DIFFERENTIAL GEOMETRY HW 4

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11.

(a): Show that $\mathbb{CP}^n$ has a differentiable structure of a manifold of real dimension $2n$ and that $\mathbb{CP}^1$ is diffeomorphic to S^2.

Proof. Form coordinate charts (U_i, x_i) by $U_i = \{(z_0 : \ldots : z_n) \in \mathbb{CP}^n | z_i = 1\}$ and $x_i(z_0 : \ldots : z_n) = \frac{1}{z_i}(z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$. It's clear that the U_i cover $\mathbb{CP}^n$. Also, if $i < j$, $z_{j-1} \neq 0$,

$$\begin{aligned} x_j \circ x_i^{-1}(z_0, \ldots, z_{n-1}) &= x_j(z_0 : \ldots : z_{i-1} : 1 : z_{i+1} : \ldots : z_{n-1}) \\ &= \frac{1}{z_{j-1}}(z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_{j-2}, z_j, \ldots, z_{n-1}), \end{aligned}$$

which is certainly differentiable. If we embed $\mathbb{CP}^1$ in $\mathbb{R}^3$ by letting $\frac{z_1}{z_0} = u + iv$ where

$$u = \frac{x}{1+z}, \quad v = \frac{y}{1+z}$$

where $x^2 + y^2 + z^2 = 1$, then the above coordinate functions are just stereographic projection, so $\mathbb{CP}^1$ is diffeomorphic to S^2. □

(b): Let $(Z, W) = z_0\bar{w}_0 + \cdots + z_n\bar{w}_n$ be the hermitian product on $\mathbb{C}^{n+1}$. Identify $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$ by putting $z_j = x_j + iy_j = (x_j, y_j)$. Show that

$$S^{2n+1} = \{N \in \mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}; (N, N) = 1\}$$

is the unit sphere in $\mathbb{R}^{2n+2}$.

Proof. Suppose $Z \in \mathbb{C}^{n+1}$ such that $(Z, Z) = 1$. Then $Z = (z_0, \ldots, z_n)$ and

$$1 = (Z, Z) = |z_0|^2 + \ldots + |z_n|^2 = x_0^2 + y_0^2 + \ldots + x_n^2 + y_n^2,$$

so $Z \in S^{2n+1}(1)$. Alternatively, if we let $(x_0, y_0, \ldots, x_n, y_n) \in S^{2n+1}(1)$, and let $z_j = x_j + iy_j$ for all j, then the above equality shows that $Z = (z_0, \ldots, z_n)$ is such that $(Z, Z) = 1$. □

(c): Show that the equivalence relation $\sim$ induces on S^{2n+1} the following equivalence relation: $Z \sim W$ if $e^{i\theta}Z = W$. Establish that there exists a differentiable map (the Hopf fibering) $f : S^{2n+1} \to \mathbb{CP}^n$ such that

$$f^{-1}([Z]) = \{e^{i\theta} N \in S^{2n+1}, N \in [Z] \cap S^{2n+1}, 0 \leq \theta \leq 2\pi\} = [Z] \cap S^{2n+1}.$$

Proof. The first fact is trivially true due to the fact that any norm 1 complex number can be written as $e^{i\theta}$. For the second, define $f : S^{2n+1} \to \mathbb{CP}^n$ by

$$(z_0, \dots, z_n) \mapsto (z_0 : \dots : z_n),$$

which is certainly differentiable. Now, $(z_0, \dots, z_n) \sim e^{i\theta}(z_0, \dots, z_n)$ for all θ between 0 and 2π, so, if $Z = (z_0, \dots, z_n)$

$$f^{-1}([Z]) = \{e^{i\theta} N \in S^{2n+1} : N \in [Z] \cap S^{2n+1}\}.$$

□

(d): Show that f is a submersion.

Proof. Since S^{2n+1} is so symmetric, we can show that f is a submersion at just a single point, say $(1, 0, \dots, 0) \in S^{2n+1} \subset \mathbb{C}^{n+1}$. Now, consider two cases; where we go off in the direction where the first coordinate starts gaining an imaginary part, and where we go off in some other direction. In the first case,

$$f(1 + idx, 0, 0, \dots, 0) = (1 + idx : 0 : \dots : 0) = (1 : 0 : \dots : 0)$$

so *df* kills this direction. The other cases are all equivalent, so we examine

$$f(1, dx, 0 \dots, 0) = (1 : dx : 0 : \dots : 0),$$

so *df* is surjective on the $2n+1$-dimensional subspace of the tangent space corresponding to these directions. Therefore, f is a submersion. □

12.

