LINKING INTEGRALS ON S^n

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1. Introduction

In [4], Gauss gave an integral formula for the linking number of two closed curves in Euclidean 3-space. Although Gauss' paper does not include a proof of the formula, Moritz Epple [3] suggests that Gauss had in mind a degree-of-map argument, though he undoubtedly also knew how to prove the formula using Ampère's Law.

Surprisingly, Gauss-type linking integrals have only recently been discovered in other spaces. In a forthcoming paper, Dennis DeTurck and Herman Gluck use an electrodynamics approach to find linking integral formulas in the 3-sphere S^3 and in hyperbolic 3-space H^3. Their basic strategy is to prove that Maxwell's equations hold in those spaces and then prove their formulas using Ampère's Law. This approach ends up being quite analytic, as they must solve for the Green and Biot-Savart operators.

In this paper, we pursue a more geometric approach to find Gauss-type linking integrals in S^n for all $n \geq 2$. Our integrals are equivalent to the integrals found via more analytic techniques in more recent work of DeTurck and Gluck [2] as well as those found via an alternative geometric technique by Greg Kuperberg [5].

1.1. Notation.

Before stating our main result (Theorem 1.1), we must first establish some notation. Throughout, we will consider oriented, smooth, closed submanifolds $K^k, L^\ell \subset S^n$ where $k + \ell = n - 1$, $k \leq \ell$ and S^n is the round n-sphere which we will usually consider as embedded in the usual way in $\mathbb{R}^{n+1}$. For reasons to be explained in Section 2, we will assume that K is disjoint both from L and from its antipodal image $-L$. Also, we think of K and L as being parametrized by local coordinates $s = (s_1, \ldots, s_k)$ and $t = (t_1, \ldots, t_\ell)$, respectively; that is, $K = \{x(s)\}$ and $L = \{y(t)\}$ for smooth functions x and y. We will denote the distance in S^n from $x(s)$ to $y(t)$ by $\alpha(s,t)$.

Also, for any $m \geq 1$ we will let $[V_1, \ldots, V_m]$ signify the determinant of the $m \times m$ matrix with columns given by the V_i. With that in mind, we will often need to make use of the expression

$$\left[x(s), y(t), \frac{\partial x(s)}{\partial s_1}, \ldots, \frac{\partial x(s)}{\partial s_k}, \frac{\partial y(t)}{\partial t_1}, \ldots, \frac{\partial y(t)}{\partial t_\ell}\right] ds \wedge dt, \tag{1}$$

which we abbreviate as $[x, y, dx, dy]$.

Finally, our linking integral will contain a convolution of sine functions, so we recall that, for any α (in practice, we will use α as defined above),

$$\sin^k * \sin^\ell(\alpha) := \int_0^{2\pi} \sin^k(\alpha-\beta)\sin^\ell(\beta)d\beta$$

1.2. **Statement of main theorem.** With all of the above notation in mind, we can finally state the main theorem:

Theorem 1.1. *With K, L, x, y and α as above,*

$$Lk(K,L) + (-1)^n Lk(K,-L) = \frac{(-1)^k}{\text{vol } S^n}\int_{K\times L}\frac{\sin^k * \sin^\ell(\alpha)}{\sin^n \alpha}[x,y,dx,dy] \tag{2}$$

where $Lk(K,L)$ denotes the linking number of K and L and $Lk(K,-L)$ denotes the linking number of K with the antipodal image of L.

Remark 1.2. The fact that the lefthand side of (2) has two terms is the price we must pay for the fact that a straightforward application of the degree-of-map argument fails in S^n.

2. The degree-of-map argument

First, recall the degree-of-map proof of the Gauss linking integral. Let K and L be smooth, oriented, disjoint, simple closed curves in $\mathbb{R}^3$ parametrized by $s,t \in [0,2\pi]$. Then $K = \{x(s)\}$ and $L = \{y(t)\}$ for smooth functions $x,y : S^1 \to \mathbb{R}^3$. Define the map $\varphi : S^1 \times S^1 \to S^2$ given by

$$\varphi(s,t) = \frac{x(s)-y(t)}{|x(s)-y(t)|}.$$

Let D be the projection of K and L to the xy-plane. Then, assuming K and L are generic with respect to projection to the xy-plane, inverse images under φ of the north pole correspond exactly to points in the knot diagram D where K crosses over L. In other words, since φ is a smooth map and (since K and L are generic with respect to projection to the xy-plane) the north pole is a regular value of φ, the degree of φ is equal to the linking number of K and L. A bit of calculation shows that, if ω is a volume form which evaluates to 1 on S^2,

$$Lk(K,L) = deg(\varphi) = \frac{1}{4\pi}\int_{K\times L}\varphi^*\omega = \frac{1}{4\pi}\int_{K\times L}\frac{dx}{ds}\times\frac{dy}{dt}\cdot\frac{x-y}{|x-y|^3}dsdt.$$

2.1. **Why doesn't this work in S^3?** Unfortunately, we can't use the same trick to find a linking integral in S^3. The key step in the above is that the configuration space of 2 distinct points in $\mathbb{R}^3$, $Conf_2(\mathbb{R}^3)$, deformation retracts to the 2-sphere (this is easy to see: translate so that the first point is at the origin, then scale until the second point has norm 1). Therefore, the map φ contains all the homotopy information of the configuration map $S^1 \times S^1 \to Conf_2(\mathbb{R}^3)$ given by

$$(s,t) \mapsto (x(s), y(t)).$$

On the other hand, the configuration space of two distinct points in S^3, $Conf_2(S^3)$, is three-dimensional. To see this, suppose $(p, q) \in Conf_2(S^3)$. Let γ be a geodesic from p to q. Then simply translate q along γ until it is antipodal to p. In this way, we see that $Conf_2(S^3)$ deformation retracts to the "anti-diagonal" S^3. The S^3 equivalent of the map φ defined above should, therefore, be a map $\phi : S^1 \times S^1 \to S^3$. However, any such map is clearly homotopically trivial and, therefore, its degree tells us nothing about the linking number.

2.2. **How to fix this argument.** Another way to think of the map $\varphi : S^1 \times S^1 \to S^2$ defined above is as a map $\varphi : S^1 \times S^1 \to U\mathbb{R}^3$, the unit tangent bundle of $\mathbb{R}^3$, given by

$$(s,t) \mapsto \left(x(s), \frac{x(s) - y(t)}{|x(s) - y(t)|} \right). \tag{3}$$

If we let $\eta = dx \wedge dy \wedge dz$ be the standard volume form on $\mathbb{R}^3$ and let ω be a volume form on S^2, then, since $U\mathbb{R}^3$ is trivial (i.e. homeomorphic to $\mathbb{R}^3 \times S^2$), $eta \wedge \omega$ is a volume form on $U\mathbb{R}^3$ which restricts to the volume form on the fiber. Then, if φ is as defined in (3),

$$Lk(K, L) = deg(\varphi)$$

If we want to define an S^3 version of φ, one reasonable thing to try is to define $f(p, q) = (p, v)$, where $v \in T_pS^3$ is the unit vector at p pointing in the direction of the geodesic from p to q. Of course, if q is antipodal to p, there are infinitely many geodesics from p to q, so this map is not well-defined.

