

DIFFERENTIAL GEOMETRY HW 5

CLAY SHONKWILER

1

Check the calculations above that the Gaussian curvature of the upper half-plane and Poincaré disk models of the hyperbolic plane is -1 .

Proof. The calculations check out. □

2

(a): Show that if we have an orthogonal parametrization of a surface (that is, $F = 0$), then the gaussian curvature K is given by

$$K = -\frac{1}{2}(EG)^{-1/2} \left[(E_v(EG)^{-1/2})_v + (G_u(EG)^{-1/2})_u \right].$$

Proof. Note that, using this formula,

$$\begin{aligned} K &= -\frac{1}{2}(EG)^{-1/2} \left[E_{vv}(EG)^{-1/2} - \frac{E_v(E_vG + EG_v)}{2(EG)^{3/2}} + G_{uu}(EG)^{-1/2} - \frac{G_u(E_uG + EG_u)}{2(EG)^{3/2}} \right] \\ &= -\frac{E_{vv}}{2EG} + \frac{E_v^2G + EE_vG_v}{4(EG)^2} - \frac{G_{uu}}{2EG} + \frac{E_uGG_u - EG_u^2}{4(EG)^2} \\ &= \frac{E_v^2}{4E^2G} + \frac{E_vG_v}{4EG^2} + \frac{E_uG_u}{4E^2G} + \frac{G_u^2}{4EG^2} + \frac{E_{vv}}{2EG} + \frac{G_{uu}}{2EG} \\ &= \frac{E_v^2 + E_uG_u}{4E^2G} + \frac{G_u^2 + E_vG_v}{4EG^2} \end{aligned}$$

Recall the Gauss formula

$$-EK = \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12,u}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{11,v}^2 - \Gamma_{11}^2\Gamma_{11}^2.$$

Using the fact that $F = 0$, this reduces to

$$-EK = \frac{-E_v^2}{4EG} + \left(\frac{G_u}{2G} \right)_u + \frac{G_u^2}{4G^2} - \frac{E_uG_u}{4EG} - \left(\frac{-E_v}{2G} \right)_v + \frac{E_vG_v}{4G^2}$$

so

$$\begin{aligned} K &= \frac{E_v^2 + E_uG_u}{4E^2G} - \frac{G_u^2 + E_vG_v}{4EG^2} + \frac{G_u^2 - GG_{uu}}{2EG_u^2} + \frac{GE_{vv} - E_vG_v}{2EG_v^2} \\ &= \end{aligned}$$

□

1

(b): Use this formula to compute the Gaussian curvature of the upper half-plane and Poincaré disk models of the hyperbolic plane.

Answer: For the upper half-plane,

$$\begin{aligned}
 K &= -\frac{1}{2}(EG)^{-1/2} \left[(E_v(EG)^{-1/2})_v + (G_u(EG)^{-1/2})_u \right] \\
 &= -\frac{v^2}{2} \left[\left(\frac{-2v^2}{v^3} \right)_v + 0 \right] \\
 &= -\frac{v^2}{2} \left[\left(\frac{-2}{v} \right)_v \right] \\
 &= -\frac{v^2}{2} \cdot \frac{2}{v^2} \\
 &= -1.
 \end{aligned}$$

For the Poincaré disk,

$$\begin{aligned}
 K &= -\frac{1}{2}(EG)^{-1/2} \left[(E_v(EG)^{-1/2})_v + (G_u(EG)^{-1/2})_u \right] \\
 &= -\frac{(1-u^2-v^2)^2}{8} \left[\left(\frac{16v(1-u^2-v^2)^2}{4(1-u^2-v^2)^3} \right)_v + \left(\frac{16u(1-u^2-v^2)^2}{4(1-u^2-v^2)^3} \right)_u \right] \\
 &= -\frac{(1-u^2-v^2)^2}{8} \left[\left(\frac{4v}{1-u^2-v^2} \right)_v + \left(\frac{4u}{1-u^2-v^2} \right)_u \right] \\
 &= -\frac{(1-u^2-v^2)^2}{8} \left[\frac{4(1-u^2-v^2) - 4v(-2v)}{(1-u^2-v^2)^2} + \frac{4(1-u^2-v^2) - 4u(-2u)}{(1-u^2-v^2)^2} \right] \\
 &= -\frac{(1-u^2-v^2)^2}{8} \left[\frac{8(1-u^2-v^2) + 8v^2 + 8u^2}{(1-u^2-v^2)^2} \right] \\
 &= -\frac{8}{8} \\
 &= -1
 \end{aligned}$$