(a): Show that, for all $0 \leq \theta \leq 2\pi$, $e^{i\theta} : S^{2n+1} \to S^{2n+1}$ is an isometry and that, therefore, it is possible to define a Riemannian metric on $\mathbb{CP}^n$ in such a way that the submersion f is Riemannian.

Proof. By 11(c) above, the map $e^{i\theta}$ is transitive on the fibers of the map $f : S^{2n+1} \to \mathbb{CP}^n$. Therefore, we can define the following metric on $\mathbb{CP}^n$: if $v, w \in T_p\mathbb{CP}^n$, let

$$\langle v, w \rangle_p = \langle \bar{v}, \bar{w} \rangle_{f^{-1}(p)}$$

where r and s are the horizontal components of $df^{-1}_{f^{-1}(p)}(v)$ and $df^{-1}_{f^{-1}(p)}(w)$, respectively. Then, by construction, this is a Riemannian submersion. □

(b): Show that, in this metric, the sectional curvature of $\mathbb{CP}^n$ is given by

$$K(\sigma) = 1 + 3\cos^2\phi,$$

where σ is generated by the orthonormal pair X, Y, $\cos\phi = \langle \bar{X}, i\bar{Y}\rangle$, and $\bar{X}$ and $\bar{Y}$ are the horizontal lifsts of X and Y, respectively. In particular, $1 \leq K(\sigma) \leq 4$.

Proof. Let $Z \in S^{2n+1}$. Then $\frac{d}{d\theta}e^{i\theta}Z|_{\theta=0} = iZ$, so $iZ \in T_Z S^{2n+1}$ and is vertical. Let $\bar{\nabla}$ be the Riemannian connection of $\mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$. Let $\alpha : (-\epsilon, \epsilon) \to S^{2n+1}$ with $\alpha(0) = Z$, $\alpha'(0) = \bar{X}$. Then

$$(\bar{\nabla}_{\bar{X}} iZ)_Z = \frac{d}{dt} iZ \circ \alpha(t)|_{t=0} = \frac{d}{dt} i\alpha(t)|_{t=0} = i\alpha'(0) = i\bar{X}.$$

Similarly, $(\bar{\nabla}_{\bar{Y}} iZ)_Z$. Therefore,

$$\begin{aligned}
\big\langle [\bar{X}, \bar{Y}], iZ \big\rangle &= \big\langle \bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X}, iZ \big\rangle \\
&= \bar{X}\langle \bar{Y}, iZ\rangle - \big\langle \bar{Y}, \bar{\nabla}_{\bar{X}} iZ \big\rangle + \bar{Y}\langle \bar{X}, iZ\rangle - \big\langle \bar{X}, \bar{\nabla}_{\bar{Y}} iZ \big\rangle \\
&= \bar{X}(0) - \langle i\bar{X}, \bar{Y}\rangle - \bar{Y}(0) + \langle i\bar{Y}, \bar{X}\rangle \\
&= 2\langle \bar{X}, i\bar{Y}\rangle \\
&= 2\cos\phi,
\end{aligned}$$

since iZ is perpindicular to $\bar{X}$ and $\bar{Y}$. Then, using the result proved in problem 5 of HW #1,

$$\begin{aligned}
K(\sigma) &= \bar{K}(\bar{\sigma}) + \frac{3}{4}\left|[\bar{X}, \bar{Y}]^v\right|^2 \\
&= 1 + \frac{3}{4}|2\cos\phi|^2 \\
&= 1 + 3\cos^2\phi.
\end{aligned}$$

□

1.

Let M be a complete Riemannian manifold, and let $N \subset M$ be a closed submanifold of M. Let $p_0 \in M$, $p_0 \notin N$, and let $d(p_0, N)$ be the distance from p_0 to N. Show that there exists a point $q_0 \in N$ such that $d(p_0, q_0) = d(p_0, N)$ and that a minimizing geodesic which joins p_0 to q_0 is orthogonal to N at q_0.