However, if we add the restriction that our closed curves K and L in S^3 must not only be smooth and disjoint, but also disjoint from the other's antipodal image, then there is no problem (locally) a map f in this way. More precisely, for $K = \{x(s)\}$ and $L = \{y(t)\}$, we define $f : S^1 \times S^1 \to US^3$ by

$$f(s, t) = (x(s), v(s, t)),$$

where $v(s, t) \in T_pS^3$ is the unit vector tangent to the unique geodesic from $x(s)$ to $y(t)$.

In fact, it is just as straightforward to define such an $f : K \times L \to US^n$ for any n.

Therefore, in analogy with the argument outlined above in the case of $\mathbb{R}^3$, our goal is to find an $SO(n+1)$-invariant $(n-1)$-form ω_{n-1} on US^n which restricts to the volume form on the S^{n-1} fiber. Then we will show that

$$Lk(K, L) + (-1)^n Lk(K, -L) = \frac{1}{\text{vol } S^{n-1}} \int_{K \times L} f^* \omega_{n-1}. \tag{4}$$

Unfortunately, as we will see, such a form only exists when n is odd. Therefore, the proof of Theorem 1.1 will split into two cases. The case where n is odd is addressed in Section 3: we will find the form ω_{n-1} in Section 3.1, prove (4) in Section 3.2 and derive (2) in Section 3.3. A completely different argument is used in Section 4 to show that, if Theorem 1.1 is true for odd spheres, it is also true for even spheres.

3. Theorem 1.1 for odd spheres

3.1. Invariant forms on the unit tangent bundle of S^n. For positive n, let US^n denote the unit tangent bundle of the sphere S^n. The goal in this section is to prove the following proposition:

Proposition 3.1. *For odd n, there exists a closed $SO(n+1)$-invariant $(n-1)$-form ω_{n-1} on US^n which restricts to the volume form on the fibers of US^n.*

Almost as important as the statement of the proposition is the formula for ω_{n-1}, which is given below in (5).

Note that the fact that n is odd is essential: a simple application of the Gysin sequence demonstrates that the deRham cohomology of US^n is given by

$$H^k_{\mathrm{dR}}(US^n) \simeq \begin{cases} H^k_{\mathrm{dR}}(S^{2n-1}) & \text{for } n \text{ odd} \\ H^k_{\mathrm{dR}}(S^{n-1}\times S^n) & \text{for } n \text{ even} \end{cases}$$

so there's no point to looking for such a closed form in US^n for n even.

Proof of Proposition 3.1. First, we establish some terminology. Let $E_1, \dots, E_{n+1}$ denote the standard basis of $\mathbb{R}^{n+1}$ and let $M_{i,j}(\theta) \in SO(n+1)$ be the one-parameter group of isometries of $\mathbb{R}^{n+1}$ which spin the x_ix_j-plane by an angle θ and leave the complement of this plane fixed. Let $E_{i,j}$ be the tangent vector to this family of rotations at the identity. Then $\{E_{i,j} | 1 \leq i \neq j \leq n+1\}$ forms a basis of the Lie algebra $\mathfrak{so}(n+1)$. Let $V_{i,j}$ denote the left-invariant vector field on $SO(n+1)$ determined by $E_{i,j}$ and let $\Phi_{i,j}$ denote the dual left-invariant 1-form.

Now, consider the point $(E_1, E_2) \in US^n$. The tangent vectors to the vertical fiber at this point have $\{E_{2,j} | 2 < j \leq n+1\}$ as a basis, while $\{E_{1,j} | 1 < j \leq n+1\}$ forms a basis for the horizontal tangent vectors. Let $\varphi_{i,j}$ be the 1-forms on $US^n = SO(n+1)/SO(n-1)$ corresponding to the $\Phi_{i,j}$ defined above (of course, $\varphi_{i,j}$ only makes sense for $i = 1, 2$). We are interested in finding a closed $SO(n+1)$-invariant $(n-1)$-form on US^n which restricts to the volume form $\varphi_{2,3} \wedge \ldots \wedge \varphi_{2,n+1}$ on the fiber.

Unfortunately the fiber form $\varphi_{2,3} \wedge \ldots \wedge \varphi_{2,n+1}$ is not closed on US^n. However, it is invariant under the $SO(n-1)$ action which fixes the x_1x_2-plane in $\mathbb{R}^{n+1}$. Moreover, we can make the fiber form closed by averaging it over the $SO(2)$-action which spins the x_1x_2-plane as follows:

(5)
$$\omega_{n-1} = \frac{1}{\int_0^{2\pi} \sin^{n-1}\beta d\beta} \int_0^{2\pi} (\cos\beta\varphi_{1,3} - \sin\beta\varphi_{2,3}) \wedge \ldots \wedge (\cos\beta\varphi_{1,n+1} - \sin\beta\varphi_{2,n+1}) d\beta,$$

Thus defined, ω_{n-1} certainly restricts to the fiber form. Also, the extra $SO(2)$ averaging makes it the pullback of an $SO(n+1)$-invariant form in the Grassmann manifold $G_2\mathbb{R}^{n+1}$ of oriented 2-planes through the origin of $\mathbb{R}^{n+1}$. Since the Grassmann manifold is a symmetric space, all invariant forms are closed, so we see that ω_{n-1} is the desired closed, $SO(n+1)$-invariant form on US^n. □

3.2. **Linking numbers via pullbacks of invariant forms.** Now that we have our closed, $SO(n+1)$-invariant $(n-1)$-form ω_{n-1} on US^n for odd n, we want to prove the following proposition:

Proposition 3.2. *Suppose that ω_{2n} is a closed, $SO(2n+2)$-invariant $(2n)$-form on US^{2n+1} such that ω_{2n} restricts to the volume form on the fiber. Moreover, suppose K^k and L^ℓ are smooth, oriented, closed submanifolds of S^{2n+1} such that $k+\ell = 2n$ and K and L are disjoint from each other and from the other's antipodal image. Let $K = \{x(s)\}$ where $s = (s_1, \ldots, s_k)$ and $L = \{y(t)\}$ where $t = (t_1, \ldots, t_\ell)$. Then*

$$Lk(K, L) - Lk(K, -L) = \frac{1}{vol\ S^{2n}} \int_{K\times L} f^*\omega_{2n} \tag{6}$$

where $Lk(\cdot,\cdot)$ denotes the linking number of two manifolds, $-L$ denotes the antipodal image of L and $f : K \times L \to US^n$ is given by

$$f(s, t) = (x(s), v(s, t))$$

where $v(s,t)$ is the unit vector in $T_{x(s)}S^{2n+1}$ tangent to the unique geodesic from $x(s)$ to $y(t)$.

