Stratification of 3×3 Matrices

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1 Warmup with 2×2 Matrices

- $\{$ Best matrices of rank 2$\} = O(2) \subset S^3(\sqrt{2})$
- $\{$ Best matrices of rank 1$\} \subset S^3(1)$

1.1 Viewing $O(2) \subset S^3(\sqrt{2})$

$$O(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \right\}$$

- Both of two components are intersections of 2-plane with $S^3(\sqrt{2})$, hence *great circles* which are orthogonal, and linked with on another in $S^3(\sqrt{2})$.
- $S^3(\sqrt{2})$ = Nbhd of $SO(2) \cup$ Nbhd of $O^-(2)(= SO(2) - O(2))$, where each tubular nbhd is a solid torus and their common boundary is a *Clifford torus* in $S^3(\sqrt{2})$, consisting of matrices of rank 1.
- The Clifford torus is a minimal surface in S^3.
- Gauss Theorem for $T^2 \subset S^3 : K(X,Y) - \underline{K}(X,Y) = 0 - 1 = -1$
$< B(X,X), B(Y,Y) > - \parallel B(X,Y) \parallel^2 = < N^*, -N^* > = -1$.

1.2 Symmetries of the stratification of 2×2 matrices

The orthogonal group $O(2)$ acts on the space of 2×2 matrices by either left or right multiplication

- left and right multiplications are isometry, orientation preserving, and preserves the various strata of $M(2)$ and their cores.
- The combined map $A \mapsto E_1 A E_2^{-1}$ gives an isometric action of $O(2) \times O(2)$ on $M(2)$.

1.3 Alexander duality

If C is a closed subset of S^n, then Alexander duality says

- The free part of $H_k(C;\mathbb{Z})$ is isomorphic to the free part of $H_{n-k-1}(S^n - C;\mathbb{Z})$, where we use reduced homology.
- The torsion part of $H_k(C;\mathbb{Z})$ is isomorphic to the torsion part of $H_{n-k-2}(S^n - C;\mathbb{Z})$.
- $H_1(O(2);\mathbb{Z}) \cong \mathbb{Z} + \mathbb{Z}$.
- $S^3(\sqrt{2}) - O(2)$ deformation retracts to the Clifford torus T^2.
- $H_1(T^2;\mathbb{Z}) \cong \mathbb{Z} + \mathbb{Z}$.
- $H_2(T^2;\mathbb{Z}) \cong \mathbb{Z}$, and the reduced 0-dim'l homology of $S^3 - T^2$ is also $\cong \mathbb{Z}$, reminding us that it is disconnected.

2 Stratification of 3×3 Matrices

- $M(3)$ = set of all 3×3 matrices $\cong \mathbb{R}^9$
- Euclidean inner product $< A, B >= trAB^t = a_{ij}b_{ij}$
- $\{$Best matrices of rank 3$\} = O(3) \subset S^8(\sqrt{3})$
- $\{$Best matrices of rank 2$\} \subset S^8(\sqrt{2})$
- $\{$Best matrices of rank 1$\} \subset S^8(1)$.

2.1 Symmetries of the stratification of 3×3 matrices

The map $A \to E_1 A E_2^{-1}$ gives an isometric action of $O(3) \times O(3)$ on $M(3)$. It is clearly transitive on $O(3) \subset S^8(\sqrt{3})$.

2.2 Viewing $O(3) \subset S^8(\sqrt{3})$

$SO(3) \subset S^8(\sqrt{3})$ is isometric to the $\mathbb{R}P^3$ which is double covered by $S^3(2\sqrt{2})$. The closed geodesics on $SO(3)$ are small circles of radius $\sqrt{2}$ on $S^8(\sqrt{3})$.