Proof. Let $a = d(p_0, N)$. Let $q_i \in N$ be a sequence of points such that $d(p_0, q_i) \to a$. Let $\epsilon > 0$ and let D be a disc centered at p_0 of radius $a + \epsilon$. Then the tail of the sequence $\{q_i\}$ is contained in $D \cap N$, which, since N is closed and D is compact, is compact. Therefore, there exists a subsequence $\{q_{i_k}\}$ converging to a point $q_0 \in D \cap N$. Since $d(p_0, q_{i_k}) \to a$, $d(p_0, q_0) = a$.

Now, since M is complete, there exists a minimizing geodesic γ from p_0 to q_0 and $\gamma(t) = \exp_{p_0} tW$ for some $W \in T_{p_0}M$. Suppose q_0 is not conjugate to p_0 along γ. Let $V \in T_{q_0}N$ and let α be a curve in N such that $\alpha(0) = q_0$

and $\alpha'(0) = V$. Then, since q_0 is not a conjugate point of p_0, $\exp_{p_0}$ is a diffeomorphism from a neighborhood of aW to a neighborhood of q_0; in particular, $\alpha(s) = \exp_{p_0} aW(s)$ for some curve $W(s)$ with $W(0) = W$. Form the variation $f(s,t) = \exp_{p_0} tW(s)$. Then, since γ is a minimizing geodesic, energy has a minimum at $(0,a)$, meaning that, if $V(t) = \frac{\partial f}{\partial s}(0,t)$,

$$\begin{aligned} 0 = \frac{1}{2}E'(0) &= -\int_0^a \left\langle V(t), \frac{D}{dt}\gamma'(t) \right\rangle dt - \langle V(0), \gamma'(0)\rangle + \langle V(a), \gamma'(a)\rangle \\ &= 0 - 0 + \langle V(a), \gamma'(a)\rangle \\ &= \langle V, \gamma'(a)\rangle \end{aligned}$$

since $\frac{D}{dt}\gamma'(t) \equiv 0$ and $V(0) = \exp_{p_0} 0W(s) = 0$. Therefore, since our choice of $V \in T_{q_0}N$ was arbitrary, we see that γ is orthogonal to N at q_0.

If q_0 is a conjugate point of p_0 along γ, then let $p_1 = \gamma(t_1)$ for $0 < t_1 < a$. γ is a distance minimizing geodesic from p_1 to q_0 and q_0 is not a conjugate point of p_1, so we can run the above argument to see that γ is orthogonal to N at q_0. □

2.

Introduce a complete Riemannian metric on $\mathbb{R}^2$. Prove that

$$\lim_{r\to\infty}\Big(\inf_{x^2+y^2\geq r^2} K(x,y)\Big) \leq 0$$

where $(x,y) \in \mathbb{R}^2$ and $K(x,y)$ is the Gaussian curvature of the given metric at (x,y).

Proof. Let $a_r = \inf_{x^2+y^2\geq r^2} K(x,y)$. Then the a_r are monotone, so either a limit L exists or $\lim_{r\to\infty} a_r = \infty$. Suppose $\lim_{r\to\infty} a_r = L > 0$. Then there exists $R > 0$ such that $a_r > \frac{L}{2}$ for all $r \geq R$. Let B_R be the ball of radius R centered at the origin. Then B_R is compact and so has finite diameter d in the given metric. Now, since $K(x,y) > \frac{L}{2}$ for all $(x,y) \notin B_R$, so $d((x,y), B_R) < \frac{2}{L}$ for all $(x,y) \notin B_R$. Therefore, the diameter of $\mathbb{R}^2$ in this metric is less than $d + \frac{2}{L}$, meaning $\mathbb{R}^2$ is compact in this metric. From this contradiction, then, we conclude that $L \leq 0$.

On the other hand, if $\lim a_r = \infty$, then there exists $R > 0$ such that $a_r > 1$ for all $r \geq R$. Again, B_R is compact with diameter d and, by the same argument, the diameter of $\mathbb{R}^2$ is less than $d + 1$, which still leads to a contradiction. □

4.