Remark: (6) is, spiritually, the exact analogue of Gauss' linking integral in $\mathbb{R}^3$, in which the linking number of two curves is given by $1/4\pi$ times the integral of the pullback of the volume form on S^2 (i.e. the fiber of the unit tangent bundle to $\mathbb{R}^3$). The assumption that K and L are disjoint not only from each other but from the other's antipodal image is required so that the vector v (and thus the map f) is well-defined.

To prove Proposition 3.2, we will construct a singular "annulus" A bounded by L and $-L$ such that $Lk(K, L) - Lk(K, -L)$ is equal to the signed intersection number of K with A. Then, we will demonstrate that the signed intersection number of K with A is equal to the right hand side of (6).

Lemma 3.3. *Let K and L be submanifolds of S^{2n+1} satisfying the hypotheses of Proposition 3.2. Then*

$$Lk(K, L) - Lk(K, -L) = K \cdot A$$

where A is a singular manifold of dimension $\ell + 1$ such that $\partial A = L - (-L)$ and $K \cdot A$ denotes the signed intersection number of K with A.

Proof. Let $h : S^{2n+1} \to \mathbb{CP}^n$ be the Hopf map and let H denote the corresponding collection of oriented great circles in S^{2n+1}. Each point $y(t)$ of L lies on some oriented Hopf circle $C(t) \in H$, as does its antipodal image

$-y(t)$. Let $S(t)$ denote the oriented semi-circle from $-y(t)$ to $y(t)$ which follows the orientation of $C(t)$. Now, define

$$A := \bigcup_t S(t),$$

oriented so that $\partial A = L - (-L)$.

Note that A can be quite singular; for example, of L is itself one of the Hopf circles, then A coincides with L. However, note that the left hand side of (6) is invariant under homotopies which keep K disjoint from both L and $-L$ and, therefore, under short homotopies. Moreover, the right hand side of (6) is invariant under short homotopies since ω_{2n} is a closed form.

Therefore, we are free to perturb L slightly so that it intersects all the Hopf circles transversally. As a result, $h|_L : L \to \mathbb{CP}^n$ is an immersion. We can perturb L a bit more so that this immersion has only finitely many double points, at all of which two branches meet transversally, and no triple points. Since the images of L, $-L$ and A under h all coincide, the result of these perturbations is that A is an immersed $(\ell+1)$-manifold with one arc of double points for each isolated double point in $h(L)$.

Now, we claim that $Lk(K, L) - Lk(K, -L) = K \cdot A$. Given the construction of A, this is straightforward. Let D be an $(\ell+1)$-cycle in S^{2n+1} bounded by $-L$. Then $A \cup D$ is an $(\ell+1)$-cycle bounded by L. Hence, by definition of the linking number,

$$Lk(K, -L) = K \cdot D$$

and

$$Lk(K, L) = K \cdot (A \cup D).$$

Therefore,

$$Lk(K, L) - Lk(K, -L) = K \cdot A,$$

as desired. □

Now, we want to show that $K \cdot A$ is equal to the right hand side of (6):

Lemma 3.4. *Under the hypotheses of Proposition 3.2,*

$$\frac{1}{vol\ S^{2n}} \int_{K \times L} f^* \omega_{2n} = K \cdot A,$$

with A defined as in Lemma 3.3.

Proof. Let H be the Hopf fibration of S^{2n+1} considered above. Let v_H be the Hopf vector field on S^{2n+1}; that is, for $p \in S^{2n+1}$, $v_H(p)$ is the unit vector at p tangent to the oriented Hopf circle through p. Then

$$S_H^{2n+1} := \{(p, v_H(p))\} \subset US^{2n+1}$$

is a section of the unit tangent bundle

$$\begin{array}{ccc} S^{2n} & \longrightarrow & US^{2n+1} \\ & & \big\downarrow \pi \\ & & S^{2n+1} \end{array}$$

and meets each fiber $\pi^{-1}(p)$ transversally at the single point $(p, v_H(p))$.

Now, each fiber $\pi^{-1}(p)$ is a cycle representing the generator of $H_{2n}(US^{2n+1};\mathbb{Z}) \simeq \mathbb{Z}$ and each section of US^{2n+1} (e.g. S^{2n+1}_H) is a cycle representing the generator of $H_{2n+1}(US^{2n+1};\mathbb{Z}) \simeq \mathbb{Z}$. Choose orientations so that the intersection number of $\pi^{-1}(p)$ with S^{2n+1}_H is $+1$.

Since ω_{2n} represents a $2n$-dimensional cohomology class of US^{2n+1} which restricts to the volume form on each fiber and since each fiber generates the $2n$-dimensional homology of US^{2n+1}, we see that, for any $2n$-cycle c_{2n},

$$\int_{c_{2n}} \frac{1}{\text{vol } S^{2n}}\omega_{2n} = c_{2n} \cdot S^{2n+1}_H$$

Applied to the $2n$-cycle $c_{2n} = f(K \times L)$, this formula tells us that

$$(7) \qquad \frac{1}{\text{vol } S^{2n}} \int_{K\times L} f^*\omega_{2n} = \int_{f(K\times L)} \frac{1}{\text{vol } S^{2n}}\omega_{2n} = f(K \times L) \cdot S^{2n+1}_H.$$

Since $f(K \times L) = \{(x(s), v(s,t))\}$, we see that $f(K \times L)$ meets S^{2n+1}_H whenever

$$v(s,t) = v_H(x(s)),$$

which occurs precisely when K intersects A. Therefore, by our choices of orientations,

$$\frac{1}{\text{vol } S^{2n}} \int_{K\times L} f^*\omega_{2n} = K \cdot A,$$

□

To complete the proof of Proposition 3.2, we simply note that combining Lemma 3.3 and Lemma 3.4 yields:

$$Lk(K,L) - Lk(K,-L) = K \cdot A = f(K \times L) \cdot S^{2n+1}_H = \frac{1}{\text{vol } S^{2n}} \int_{K\times L} f^*\omega_{2n}.$$

3.3. **Deriving the convolution formula.** With Proposition 3.2, it now simply remains to show that the right hand side of (6) is equal to the right hand side of (2) when n is odd in order to complete the proof of Theorem 1.1 for odd n. In other words, we want to prove the following proposition:

Proposition 3.5. *Suppose n is odd. Let $K^k, L^\ell \subset S^n$ satisfy the conditions of Theorem 1.1. Let ω_{n-1} be given by (5) and let f be as defined in Proposition 3.2. Then*

$$(8) \qquad \frac{1}{vol\ S^{n-1}} \int_{K\times L} f^*\omega = \frac{(-1)^k}{vol\ S^n} \int_{K\times L} \frac{sin^k * \sin^\ell(\alpha)}{\sin^n \alpha}[x, y, dx, dy].$$

Proof. Recall that $K = \{x(s)\}$, that $L = \{y(t)\}$ where $s = (s_1, \ldots, s_k)$ and $t = (t_0, \ldots, t_\ell)$ and that $v(s,t)$ is the unit vector at $x(s)$ pointing to $y(t)$. Since ω_{n-1} is $SO(n+1)$-invariant, we are free work at the point $(s_0, t_0) = (0,0)$ and to assume that $x(0) = E_1$ and $v(0,0) = E_2$ (where the E_i are the standard basis for $\mathbb{R}^{n+1}$).