♣

3

(a): Check that $ds^2 = dr^2 + r^2d\theta^2$ describes the Euclidean metric in \mathbb{R}^2 , and compute that the Gaussian curvature is $K \equiv 0$.

Proof. The Euclidean metric on \mathbb{R}^2 is given by the parametrization

$$X(r, \theta) = (x, y) = (r \cos \theta, r \sin \theta).$$

Hence,

$$\begin{aligned}
 X_r &= (\cos \theta, \sin \theta) \\
 X_\theta &= (-r \sin \theta, r \cos \theta),
 \end{aligned}$$

so

$$\begin{aligned}\langle X_r, X_r \rangle &= \cos^2 \theta + \sin^2 \theta = 1 \\ \langle X_r, X_\theta \rangle &= -r \sin \theta \cos \theta + r \sin \theta \cos \theta = 0 \\ \langle X_\theta, X_\theta \rangle &= r^2 \sin^2 \theta + r^2 \cos^2 \theta = r^2.\end{aligned}$$

Thus, $E = 1$, $F = 0$ and $G = r^2$, so $ds^2 = dr^2 + r^2 d\theta^2$.

Furthermore, using the formula from 2(a),

$$\begin{aligned}K &= -\frac{1}{2}(EG)^{-1/2} \left[(E_\theta(EG)^{-1/2})_\theta + (G_r(EG)^{-1/2})_r \right] \\ &= -\frac{1}{2r} \left[0 + \left(\frac{2r}{r} \right)_r \right] \\ &= 0.\end{aligned}$$

□

(b): Check that $ds^2 = dr^2 + \sin^2 r d\theta^2$ describes the round metric on the 2-sphere S^2 , and compute that the Gaussian curvature is $K \equiv 1$. Note that r measures distance along the sphere.

Proof. The parametrization for the round 2-sphere is given by

$$X(r, \theta) = (x, y, z) = (\cos \theta \sin r, \sin \theta \sin r, \cos r).$$

Note that this really is the round sphere, since,

$$x^2 + y^2 + z^2 = \cos^2 \theta \sin^2 r + \sin^2 \theta \sin^2 r + \cos^2 r = \sin^2 r + \cos^2 r = 1.$$

Now,

$$\begin{aligned}X_r &= (\cos \theta \cos r, \sin \theta \cos r, -\sin r) \\ X_\theta &= (-\sin \theta \sin r, \cos \theta \sin r, 0),\end{aligned}$$

so

$$\begin{aligned}\langle X_r, X_r \rangle &= \cos^2 \theta \cos^2 r + \sin^2 \theta \cos^2 r + \sin^2 r = 1 \\ \langle X_r, X_\theta \rangle &= -\sin \theta \cos \theta \sin r \cos r + \sin \theta \cos \theta \sin r \cos r = 0 \\ \langle X_\theta, X_\theta \rangle &= \sin^2 \theta \sin^2 r + \cos^2 \theta \sin^2 r = \sin^2 r.\end{aligned}$$

Hence, $E = 1$, $F = 0$ and $G = \sin^2 r$, so $ds^2 = dr^2 + \sin^2 r d\theta^2$.