Let M^n be an orientable Riemannian manifold with positive curvature and even dimension. Let γ be a closed geodesic in M, that is, γ is an immersion of the circle S^1 in M that is geodesic at all of its points. Prove that γ is homotopic to a closed curve whose length is strictly less than that of γ.

Proof. Let $T_pM^\perp$ denote the $n-1$-dimensional subspace of T_pM orthogonal to $\gamma'(t)$ where $\gamma(t) = p$. Then parallel transport keeps $T_pM^\perp$ orthogonal to $\gamma'(t)$. If $\gamma(0) = \gamma(a) = p_0$, then parallel transport gives an isomorphism $T_{p_0}M^\perp \to T_{p_0}M^\perp \simeq \mathbb{R}^{n-1}$. Since n is even, this is an odd-dimensional vector space. The determinant of this isomorphism is 1, so one of its eigenvalues must be 1; let V denote the corresponding eigenvector orthonormal to $\gamma'(0)$. Therefore, the parallel transport $V(t)$ of V along γ is parallel. Then the second variation of energy of the variation associated to $V(t)$ is

$$\frac{1}{2}E''(0) = -\int_0^a \langle V(t), V''(t) + R(\gamma', V)\gamma' \rangle \, dt = -\int_0^a \langle R(\gamma', V)\gamma', V \rangle = -\int_0^a K(\gamma', V) < 0$$

since $V'(t) \equiv 0$ (since $V(t)$ is parallel) and $K(\gamma', V) > 0$. Therefore, by sliding γ in the direction of $V(t)$, we decrease the energy and, therefore, the length of the curve. Therefore, we conclude that γ is homotopic to a closed curve of length strictly less than that of γ. □

5.

Let N_1 and N_2 be two closed disjoint submanifolds of a compact Riemannian manifold.

(a): Show that the distance between N_1 and N_2 is assumed by a geodesic perpendicular to both N_1 and N_2.

Proof. Let p_i be a sequence of points in N_1 such that $d(N_2, p_i) \to d(N_2, N_1)$. Since M is compact, so is $M \cap N_1$, so there is a convergent subsequence converging to p_0 such that $d(N_2, p_0) = d(N_2, N_1)$. Now, let q_i be a sequence in N_2 such that $d(p_0, q_i) = d(p_0, N_2) = d(N_1, N_2)$. Again, M is compact, so $M \cap N_2$ is compact, so a subsequence converges to q_0 where $d(p_0, q_0) = d(N_1, N_2)$. Let γ be the distance-minimizing geodesic from p_0 to q_0. By problem 1 above, γ is perpendicular to N_2 at q_0 and, by reversing the role of p_0 and q_0, γ is perpindicular to N_1 at p_0. □

(b): Show that, for any orthogonal variation $h(t, s)$ of γ, with $h(0, s) \in N_1$ and $h(\ell, s) \in N_2$, we have the following expression for the formula for the second variation

$$\frac{1}{2}E''(0) = I_\ell(V, V) + \langle V(\ell), S^{(2)}_{\gamma'(\ell)} V(\ell) \rangle - \langle V(0), S^{(1)}_{\gamma'(0)} V(0) \rangle.$$

where V is the variational vector and $S^{(i)}_{\gamma'}$ is the linear map associated to the second fundamental form of N_i in the direction of γ', $i = 1, 2$.

Proof. By the formula for the second variation of energy,

$$\frac{1}{2}E''(0) = I_\ell(V, V) - \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \gamma' \right\rangle (0, 0) + \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \gamma' \right\rangle (0, a).$$

Now,

$$\begin{aligned}\langle V(0), S^{(1)}_{\gamma'(0)}\rangle &= \langle B(V(0),V(0)), \gamma'(0)\rangle \\ &= \left\langle \bar{\nabla}_{V(0)}V(0), \gamma'(0)\right\rangle - \left\langle \nabla_{V(0)}V(0), \gamma'(0)\right\rangle \\ &= \left\langle \bar{\nabla}_{V(0)}V(0), \gamma'(0)\right\rangle - 0 \\ &= \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \gamma'\right\rangle (0,0),\end{aligned}$$

where the 0 in the third line is due to the fact that N_1 is perpindicular to $\gamma'(0)$. A similar calculation shows that $\langle V(\ell), S^{(2)}_{\gamma'(\ell)}V(\ell)\rangle = \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \gamma'\right\rangle (0,a)$. The desired equality follows. □

6.

Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with curvature $K \leq 0$. Let $\gamma : (-\infty, \infty) \to \widetilde{M}$ be a normalized geodesic and let $p \in \widetilde{M}$ be a point which does not belong to γ. Let $d(s) = d(p, \gamma(s))$.

(a): Consider the minimizing geodesic $\sigma_s : [0, d(s)] \to \widetilde{M}$ joining p to $\gamma(s)$, that is, $\sigma_s(0) = p$, $\sigma_s(d(s)) = \gamma(s)$. Consider the variation $h(t,s) = \sigma_s(t)$, and show that:
- **(i):** $\frac{1}{2}E'(s) = \langle \gamma'(s), \sigma_s'(d(s))\rangle$,
- **(ii):** $\frac{1}{2}E''(s) > 0$.

Proof. Not quite sure how to prove the this as stated, so modify to a situation where $\gamma : [0,1] \to \widetilde{M}$ is a minimizing (but not normalized) geodesic from p to $\gamma(s)$; as in the statement, let $h(s,t) = \sigma_s(t)$. By Hadamard's Theorem, $\exp_p$ is a diffeomorphism, so $h(s,t) = \exp_p tW(s)$ where W is the curve in $T_p\widetilde{M}$ diffeomorphic to $\gamma(s)$ under $\exp_p$. Now,

$$\begin{aligned}\frac{1}{2}E'(s) &= \int_0^1 \left\langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle dt \\ &= \int_0^1 \left\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle dt \\ &= \int_0^1 \frac{d}{dt}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle dt - \int_0^1 \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\right\rangle dt \\ &= \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle\Big]_0^1 - 0 \\ &= \left\langle \frac{\partial f}{\partial s}(s,1), \frac{\partial f}{\partial t}(s,1)\right\rangle - \left\langle \frac{\partial f}{\partial s}(s,0), \frac{\partial f}{\partial t}(s,0)\right\rangle \\ &= \langle \gamma'(s), \sigma_s'(1)\rangle\end{aligned}$$

because $\frac{\partial f}{\partial t}$ is a geodesic and so $\frac{D}{dt}\frac{\partial f}{\partial t} = 0$ and $\frac{\partial f}{\partial s}(s, 0) = 0$ since the variation is fixed at p_0.

As for (ii),

$$\begin{aligned}
\frac{1}{2}E''(s) &= \frac{\partial}{\partial s}\left(\frac{1}{2}E'(s)\right) \\
&= \int_0^1 \left\langle \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right\rangle dt + \int_0^1 \left\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{D}{ds}\frac{\partial f}{\partial t}\right\rangle dt \\
&= \int_0^1 \left\langle \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial s} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle dt + \int_0^1 \left|\frac{D}{dt}\frac{\partial f}{\partial s}\right|^2 dt
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^1 \left\langle \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle dt &= \int_0^1 \frac{\partial}{\partial t}\left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle dt - \int_0^1 \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\right\rangle dt \\
&= \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle\Bigg]_0^1 \\
&= \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}(s,1), \frac{\partial f}{\partial t}(s,1)\right\rangle - \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}(s,0), \frac{\partial f}{\partial t}(s,0)\right\rangle \\
&= 0
\end{aligned}$$

since $\frac{\partial f}{\partial s}(s, 1) = \gamma'(s)$ and $\frac{\partial f}{\partial s}(s, 0) \equiv 0$. Therefore,

$$\frac{1}{2}E''(s) = -\int_0^1 \left\langle R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle dt + \int_0^1 \left|\frac{D}{dt}\frac{\partial f}{\partial s}\right|^2 dt > 0$$

since curvature is ≤ 0 and $\frac{\partial f}{\partial s}$ is non-constant. □

(b): Conclude from (i) that s_0 is a critical point of d if and only if $\langle\gamma'(s_0), \sigma_s'(d(s_0))\rangle = 0$. Conclude from (ii) that d has a unique critical point, which is a minimum.