Although $x(s)$ does not depend on t, assume, for the moment, that x is a function of s and t. Then, to first order,

$$\begin{aligned} x(s,t) &= E_1 + (a_{2,1}s_1 + \ldots + a_{2,k}s_k + b_{2,1}t_1 + \ldots + b_{2,\ell}t_\ell)E_2 \\ &\quad + \ldots + (a_{n+1,1}s_1 + \ldots + a_{n+1,k}s_k + b_{n+1,1}t_1 + \ldots + b_{n+1,\ell}t_\ell)E_{n+1} \\ v(s,t) &= (c_{1,1}s_1 + \ldots + c_{1,k}s_k + d_{1,1}t_1 + \ldots + d_{1,\ell}t_\ell)E_1 + E_2 \\ &\quad +(c_{3,1}s_1 + \ldots + c_{3,k}s_k + d_{3,1}t_1 + \ldots + d_{3,\ell}t_\ell)E_3 + \ldots \\ &\quad +(c_{n+1,1}s_1 + \ldots + c_{n+1,k}s_k + d_{n+1,1}t_1 + \ldots + d_{n+1,\ell}t_\ell)E_{n+1}. \end{aligned}$$

Since $v(s,t) \in T_{x(s,t)}S^n$, $\langle x(s,t), v(s,t)\rangle = 0$ (where this is the inner product in $\mathbb{R}^{n+1}$). Differentiating this relation with respect to the s_i and the t_j yields to following system of equations:

$$\begin{aligned} a_{2,i} + c_{1,i} = 0 \\ b_{2,j} + d_{1,j} = 0 \end{aligned}$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, \ell$.

From their first-order expansions, we see that differentiating x and v gives:

$$\begin{aligned} \frac{\partial x}{\partial s_i} &= a_{2,i}E_2 + \ldots + a_{n+1,i}E_{n+1} \\ \frac{\partial v}{\partial s_i} &= c_{1,i}E_1 + c_{3,i}E_3 + \ldots + c_{n+1,i}E_{n+1} \\ \frac{\partial x}{\partial t_j} &= b_{2,j}E_2 + \ldots + b_{n+1,j}E_{n+1} \\ \frac{\partial v}{\partial t_j} &= d_{1,j}E_1 + d_{3,j}E_3 + \ldots + d_{n+1,j}E_{n+1}. \end{aligned}$$

Therefore,

$$f^*\omega_{n-1}\left(\frac{partial}{\partial s_1}, \ldots, \frac{partial}{\partial s_k}, \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_\ell}\right) = \omega_{n-1}(A_1, \ldots, A_k, B_1, \ldots, B_\ell)$$

where

$$\begin{aligned} A_i &= \left(\frac{\partial x}{\partial s_i}, \frac{\partial v}{\partial s_i}\right) \\ &= (a_{2,i}E_2 + \ldots + a_{n+1,i}E_{n+1}, c_{1,i}E_1 + c_{3,i}E_3 + \ldots + c_{n+1,i}E_{n+1}) \\ &= (a_{2,i}E_2 + \ldots + a_{n+1,i}E_{n+1}, -a_{2,i}E_1 + c_{3,i}E_3 + \ldots + c_{n+1,i}E_{n+1}) \\ &= a_{2,i}(E_2, -E_1) + a_{3,i}(E_3, 0) + \ldots + a_{n+1,i}(E_{n+1}, 0) + c_{3,i}(0, E_3) + \ldots + c_{n+1,i}(0, E_{n+1}) \\ &= a_{2,i}e_{1,2} + a_{3,i}e_{1,3} + \ldots + a_{n+1,i}e_{1,n+1} + c_{3,i}e_{2,3} + \ldots + c_{n+1,i}e_{2,n+1} \end{aligned}$$

and, similarly,

$$\begin{aligned} B_j &= \left(\frac{\partial x}{\partial t_j}, \frac{\partial v}{\partial t_j}\right) \\ &= b_{2,j}e_{1,2} + b_{3,j}e_{1,3} + \ldots + b_{n+1,j}e_{1,n+1} + d_{3,j}e_{2,3} + \ldots + d_{n+1,j}e_{2,n+1} \end{aligned}$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, \ell$.

Of course, $x(s,t) = x(s)$ so $\frac{\partial x}{\partial t_j} = 0$, meaning that $b_{\nu,\eta} = 0$ for all $\nu = 2,\dots,n+1$ and $\eta = 1,\dots,k$. Therefore,

$$\begin{aligned}
&\omega_{n-1}(A_1,\dots,A_k,B_1,\dots,B_\ell)\\
= \ &\omega_{n-1}(a_{2,1}e_{1,2} + a_{3,1}e_{1,3} + \dots + a_{n+1,1}e_{1,n+1} + c_{3,1}e_{2,3} + \dots + c_{n+1,1}e_{2,n+1},\\
&\qquad \vdots \qquad\qquad\qquad \vdots\\
&a_{2,k}e_{1,2} + a_{3,k}e_{1,3} + \dots + a_{n+1,k}e_{1,n+1} + c_{3,k}e_{2,k} + \dots + c_{n+1,k}e_{2,n+1},\\
&\qquad d_{3,1}e_{2,3} + \dots + d_{n+1,1}e_{2,n+1},\\
&\qquad \vdots\\
&\qquad d_{3,\ell}e_{2,3} + \dots + d_{n+1,\ell}e_{2,n+1}).
\end{aligned} \tag{9}$$

Having analyzed $f^*\omega_{n-1}$ as much as we can without actually writing down ω_{n-1}, we now recall from (5) that