2.3 Tubular neighborhood of $O(3)$ in $S^8(\sqrt{3})$

- Left (or right) action of $SO(3)$ on $S^8(\sqrt{3})$ acts simply transitively on each of $SO(3)$ and $O^-(3)$.
- The tubular nbhds of $SO(3)$ and of $O^-(3)$ in $S^8(\sqrt{3})$ are product bundles, with fibre a 5-cell.
- The largest possible tubular nbhds of $SO(3)$ and $O^-(3)$ in $S^8(\sqrt{3})$ are the two components of $GL(3,\mathbb{R}) \cap S^8(\sqrt{3})$.
- The fibres will be open 5-cells lying on the great 5-spheres which meet $O(3)$ orthogonally.
- But these 5-cells will not be round as is the case of 2×2 matrices. Instead, they will be elongated in some directions, and foreshortened in others.

2.4 Shape of the 5-cell fibres

Consider the great 5-sphere in $S^8(\sqrt{3})$ orthogonal to $SO(3)$ at the identity. Note that its tangent space at the identity consists of symmetric 3×3 matrices with trace 0. We want to see how far one can go out from the identity along this 5-sphere before coming to a singular matrix.

- The isotropy subgroup of the identity in the symmetry group $SO(3) \times SO(3)$ consists of $SO(3)$ acting by conjugation, so, under this action, any traceless symmetric matrix can be written in the normal form $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and $\lambda_1 + \lambda_2 + \lambda_3$.
- Scaling so that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 3$, the great circle through the identity in the direction of this matrix is given by $\mathrm{diag}(\cos t + \lambda_1 \sin t, \cos t + \lambda_2 \sin t, \cos t + \lambda_3 \sin t)$, so this great circle first hits a singular matrix when $t = \cot^{-1} |\lambda_3|$, which, depending on the λ_i, varies from 0.196π to 0.304π.
- Each such value corresponds to an orbit of the conjugation action of $SO(3)$ on a 4-sphere in the space of symmetric traceless 3×3 matrices.
- Orbit structure: note that each orbit contains a traceless matrix $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3$. When the λ_i are distinct, the subgroup of $SO(3)$ fixing this matrix is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, so the orbit is diffeomorphic to $SO(3)/\mathbb{Z}/2 \oplus \mathbb{Z}/2$. The middle orbit occurs when $\lambda_2 = 0$

and is minimal in the corresponding 4-sphere. There are two singular orbits diffeomorphic to a round $\mathbb{RP}^2$ which occur when $\lambda_1 = \lambda_2$ and $\lambda_2 = \lambda_3$; these are also minimal in the corresponding 4-sphere.

2.5 $O(3)$ is a minimal submanifold of $S^8(\sqrt{3})$

Straightforward covariant derivative computations demonstrate that $O(3) \subset S^8(\sqrt{3})$ has vanishing mean curvature at the identity in $SO(3)$; since the isometry group $O(3) \times O(3)$ acts transitively, this suffices to show that $O(3)$ is a minimal submanifold of $S^8(\sqrt{3})$.

2.6 3×3 matrices of rank 2

- The set of rank ≤ 2 matrices is an 8-dimensional space separating $M(3)$ into the two open components of $GL(3, \mathbb{R})$, so $\{\text{rank} \leq 2\}$ meets S^8(any radius) in a closed 7-dimensional set separating that 8-sphere into two open components.
- $\{\text{rank} \leq 2\} \cap S^8$ is singular along its subset $\{\text{rank} = 1\} \cap S^8$.
- On the other hand, $\{\text{rank} = 2\}$ is an 8-dimensional submanifold of $M(3)$ and so $\{\text{rank} = 2\} \cap S^8$ is a 7-dimensional submanifold of S^8. This is easy to see, since it is the pre-image of 0 under the determinant map, which has critical points only on rank ≤ 1 matrices.

2.7 {Best of rank 2}

The best 3×3 matrices of rank 2 consist of orthogonal projections of $\mathbb{R}^3$ to a 2-plane, followed by an orthogonal transformation to another 2-plane.

- Dimension-counting, we see that {Best of rank 2} is a 5-dimensional manifold, which is acted on transitively by the identity component $SO(3) \times SO(3)$ of the symmetry group of our stratification.
- Considering $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The isotropy subgroup in $SO(3)$ consists of

  $$\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  acting simultaneously from the left and the right.