Proof. Note that

$$E(s) = \int_0^1 \left|\frac{\partial f}{\partial t}\right|^2 dt = \int_0^1 \left|\sigma_s'(t)\right|^2 dt = (L(\sigma_s))^2 = (d(s))^2.$$

Hence, s_0 is a critical point of d if and only if it is a critical point of E. By (i) above, this occurs if and only if $\langle\gamma'(s_0), \sigma_s'(d(s_0))\rangle = 0$. By (ii) $E''(s)$ is always positive, such a critical point must be a minimum and, moreover, unique. □

(c): From (b), it follows that if $\widetilde{M}$ is complete, simply connected and has curvature $K \leq 0$, then a point off the geodesic γ of $\widetilde{M}$ can be connected by a unique perpendicular to γ. Show by examples that the condition on the curvature and the condition of simple connectivity are essential to the theorem.

Proof. Let p be a point off γ; then the minimizing geodesic from p to γ is perpindicular to γ by (i) and is unique by (b). To see why the nypotheses are necessary, consider $S^2(1)$ and a cylinder. On $S^2(1)$, $K = 1 > 0$. If we let γ be the equator and p be the north pole, then there are infinitely many minimizing geodesics from p to γ, namely every great circle originating at the north pole. The cylinder is not simply connected, and the following picture demonstrates that there isn't a unique geodesic connecting p and γ:

□

11.

(a): Define "cut locus" of a Riemannian manifold.

Answer: Intuitively, a cut point of a point p on a Riemannian manifold M is the point along some geodesic γ through p where γ ceases to be a distance minimizing geodesic. Visually, this often means that γ intersects some other geodesic through p. The cut locus is the union of the cut points associated to each geodesic through p.

More rigorously, if M is a complete Riemannian manifold and $p \in M$, let $\gamma : [0, \infty) \to M$ be a normalized geodesic with $\gamma(0) = p$. If $t > 0$ is sufficiently small, $d(\gamma(0), \gamma(t)) = t$; i.e., $\gamma([0, t])$ is a minimizing geodesic. If $\gamma([0, t_1]0)$ is not minimizing, then the same is true for all $t > t_1$. By continuity, the set of numbers $t > 0$ for which $d(\gamma(0), \gamma(t)) = t$ is of the form $[0, t_0]$ or $[0, \infty)$. In the first case, $\gamma(t_0)$ is called the cut point of p along γ; in the second, there is no cut point. The cut locus of p, $C_m(p)$ is the union of all cut points of p along all geodesics starting at p.

♣

(b): Define "conjugate locus" along a geodesic.

Answer: Intuitively, a conjugate point of a point p is a point where geodesics through p come together. The conjugate locus is the union of all conjugate points.

Rigorously, let $\gamma : [0, a] \to M$ be a geodesic starting at $\gamma(0) = p$. The point $\gamma(t_0)$ is conjugate to p along γ for $t_0 \in (0, a]$ if there exists a Jacobi field J along γ (not identically zero) with $J(0) = 0 = J(t_0)$.

The conjugate locus is the union of all conjugate points to p along all geodesics starting at p.

$\clubsuit$

(c): Determine the cut and conjugate points on S^2 and $\mathbb{RP}^2$.

Answer: On S^2, if we let p be the north pole (rotating if necessary), then geodesics starting at p are minimizing until they reach the south pole (where all great circles through the north pole come together), so the cut locus is just the south pole. Since all geodesics starting at p come together at the south pole, the south pole is also a conjugate point of p. Moreover, if we continue these geodesics, they come back together again at p, then again at the south pole, and so on, so the conjugate locus of p is $\{p, -p\}$.

If we think of $\mathbb{RP}^2$ as a quotient of S^2, then the conjugate points of $p \in \mathbb{RP}^2$ are just the images under the quotient map of the conjugate points of $\widetilde{p}$ in the fiber over p in S^2. Since we know the conjugate points of $\widetilde{p}$ are just $\widetilde{p}$ and $-\widetilde{p}$, which both map to p under the quotient map, the conjugate locus of $p \in \mathbb{RP}^2$ is just $\{p\}$. On the other hand, if we think of the upper hemisphere model of $\mathbb{RP}^2$, points on the equator are identified with their antipodes and so geodesics through the north pole and antipodal points on the equator are both minimizing geodesics to the same point in the quotient, so the cut locus of a point p is the entire equator in $\mathbb{RP}^2$ (where we identify p with the north pole).