Furthermore, using the formula from 2(a),

$$\begin{aligned}K &= -\frac{1}{2}(EG)^{-1/2} \left[(E_\theta(EG)^{-1/2})_\theta + (G_r(EG)^{-1/2})_r \right] \\ &= -\frac{1}{2 \sin r} \left[0 + \left(\frac{2 \sin r \cos r}{\sin r} \right)_r \right] \\ &= -\frac{1}{2 \sin r} (-2 \sin r) \\ &= 1.\end{aligned}$$

□

(c): Show that $ds^2 = dr^2 + \sinh^2 r d\theta^2$ describes a metric with Gaussian curvature $K \equiv -1$.

Proof. With this fundamental form, $E = 1$, $F = 0$, $g = \sinh^2 r$. Using the formula from 2(a),

$$\begin{aligned} K &= -\frac{1}{2}(EG)^{-1/2} \left[(E_\theta(EG)^{-1/2})_\theta + (G_r(EG)^{-1/2})_r \right] \\ &= -\frac{1}{2 \sinh r} \left[0 + \left(\frac{2 \sinh r \cosh r}{\sinh r} \right)_r \right] \\ &= -\frac{1}{2 \sinh r} (2 \sinh r) \\ &= -1. \end{aligned}$$

□

4

Find as many examples you can of complete surfaces with Gaussian curvature $K \equiv 0$. Are they all locally homogeneous? Homogeneous?

Examples: Obviously, \mathbb{R}^2 is a complete surface with Gaussian curvature zero. The infinite cylinder is also a complete surface and has Gaussian curvature zero. Similarly, thinking of the infinite Möbius strip as the strip $\{(x, y) \in \mathbb{R}^2 \mid -1 \leq x < 1\}$ with the endlines identified oppositely, this is a complete surface. Under the metric it inherits from \mathbb{R}^2 , it has Gaussian curvature zero.

Similarly, the torus

is a complete surface and inherits a flat metric from \mathbb{R}^2 . By adjusting the orientations of the edges of this square, we get the Klein bottle and \mathbb{RP}^2 ,

both of which are complete surfaces and inherit a flat metric from \mathbb{R}^2 :

Of course, we can get different versions of all these spaces (homeomorphic but not diffeomorphic) by taking them to be quotients of \mathbb{R}^2 by different lattices. All of the surfaces we've described are homogeneous, since each is covered by \mathbb{R}^2 and so we can p to q via an isometry by taking a point in the fiber over p to a point in the fiber over q by an isometry of \mathbb{R}^2 ; passing to the quotient is still an isometry.



6

Show that on a surface of constant curvature K , the length of the “circle” of radius r is

$$L = 2\pi r - (\pi/3)Kr^3 + \dots$$

Proof. If $K \equiv 0$, this is trivial. Suppose $K \neq 0$. By definition, if C is the circle,

$$L = \int_C ds = \int_C \sqrt{dr^2 + \left(\frac{1}{\sqrt{K}} \sin(\sqrt{K}r) d\theta\right)^2}.$$

Since r is constant, the dr term contributes nothing, so this reduces to

$$\begin{aligned} L &= \int_0^{2\pi} \frac{1}{\sqrt{K}} \sin(\sqrt{K}r) d\theta = \frac{2\pi}{\sqrt{K}} \sin(\sqrt{K}r) = \frac{2\pi}{\sqrt{K}} \left[\sqrt{K}r - \frac{(\sqrt{K}r)^3}{3!} + \dots \right] \\ &= 2\pi r - \frac{\pi}{3}Kr^3 + \dots \end{aligned}$$

□

7

Start with the disk model of the hyperbolic plane,

$$ds^2 = 4(dx^2 + dy^2)/(1 - x^2 - y^2)^2,$$

switch to geodesic polar coordinates, and show directly that

$$ds^2 = dr^2 + \sinh^2 r d\theta^2.$$

Proof. We know that, in geodesic polar coordinates, $ds^2 = dr^2 + G(r, \theta)^2 d\theta^2$. Now, since rotations of the Poincaré disk preserve the metric and every point in the disk can be rotated to a point on the non-negative x -axis, we may as well compute ds^2 at a point on the non-negative x -axis. On the positive x -axis, the positive y direction corresponds to the positive θ direction, so the component of ds in the $d\theta$ direction is given by