- Hence, {Best of rank 2} is homeomorphic to $SO(3) \times SO(3)$ modulo this diagonal $SO(2)$ subgroup.

2.8 {Best of rank 2} is a minimal submanifold of $S^8(\sqrt{2})$

To see that $M^5 = \{\text{Best of rank 2}\}$ is a minimal submanifold of $S^8(\sqrt{2})$, a straightforward computation shows that the isotropy subgroup of $P \in M^5$ fixes no nonzero normal vector. This implies that the mean curvature vector at P must be zero (since it *is* fixed by the isotropy subgroup) and so, by symmetry, this is true at every point of M^5.

2.9 3×3 matrices of rank 1

- The best 3×3 matrices of rank 1 consist of orthogonal projections of $\mathbb{R}^3$ to a line, followed by an orthogonal transformation of this line to another line. In fact, the best of rank 1 are precisely the intersection {All of rank 1} $\cap S^8(1)$.

- The identity component $SO(3) \times SO(3)$ of symmetries acts transitively on the best of rank 1, with isotropy subgroup $SO(2) \times SO(2)$.

2.10 A way to describe the best 3×3 matrices of rank 1

- If $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$, then $(y_i x_j)$ takes X to Y and kills $X^\perp$, so this is one of the best matrices of rank 1.

- All the best of rank 1 appear in this way twice, since taking X to Y is the same as taking $-X$ to $-Y$. Hence, $M^4 = \{\text{Best of rank } 1\}$ is homeomorphic to $(S^2 \times S^2)/\{(X, Y) \sim (-X, -Y)\}$, which is orientable.

- It's easy to see that $S^2 \times S^2 \to \{\text{Best of rank 1}\} \subset S^8(1)$ is a local isometry.

2.11 The singularity at M^4

- A geodesic through $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ orthogonal there to $M^4 =$ {best of rank 1} $=$ {rank 1}$\cap S^8(1)$ has the form $\gamma(t) = \begin{pmatrix} \cos t & 0 & 0 \\ 0 & a \sin t & b \sin t \\ 0 & c \sin t & d \sin t \end{pmatrix}$.

- If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has rank 1, then $\gamma(t)$ has rank 2 for small $t > 0$.

- from our study of 2×2 matrices, we know the matrices of rank 1 form a cone over the Clifford torus, so the tubular neighborhood of {Best of rank 1} in {Best of rank 2 } $\cap S^8(1)$ is a bundle over M^4 whose normal fiber is a cone over the Clifford torus.

2.12 {Best of rank 1} is a minimal submanifold of $S^8(1)$

This follows from a similar argument to that given above in the case of {Best of rank 2}: since the subgoup of the isometry group of our stratification that fixes a point of {Best of rank 1} fixes none of its normal vectors in $S^8(1)$ except 0, the mean curvature vector must be zero and so, by symmetry, the mean curvature is zero on all of {Best of rank 1}.

2.13 Topology of {Best of rank 1}

- Since M^4 = {Best of rank 1} is orientable and double covered by $S^2 \times S^2$

- Thus, $\pi_1(M^4) \simeq \mathbb{Z}/2$, and so $H_1(M^4) \simeq \mathbb{Z}/2$.

- Using Poincaré duality, we see that $H_0 \simeq \mathbb{Z}$, $H_1 \simeq \mathbb{Z}/2$, $H_2 \simeq$ free part $+ \mathbb{Z}/2$, $H_3 \simeq 0$, $H_4 \simeq \mathbb{Z}$.

2.14 Theorem of Wu-Yi Hsiang

Theorem 2.1. *(Wu-Yi Hsiang, 1966) Let G be a compact connected group of isometries of a Riemannian manifold M. Then any orbit of G, whose volume is extremal among nearby orbits of the same type, is a minimal submanifold of M.*

This theorem can be used to give alternate proofs of the minimality of many of the above submanifolds.