$\clubsuit$

(d): Determine the same on the flat torus.

Answer: Consider the point p as in the below picture (translating if necessary):

There are no conjugate points, but at all points on the "boundary" geodesics cease to be uniquely minimizing, so the entire "boundary" forms the cut locus of p. Topologically, this is just $S^1 \vee S^1$.

$\clubsuit$

(e): Prove that if M is closed, then the cut locus is compact. Give an example where M is not closed, and the cut locus is not compact.

Proof. This is Corollary 2.10 in Chapter 13 of do Carmo. □

(f): How does the topology of the cut locus reflect the topology of the manifold.

Answer: Let M^n be complete, with $p \in M$ and $q \in M - C_m(p)$. Then there exists a unique minimizing geodesic connecting p and q, so we see that $M - C_m(p)$ can be contracted along minimizing geodesics down to p. Hence, $M - C_m(p)$ is homeomorphic to an n-cell. Thus, we can build M by attaching an n-cell to $C_m(p)$ in some possibly weird way. Note also that, by the same sort of argument, $M - \{p\}$ can be contracted down to $C_m(p)$ along minimizing geodesics.

In terms of homology, since attaching an n-cell to some $(n-1)$-dimensional space affects at most the nth and $(n-1)$st homologies, we know that, except for the top two homology groups, the homology of M and the homology of $C_m(p)$ are the same. In fact, attaching an n-cell at worst introduces one relation to the $(n-1)$ homology, so $H_{n-1}(M)$ is $H_{n-1}(C_m(p))$ with possibly a relation added.

♣

(g): Is there a Riemannian metric on S^2 whose cut locus is a circle? On $\mathbb{CP}^2$?

Answer: For the S^2 case, the answer is no. Part (f) above demonstrates that, if $C_m(p) \approx S^1$, then $S^2 - C_m(p)$ must be homeomorphic to a disc. However, by the Jordan curve theorem we know that the complement of an embedded S^1 in S^2 is two disjoint discs.

For the $\mathbb{CP}^2$ case, the answer is also no. To see why, note that, if there were such a cut locus, then, by (f), $\mathbb{CP}^2$ could be built from a 4-cell attached to a 1-dimensional space. However, such a space must have trivial second homology group, and we know that $H_2(\mathbb{CP}^2) \simeq \mathbb{Z}$.

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(h): Give an example where the geodesic is not minimizing up to the first conjugate point.

Answer: Consider $\mathbb{RP}^2$ with the round metric. As we saw in (c) above, along any geodesic through p, the only conjugate point is p itself. However, before a geodesic γ can traverse back to p, it must pass through the equator, where it ceases to be minimizing, as we saw in (c).

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(i): Give examples of cut and conjugate loci on surfaces with constant curvature.

Answer: For surfaces of zero and negative curvature, there are no conjugate points. For orientable compact surfaces of constant negative curvature, the cut locus is just $\bigvee_{2g} S^1$, where g is the genus of the surface. To see why, consider the example of the double torus:

As with the flat torus, the "boundary" forms the cut locus of p and this boundary is just $S^1 \vee S^1 \vee S^1 \vee S^1$. This is, perhaps, more clear on a picture that is topologically, but not metrically, accurate:

For non-orientable surfaces, the idea is the same and the result is that the cut locus is $\bigvee_{g+1} S^1$ where g is the genus.

For flat surfaces, we've already seen the flat torus in (d) above and the Klein bottle, by an entirely similar argument, has cut locus $S^1 \vee S^1$. Another example is the cylinder:

The cut locus is the entire vertical line antipodal to p.

For surfaces of positive curvature, we've already seen the sphere and $\mathbb{RP}^2$, which are really our only examples of surfaces with constant positive curvature.

♣

(j): Same on surfaces of positive curvature.

Answer: For another example, consider the paraboloid:

As with the cylinder, the cut locus is the "antipodal" curve to p, but not extending all the way to the base. If p is the base point, then it has no cut locus since there is a minimizing geodesic to all points from p.

On an ellipsoid, the cut locus looks something like a slit "antipodal" to p (similarly to on the paraboloid):

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