$$(1) \quad \frac{2}{1-x^2} \frac{dy}{d\theta} d\theta$$

since the component in the y -direction is given by $\frac{2}{1-x^2} dy$. Now, $\frac{dy}{d\theta}$ gives the rate of change of y with respect to θ ; since the y and θ directions correspond, this will simply be a scalar. The multiplier will be r and, since we're on the x -axis, $r = x$, so $\frac{dy}{d\theta} = x$. Hence, (1) reduces to

$$(2) \quad \frac{2x}{1-x^2} d\theta.$$

Now, since

$$\begin{aligned} r &= \int_0^x \frac{2}{1-t^2} dt = 2 \int_0^x \left[\frac{1/2}{1-t} + \frac{1/2}{1+t} \right] dt \\ &= 2 \left[\frac{1}{2} \log |1-x| + \frac{1}{2} \log |1+x| \right] \\ &= \log \left| \frac{x+1}{x-1} \right| \end{aligned}$$

and $0 \leq x < 1$,

$$r = \log \left(\frac{x-1}{x+1} \right).$$

Therefore,

$$\sinh r = \frac{e^r - e^{-r}}{2} = \frac{\frac{x-1}{x+1} - \frac{x+1}{x-1}}{2} = \frac{-4x}{2(x^2-1)} = \frac{2x}{1-x^2}.$$

Hence, (2) is equal to $\sinh r$ and so, from (1), the component of ds in the $d\theta$ direction is $\sinh r d\theta$. In turn, this implies that $G(r, \theta) = \sinh r$, so

$$ds^2 = dr^2 + \sinh^2 r d\theta^2.$$

□

Show that $w = f(z) = \frac{z-i}{z+i}$ is an isometry from $\mathbb{H} \rightarrow D$.

Proof. First, note that, if $x \in \mathbb{R}$, then

$$\begin{aligned} |f(x)| &= \left| \frac{x-i}{x+i} \right| = \left| \frac{(x-i)^2}{x^2+1} \right| = \left| \frac{x^2-1}{x^2+1} - i \frac{2x}{x^2+1} \right| \\ &= \left(\frac{x^2-1}{x^2+1} \right)^2 + \frac{4x^2}{(x^2+1)^2} \\ &= \frac{x^2+2x+1}{(x^2+1)^2} \\ &= 1, \end{aligned}$$

so f maps the real line to the unit circle. Then, since

$$f(i) = \frac{i-i}{i+i} = 0 \in D,$$

we see that $f : \mathbb{H} \rightarrow D$. Since, as we know from complex analysis, LFTs are biholomorphic where they are defined and f is only undefined at $-i$, we see that $f : \mathbb{H} \rightarrow D$ is a diffeomorphism. Thus, to show that f is an isometry, we need only show that f preserves the first fundamental form. To that end, note that

$$dz = dx + idy \text{ and } d\bar{z} = dx - idy,$$

so $|dz|^2 = dzd\bar{z} = dx^2 + dy^2$, meaning that

$$|ds|_{\mathbb{H}} = \frac{|dz|}{\Im(z)}.$$

Also, $|w|^2 = w\bar{w} = x^2 + y^2$, so

$$|ds|_D = \frac{2|dw|}{1-|w|^2}.$$

Now,

$$\begin{aligned} 1 - |w|^2 &= 1 - \left| \frac{z-i}{z+i} \right|^2 = 1 - \frac{(z-i)(\overline{z-i})}{(z+i)(\overline{z+i})} = 1 - \frac{(z-i)(\bar{z}+i)}{|z+i|^2} \\ &= 1 - \frac{z\bar{z} - i\bar{z} + iz + 1}{|z+i|^2} \\ &= 1 - \frac{|z|^2 + 1 - 2\Im(z)}{|z+i|^2} \\ &= \frac{|z+i|^2 - |z|^2 - 1 + 2\Im(z)}{|z+i|^2} \\ &= \frac{z\bar{z} + i\bar{z} - iz + 1 - z\bar{z} - 1 + 2\Im(z)}{|z+i|^2} \\ &= \frac{4\Im(z)}{|z+i|^2}. \end{aligned}$$