$$\begin{aligned}
\omega_{n-1} \ &:= \ \frac{1}{\int_0^{2\pi}\sin^{k+\ell}\beta d\beta}\int_0^{2\pi}(\cos\beta\varphi_{1,3} - \sin\beta\varphi_{2,3})\wedge\dots\wedge(\cos\beta\varphi_{1,n+1} - \sin\beta\varphi_{2,n+1})d\beta\\
&= \ \frac{1}{\int_0^{2\pi}\sin^{n-1}\beta d\beta}\Bigg[\int_0^{2\pi}\cos^{n-1}\beta\varphi_{1,3}\wedge\dots\wedge\varphi_{1,n+1}d\beta\\
&\qquad - \int_0^{2\pi}\cos^{n-2}\beta\sin\beta\xi_{n-2}d\beta\\
&\qquad + \int_0^{2\pi}\cos^{n-3}\beta\sin^2\beta\xi_{n-3}d\beta\\
&\qquad - \dots\\
&\qquad + \int_0^{2\pi}\cos^2\beta\sin^{n-3}\beta\xi_2 d\beta\\
&\qquad - \int_0^{2\pi}\cos\beta\sin^{n-2}\beta\xi_1 d\beta\\
&\qquad + \int_0^{2\pi}\sin^{n-1}\beta\varphi_{23}\wedge\dots\wedge\varphi_{2,n+1}d\beta\Bigg]\\
&= \ \frac{1}{\int_0^{2\pi}\sin^{n-1}\beta d\beta}\left[\sum_{i=0}^{n-1}(-1)^i\int_0^{2\pi}\cos^{n-1-i}\beta\sin^i\beta\xi_{n-1-i}d\beta\right]\\
&= \ \frac{1}{\int_0^{2\pi}\sin^{n-1}\beta dt}\left[\sum_{i=0}^{n-1}(-1)^{n-1-i}\int_0^{2\pi}\cos^i\beta\sin^{n-1-i}\beta\xi_i d\beta\right],
\end{aligned} \tag{10}$$

where ξ_a is the sum of all possible simple $(n-1)$-forms in the $\varphi_{i,j}$ with a terms of the form $\varphi_{1,i}$ and $n-1-a$ terms of the form $\varphi_{2,j}$.

Note that, in (9), there are only k entries with terms of the form $e_{1,i}$, so $\xi_\eta(A_1,\dots,A_k,B_1,\dots,B_\ell) = 0$ for all $\eta > k$.

Now, $\xi_0(A_1, \ldots, A_k, B_1, \ldots, B_\ell) = \varphi_{2,3} \wedge \ldots \varphi_{2,n+1}(A_1, \ldots, A_k, B_1, \ldots, B_\ell)$ is simply

$$\det \begin{pmatrix} c_{3,1} & \cdots & c_{n+1,1} \\ \vdots & & \vdots \\ c_{3,k} & \cdots & c_{n+1,k} \\ d_{3,1} & \cdots & c_{n+1,1} \\ \vdots & & \vdots \\ d_{3,\ell} & \cdots & d_{n+1,\ell} \end{pmatrix},$$

which is equal to

$$\left[x, v, \frac{\partial v}{\partial s_1}, \ldots, \frac{\partial v}{\partial s_k}, \frac{\partial v}{\partial t_1}, \ldots, \frac{\partial v}{\partial t_\ell}\right].$$

In turn, since $v = \frac{y - \cos\alpha x}{\sin\alpha}$, this is just

$$(-1)^k \frac{\cos^k \alpha}{\sin^n \alpha} \left[x, y, \frac{\partial x}{\partial s_1}, \ldots, \frac{\partial x}{\partial s_k}, \frac{\partial y}{\partial t_1}, \ldots, \frac{\partial y}{\partial t_\ell}\right]. \tag{11}$$

Similarly, $\xi_2(A_1, \ldots, A_k, B_1, \ldots, B_\ell)$ is given by

(12)

$$\det \begin{pmatrix} a_{3,1} & \cdots & a_{n+1,1} \\ a_{3,2} & \cdots & a_{n+1,2} \\ c_{3,3} & \cdots & c_{n+1,3} \\ \vdots & & \vdots \\ c_{3,k} & \cdots & c_{n+1,k} \\ d_{3,1} & \cdots & d_{n+1,1} \\ \vdots & & \vdots \\ d_{3,k} & \cdots & d_{n+1,\ell} \end{pmatrix} + \det \begin{pmatrix} a_{3,1} & \cdots & a_{n+1,1} \\ c_{3,2} & \cdots & c_{n+1,2} \\ a_{3,3} & \cdots & a_{n+1,3} \\ c_{3,4} & \cdots & c_{n+1,4} \\ \vdots & & \vdots \\ c_{3,k} & \cdots & c_{n+1,k} \\ d_{3,1} & \cdots & d_{n+1,1} \\ \vdots & & \vdots \\ d_{3,k} & \cdots & d_{n+1,\ell} \end{pmatrix} + \ldots + \det \begin{pmatrix} c_{3,1} & \cdots & c_{n+1,1} \\ \vdots & & \vdots \\ c_{3,k-2} & \cdots & c_{n+1,k-2} \\ a_{3,k-1} & \cdots & a_{n+1,k-1} \\ a_{3,k} & \cdots & a_{n+1,k} \\ d_{3,1} & \cdots & d_{n+1,1} \\ \vdots & & \vdots \\ d_{3,k} & \cdots & d_{n+1,\ell} \end{pmatrix}$$

(i.e. all determinants where there are two rows of a's in the first k rows and the rest of the first k rows consist of c's). In turn, the first term of (12) is just

$$\left[x, v, \frac{\partial x}{\partial s_1}, \frac{\partial x}{\partial s_2}, \frac{\partial v}{\partial s_3}, \ldots, \frac{\partial v}{\partial s_k}, \frac{\partial v}{\partial t_1}, \ldots, \frac{\partial v}{\partial t_\ell}\right],$$

which, using the definition of v, reduces to

$$(-1)^{k-2} \frac{\cos^{k-2} \alpha}{\sin^{n-2} \alpha} \left[x, y, \frac{\partial x}{\partial s_1}, \ldots, \frac{\partial x}{\partial s_k}, \frac{\partial y}{\partial t_1}, \ldots, \frac{\partial y}{\partial t_\ell}\right].$$

The other terms in (12) reduce to the same thing, so

$$\begin{aligned} &\xi_2(A_1, \ldots, A_k, B_1, \ldots, B_\ell) \\ &\qquad = \binom{k}{2} (-1)^{k-2} \frac{\cos^{k-2} \alpha}{\sin^{n-2} \alpha} \left[x, y, \frac{\partial x}{\partial s_1}, \ldots, \frac{\partial x}{\partial s_k}, \frac{\partial y}{\partial t_1}, \ldots, \frac{\partial y}{\partial t_\ell}\right] \end{aligned} \tag{13}$$

