial \Psi d_\partial. \tag{34}$$

This last expression makes it clear that Θ is a cochain map:

$$\Theta^2 = d_\partial \Psi d_\partial d_\partial \Psi d_\partial = 0.$$

Proposition 5.3. *The cohomology of the cochain complex* $(\Omega^*(\partial M), \Theta)$ *is given, up to isomorphism, by*

$$H^k(\Omega^*(\partial M), \Theta) \simeq H^{k+1}(M, \partial M; \mathbb{R}) \oplus H^k(M; \mathbb{R}).$$

Notice that $(\Omega^*(\partial M), \Theta)$ has the same cohomology as $(\Omega^*(\partial M), \widetilde{\Psi})$.

We omit the proof of Proposition 5.3, which is somewhat long and technical, though not particularly difficult. Two perhaps surprising consequences are:

(i) Since Θ has the same cohomology as $\widetilde{\Psi}$, the homology of Ψ can be completely recovered from that of Θ. However, by (34), $\Theta = d_\partial \Psi d_\partial$, so pre- and post-composing Ψ by d_∂ does not change the (co)homology.

(ii) By (32) and (33),

$$\Theta = \pm d_\partial \Phi^2 = \pm \Phi^2 d_\partial.$$

Hence, the homology of Ψ is completely determined by the operator Φ, and the results of Corollaries 3 and 4 depend only on Φ. In that spirit, the following is a restatement of the $k = 0$ case of Corollary 3.

Corollary 5. *A copy of the cohomology group $H^{n-1}(M; \mathbb{R})$ is distinguished by the operator Φ inside $\Omega^0(\partial M)$, the space of smooth functions on ∂M. Specifically,*

$$\ker(d_\partial \Phi^2) / \ker \Phi \simeq H^{n-1}(M; \mathbb{R}).$$

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