In fact, in general,

$$\xi_i(A_1,\dots,A_k,B_1,\dots,B_k) = \binom{k}{i}(-1)^{k-i}\frac{\cos^{k-i}\alpha}{\sin^{n-i}\alpha}\left[x,y,\frac{\partial x}{\partial s_1},\dots,\frac{\partial x}{\partial s_k},\frac{\partial y}{\partial t_1},\dots,\frac{\partial y}{\partial t_\ell}\right] \tag{14}$$

for $i \le k$ and $\xi_i(A_1,\dots,A_k,B_1,\dots,B_\ell) = 0$ for $i > k$

Therefore, combining (10) and (14),

$$\omega_{n-1}(A_1,\dots,A_k,B_1,\dots B_\ell) = \frac{1}{\int_0^{2\pi}\sin^{n-1}\beta d\beta}\varphi_{k,\ell}(\alpha)\left[x,y,\frac{\partial x}{\partial s_1},\dots,\frac{\partial x}{\partial s_k},\frac{\partial y}{\partial t_1},\dots,\frac{\partial y}{\partial t_\ell}\right] \tag{15}$$

where

$$\begin{aligned}\varphi_{k,\ell}(\alpha) &= \sum_{i=0}^{k}(-1)^{k-i}(-1)^{n-1-i}\binom{k}{i}\int_0^{2\pi}\cos^i\beta\sin^{n-1-i}\beta d\beta\frac{\cos^{k-i}\alpha}{\sin^{n-i}\alpha}\\ &= (-1)^k\int_0^{2\pi}\sum_{i=0}^{k}\left(\binom{k}{i}\sin^i\alpha\cos^{k-i}\alpha\sin^{n-1-i}\beta\cos^i\beta\right)d\beta\frac{1}{\sin^n\alpha}\\ &= (-1)^k\int_0^{2\pi}\sum_{i=0}^{k}\left(\binom{k}{i}\sin^i\alpha\cos^{k-i}\alpha\sin^{2k-i}\beta\cos^i\beta\right)\sin^{\ell-k}\beta d\beta\frac{1}{\sin^n\alpha}\end{aligned}$$

where we've used the facts that $n-1 = k+\ell$ and $n-1$ is even. In turn, note that the sum is simply the binomial expansion of $(\sin\alpha\sin\beta\cos\beta + \cos\alpha\sin^2\beta)^k$. Since

$$\sin\alpha\sin\beta\cos\beta + \cos\alpha\sin^2\beta = \sin(\alpha+\beta)\sin\beta,$$

and

$$\int_0^{2\pi}\sin^k(\alpha+\beta)\sin^\ell\beta d\beta = \int_0^{2\pi}\sin^k(\alpha-\beta)\sin^\ell\beta = 2\sin^k * \sin^\ell,$$

we see that (15) is equal to:

$$\begin{aligned}&\frac{(-1)^k\int_0^{2\pi}\left(\sin\alpha\sin\beta\cos\beta - \cos\alpha\sin^2\beta\right)^k\sin^{\ell-k}\beta d\beta}{\int_0^{2\pi}\sin^{n-1}\beta d\beta}\frac{1}{\sin^n\alpha}\left[x,y,\frac{\partial x}{\partial s_1},\dots,\frac{\partial x}{\partial s_k},\frac{\partial y}{\partial t_1},\dots,\frac{\partial y}{\partial t_\ell}\right]\\ &= \frac{(-1)^k\cdot 2\sin^k * \sin^\ell(\alpha)}{\int_0^{2\pi}\sin^{n-1}\beta d\beta}\frac{1}{\sin^n\alpha}\left[x,y,\frac{\partial x}{\partial s_1},\dots,\frac{\partial x}{\partial s_k},\frac{\partial y}{\partial t_1},\dots,\frac{\partial y}{\partial t_\ell}\right]\\ &= \frac{(-1)^k\sin^k * \sin^\ell(\alpha)}{\int_0^{\pi}\sin^{n-1}\beta d\beta}\frac{1}{\sin^n\alpha}\left[x,y,\frac{\partial x}{\partial s_1},\dots,\frac{\partial x}{\partial s_k},\frac{\partial y}{\partial t_1},\dots,\frac{\partial y}{\partial t_\ell}\right].\end{aligned}$$

Thus we arrive at the desired (8):

$$\frac{1}{\text{vol } S^{n-1}}\int_{K\times L} f^*\omega_{n-1} = \frac{(-1)^k}{\text{vol } S^n}\int_{K\times L}\frac{\sin^k * \sin^\ell(\alpha)}{\sin^n\alpha}[x,y,dx,dy],$$

completing the proof of Proposition 3.5. ◻

Now, combining Proposition 3.2 with Proposition 3.5 immediately gives Theorem 1.1 for odd n.

4. Moving from odd to even spheres

Now that we've proved Theorem 1.1 for odd spheres, it remains only to show that it is also true for even spheres. To that end, it is sufficient to prove the following proposition:

Proposition 4.1. *The truth of Theorem 1.1 for a given odd value of n implies that (2) holds for oriented smooth closed submanifolds K^k, L^ℓ of S^{n-1} where $k+\ell = n-2$ and $k \leq \ell$.*

We will prove this proposition by taking K and L in S^{n-1}, embedding them in S^n by suspending L and then showing that the linking formula which holds in S^n, when applied to K and the suspension of L, gives the desired formula for linking in S^{n-1}.

4.1. **The setup.** In general, let Σ denote suspension. That is, for a manifold X, ΣX is the suspension of X. We know that, for any n, ΣS^{n-1} is homeomorphic to S^n. Given an orientation of S^{n-1}, we will orient $\Sigma S^{n-1} = S^n$ by using the orientation of S^{n-1} followed by the orientation of the quarter circle from S^{n-1} to the north pole of S^n.

For a smooth submanifold $L \subset S^{n-1}$, a copy of ΣL sits inside $\Sigma S^{n-1} = S^n$ by adding the quarter circles from each point of L to the north and south poles of S^n. ΣL is smooth in S^n except at the north and south poles and we orient ΣL by using the orientation of L followed by that of the quarter circle from a point of L to the north pole of S^n.

In such a way, ΣL is a cycle so linking numbers make sense. Moreover, since ΣL is smooth except at two points, our linking integrals will make sense as well.

Moreover, for $K^k, L^\ell \subset S^{n-1}$,

$$Lk(K, L) = Lk(K, \Sigma L) \tag{16}$$

where on the lefthand side we're taking the linking number in S^{n-1} and on the righthand side we're taking linking in S^n.

Using a minus sign to denote antipodal images, $\Sigma(-L)$ and $-\Sigma L$ coincide as subsets of S^n but have opposite orientations. Therefore, from (16),

$$Lk(K, -L) = Lk(K, \Sigma(-L)) = -Lk(K, -\Sigma L).$$

Thus, it follows that

$$Lk(K, L) + (-1)^{n-1} Lk(K, -L) = Lk(K, \Sigma L) + (-1)^n Lk(K, -\Sigma L), \tag{17}$$

where, again, the lefthand side deals with linking in S^{n-1} and the righthand side with linking in S^n.

The strategy is to use (17) to prove Proposition 4.1.

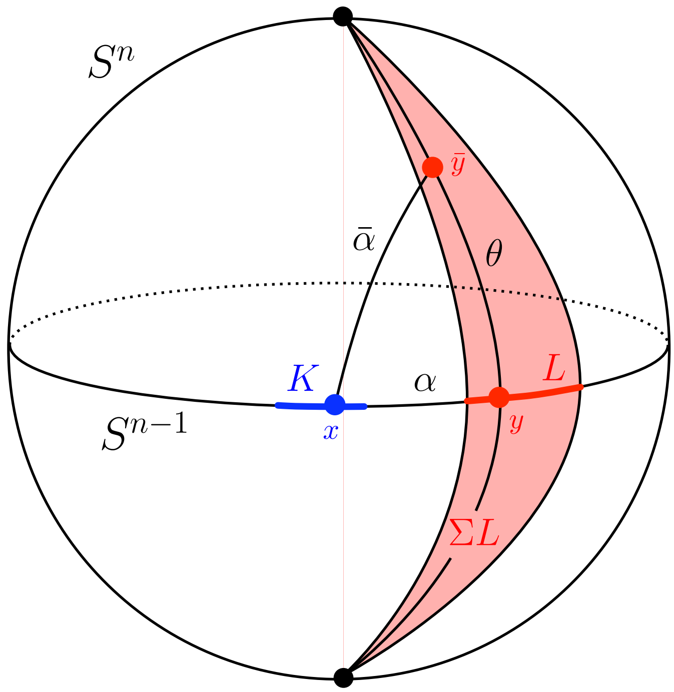

FIGURE 1. The suspension of L in S^n

A sketch of the situation is given in Figure 1. In the sketch, α is the S^{n-1} distance from $x(s) \in K$ to $y(t) \in L$, $\bar{\alpha}$ is the S^n distance from $x(s) \in K$ to a point $\bar{y} \in \Sigma L$ and θ ranges from $-\pi/2$ at the south pole to $\pi/2$ at the north pole.

Note that, from the spherical law of cosines, we have

$$\cos\bar{\alpha} = \cos\theta\cos\alpha \tag{18}$$

4.2. **Proof of Proposition 4.1.** Let $K = \{x(s)\}$ and $L = \{y(t)\}$ be parametrizations of K and L, where $s = (s_1, \dots, s_k)$ and $t = (t_1, \dots, t_\ell)$. Then

$$\bar{x}(s) = (x(s), 0)$$

and

$$\bar{y}(t, \theta) = (\cos\theta y(t), \sin\theta)$$

are parametrizations of K and ΣL, respectively, in S^n, where $\theta \in [-\pi/2, \pi/2]$.

Let $\bar{\alpha}$ be the distance in S^n from $\bar{x}$ to $\bar{y}$. The fact that Theorem 1.1 holds for n combined with (17) tells us that:

$$Lk(K, L) + Lk(K, -L) = \frac{(-1)^k}{\text{vol } S^n} \int_{K\times\Sigma L} \frac{\sin^k * \sin^{\ell+1}(\bar{\alpha})}{\sin^n \bar{\alpha}} [\bar{x}, \bar{y}, d\bar{x}, d\bar{y}]. \tag{19}$$

First, let's analyze the term $[\bar{x}, \bar{y}, d\bar{x}, d\bar{y}]$. By definition,

$$[\bar{x}, \bar{y}, d\bar{x}, d\bar{y}] = \left[\bar{x}, \bar{y}, \frac{\partial\bar{x}}{\partial s_1}, \dots, \frac{\partial x}{\partial s_k}, \frac{\partial\bar{y}}{\partial t_1}, \dots, \frac{\partial\bar{y}}{\partial t_\ell}, \frac{\partial\bar{y}}{\partial\theta}\right] dsdtd\theta.$$

Now, the determinant on the right is equal to the determinant of the matrix

$$\begin{pmatrix} x_1 & \dots & x_n & 0 \\ \cos\theta y_1 & \dots & \cos\theta y_n & \sin\theta \\ \left(\frac{\partial x}{\partial s_1}\right)_1 & \dots & \left(\frac{\partial x}{\partial s_1}\right)_n & 0 \\ \vdots & & \vdots & \vdots \\ \left(\frac{\partial x}{\partial s_k}\right)_1 & \dots & \left(\frac{\partial x}{\partial s_k}\right)_n & 0 \\ \cos\theta\left(\frac{\partial y}{\partial t_1}\right)_1 & \dots & \cos\theta\left(\frac{\partial y}{\partial t_1}\right)_n & 0 \\ \vdots & & \vdots & \vdots \\ \cos\theta\left(\frac{\partial y}{\partial t_\ell}\right)_1 & \dots & \cos\theta\left(\frac{\partial y}{\partial t_\ell}\right)_n & 0 \\ -\sin\theta y_1 & \dots & -\sin\theta y_n & \cos\theta \end{pmatrix},$$

where, e.g., $\left(\frac{\partial x}{\partial s_1}\right)_3$ denotes the 3rd coordinate of the vector $\frac{\partial x}{\partial s_1} \in \mathbb{R}^n$.

In turn, expanding along the last column, we see that this determinant is equal to

$$\begin{aligned} (20) \quad & \sin\theta\left[x, \frac{\partial x}{\partial s_1}, \dots, \frac{\partial x}{\partial s_k}, \cos\theta\frac{\partial y}{\partial t_1}, \dots, \cos\theta\frac{\partial y}{\partial t_\ell}, -\sin\theta y\right] \\ & + \cos\theta\left[x, \cos\theta y, \frac{\partial x}{\partial s_1}, \dots, \frac{\partial x}{\partial s_k}, \cos\theta\frac{\partial y}{\partial t_1}, \dots, \cos\theta\frac{\partial y}{\partial t_\ell}\right] \\ = & \sin^2\theta\cos^\ell\theta\left[x, y, \frac{\partial x}{\partial s_1}, \dots, \frac{\partial x}{\partial s_k}, \frac{\partial y}{\partial t_1}, \dots, \frac{\partial y}{\partial t_\ell}\right] + \cos^2\theta\cos^\ell\theta\left[x, y, \frac{\partial x}{\partial s_1}, \dots, \frac{\partial x}{\partial s_k}, \frac{\partial y}{\partial t_1}, \dots, \frac{\partial y}{\partial t_\ell}\right] \\ & = \cos^\ell\theta\left[x, y, \frac{\partial x}{\partial s_1}, \dots, \frac{\partial x}{\partial s_k}, \frac{\partial y}{\partial t_1}, \dots, \frac{\partial y}{\partial t_\ell}\right]. \end{aligned}$$

Therefore, (19) simplifies as

(21)

$$Lk(K,L) + Lk(K,-L) = \frac{(-1)^k}{\text{vol } S^n}\int_{K\times L}\left(\int_{\theta=-\pi/2}^{\theta=\pi/2}\frac{\sin^k * \sin^{\ell+1}(\bar{\alpha})}{\sin^n\bar{\alpha}}\cos^\ell\theta d\theta\right)[x, y, dx, dy]$$

Now, let's shift our focus to the convolution in the above formula. By definition,

$$\begin{aligned} \sin^k * \sin^{\ell+1}(\bar{\alpha}) &= \int_0^\pi \sin^k(\bar{\alpha}-\beta)\sin^{\ell+1}\beta d\beta \\ &= \int_0^\pi (\sin\bar{\alpha}\cos\beta - \cos\bar{\alpha}\sin\beta)^k \sin^{\ell+1}\beta d\beta \\ &= \sum_{i=0}^k \binom{k}{i}(-1)^i\int_0^\pi \sin^{k-i}\bar{\alpha}\cos^i\bar{\alpha}\sin^{\ell+i+1}\beta\cos^{k-i}\beta d\beta \\ (22) \qquad &= \sum_{i=0}^k \binom{k}{i}(-1)^i\int_0^\pi \sin^{k-i}\bar{\alpha}\cos^i\theta\cos^i\alpha\sin^{\ell+i+1}\beta\cos^{k-i}\beta d\beta, \end{aligned}$$

using (18). Plugging this into (21) gives

$$Lk(K,L)+Lk(K,-L)=\frac{(-1)^k}{\text{vol } S^n}\int_{K\times L}\psi(\bar{\alpha})[x,y,dx,dy] \tag{23}$$

where

$$\begin{aligned}\psi(\bar{\alpha}) &= \int_{\theta=-\pi/2}^{\theta=\pi/2}\sum_{i=0}^{k}\binom{k}{i}(-1)^i\frac{\cos^{\ell+i}\theta\cos^i\alpha}{\sin^{\ell+i+2}\bar{\alpha}}\left[\int_{\beta=0}^{\beta=\pi}\sin^{\ell+i+1}\beta\cos^{k-i}\beta d\beta\right]d\theta \\ &= \sum_{i=0}^{k}\binom{k}{i}(-1)^i\left[\int_{\theta=-\pi/2}^{\theta=\pi/2}\frac{\cos^{\ell+i}\theta\cos^i\alpha}{(1-\cos^2\theta\cos^2\alpha)^{\frac{\ell+i+2}{2}}}d\theta\right]\left[\int_{\beta=0}^{\beta=\pi}\sin^{\ell+i+1}\beta\cos^{k-i}\beta d\beta\right]d\theta\end{aligned} \tag{24}$$

since

$$\sin\bar{\alpha}=\sqrt{1-\cos^2\bar{\alpha}}=\sqrt{1-\cos^2\theta\cos^2\alpha}. \tag{25}$$

To simplify a bit more, we make use of the following lemma:

Lemma 4.2. *For m odd,*

$$\int_{\theta=-\pi/2}^{\theta=\pi/2}\frac{\cos^{m-2}\theta}{(1-\cos^2\theta\cos^2\alpha)^{m/2}}d\theta=\frac{1}{\sin^{m-1}\alpha}\int_0^{\pi}\sin^{m-2}\varphi d\varphi$$

The proof of this lemma consists entirely of freshman calculus substitutions, so we omit it.

Since $\ell+i$ and $k-i$ have opposite signs and the β integral in (24) is zero when $k-i$ is odd, Lemma 4.2 implies that (23) is equal to

$$\frac{(-1)^k}{\text{vol } S^n}\int_{K\times L}\sum_{i=0}^{k}\binom{k}{i}(-1)^i\frac{\cos^i\alpha\int_0^{\pi}\sin^{\ell+i}\varphi d\varphi}{\sin^{\ell+i+1}}\int_{\beta=0}^{\beta=\pi}\sin^{\ell+i+1}\beta\cos^{k-i}\beta d\beta[x,y,dx,dy]. \tag{26}$$

On the other hand, we want to show that (26) is equal to

$$\begin{aligned}&\frac{(-1)^k}{\text{vol } S^{n-1}}\int_{K\times L}\frac{\sin^k*\sin^{\ell}(\alpha)}{\sin^{n-1}\alpha}[x,y,dx,dy] \\ &= \frac{(-1)^k}{\text{vol } S^{n-1}}\int_{K\times L}\frac{\int_0^{\pi}(\sin\alpha\cos\beta-\cos\alpha\sin\beta)^k\sin^{\ell}\beta d\beta}{\sin^{n-1}\alpha}[x,y,dx,dy] \\ &= \frac{(-1)^k}{\text{vol } S^{n-1}}\int_{K\times L}\sum_{i=0}^{k}\binom{k}{i}(-1)^i\frac{\cos^i\alpha}{\sin^{\ell+i+1}\alpha}\left(\int_{\beta=0}^{\beta=\pi}\sin^{\ell+i}\beta\cos^{k-i}\beta d\beta\right)[x,y,dx,dy]\end{aligned} \tag{27}$$

Since we want to show that (26) is equal to (27), the nicest possible situation would be if the terms in the sums were equal term-by-term. In fact, as the following lemma demonstrates, this is true.

Lemma 4.3. *For any choices of k, ℓ and i such that k and ℓ have opposite signs,*

$$\frac{\int_0^\pi \sin^{\ell+i}\varphi d\varphi}{vol\ S^n}\int_{\beta=0}^{\beta=\pi}\sin^{\ell+i+1}\beta\cos^{k-i}\beta d\beta = \frac{1}{vol\ S^{n-1}}\int_{\beta=0}^{\beta=\pi}\sin^{\ell+i}\beta\cos^{k-i}\beta d\beta. \tag{28}$$

Proof. Note that proving (28) is equivalent to proving that

$$\left(\int_0^\pi \sin^{\ell+i}\varphi d\varphi\right)\left(\int_0^\pi \sin^{\ell+i+1}\beta\cos^{k-i}\beta d\beta\right) = \frac{\text{vol } S^n}{\text{vol } S^{n-1}}\int_0^\pi \sin^{\ell+i}\beta\cos^{k-i}\beta d\beta. \tag{29}$$

Recall that, for any positive integers ν, κ, λ with λ even,

$$\text{vol } S^\nu = \text{vol } S^{\nu-1}\int_0^\pi \sin^{\nu-1}\theta d\theta$$

and

$$\text{vol } S^{\kappa+\lambda+1} = \frac{\text{vol } S^\kappa \text{vol } s^\lambda}{2}\int_0^\pi \sin^\kappa\theta\sin^\lambda\theta d\theta.$$

Therefore, (29) becomes

$$\frac{\text{vol } S^{\ell+i+1}}{\text{vol } S^{\ell+i}}\frac{\text{vol } S^{k+\ell+2}}{2\text{vol } S^{\ell+i+1}\text{vol } S^{k-i}} = \frac{\text{vol } S^n}{\text{vol } S^{n-1}}\frac{\text{vol } S^{k+\ell+1}}{2\text{vol } S^{\ell+i}\text{vol } S^{k-i}},$$

which is clearly true, since $k+\ell+1 = n-1$ and $k+\ell+2 = n$. □

With Lemma 4.3 in hand, we see that (26) and (27) are equal term-by-term, which demonstrates that

$$Lk(K,L) + (-1)^{n-1}Lk(K,-L) = \frac{1}{\text{vol } S^{n-1}}\int_{K\times L}\frac{\sin^k * \sin^\ell(\alpha)}{\sin^{n-1}\alpha}[x,y,dx,dy],$$

completing the proof of Proposition 4.1.

Thus, we have showed that (2) holds in both odd and even spheres, completing the proof of Theorem 1.1.

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