Now,

$$|dw| = |df(z)| = |f'(z)dz| = |f'(z)||dz| = \left| \frac{2i}{(z+i)^2} \right| |dz| = \frac{2|dz|}{|z+i|^2},$$

so

$$|ds|_D = \frac{2|dw|}{1-|w|^2} = \frac{\frac{4|dz|}{|z+i|^2}}{\frac{4\Im(z)}{|z+i|^2}} = \frac{4|dz|}{4\Im(z)} = |dz|_{\mathbb{H}}.$$

Therefore, since $f(z) = w$ preserves the first fundamental form, we see that $f : \mathbb{H} \rightarrow D$ is an isometry. \square

9

Show that compositions of the above isometries give *all* isometries of \mathbb{H}^2 .

Proof. Suppose $h : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is an isometry. Let G be the group generated by the translations, dilations, rotations and flips described above. Since G acts transitively on the points of \mathbb{H}^2 , we can conjugate h by an element of G such that h fixes a point $a \in \mathbb{H}^2$; that is, up to conjugation by G , h fixes a . Again, since G is transitive, we may as well assume $a = (0, 1) \in \mathbb{H}$, which we identify with $i \in \mathbb{C}$. Let $\gamma : (-\epsilon, \epsilon) \rightarrow \mathbb{H}^2$ be a geodesic passing through i such that $\gamma(0) = i$. Since h is an isometry, $h \circ \gamma$ is also a geodesic passing through a , since $(h \circ \gamma)(0) = h(\gamma(0)) = h(i) = i$. Now, if $f : \mathbb{H} \rightarrow D$ is the isometry from problem 8, then $f(i) = 0$, the center of D , so any rotation of D fixes $f(i)$. Let θ be the angle between $\gamma'(0)$ and $dh|_i \gamma'(0)$. Let r be a rotation of D by $-\theta$. Then the composition $f^{-1} \circ r \circ f \circ h$ takes $\gamma'(0)$ to either $\gamma'(0)$ or $-\gamma'(0)$; that is, up to multiplication by an element of G , $h(\gamma) = \gamma$ (by uniqueness of geodesics), so we may as well assume $h(\gamma) = \gamma$. Since h fixes i and preserves γ , there are only two possibilities for what h can be: the identity and the flip $(x, y) \mapsto (-x, y)$. However, both of these isometries are already in G , so we see that $h \in G$. Thus, compositions of the translations, dilations, rotations and flips described above give all isometries of G . \square

12

Show that a LFT takes the upper half-plane \mathbb{H} to itself.

Proof. Let $f : z \mapsto \frac{az+b}{cz+d}$. Note that f is defined on all of \mathbb{C} except when $z = -\frac{d}{c} \notin \mathbb{H}$, so f is defined on all of \mathbb{H} . Now, if $z = x + iy$,

$$\begin{aligned} f(z) = f(x + iy) &= \frac{a(x + iy) + b}{c(x + iy) + d} \\ &= \frac{ax + b + iay}{cx + d + icy} \cdot \frac{cx + d - icy}{cx + d - icy} \\ &= \frac{acx^2 + (ad + bc)x + acy^2 + bd + i(acxy - acxy + (ad - bc)y)}{(cx + d)^2 + c^2y^2} \\ &= \frac{acx^2 + (ad + bc)x + acy^2 + bd}{(cx + d)^2 + c^2y^2} + i \frac{y}{(cx + d)^2 + c^2y^2}, \end{aligned}$$

which is in \mathbb{H} , since $\frac{y}{(cx+d)^2+c^2y^2} > 0$. Therefore, we see that $f : \mathbb{H} \rightarrow \mathbb{H}$. \square

13

Show that the identity matrix I and its negative $-I$ are the only matrices in $SL(2, \mathbb{R})$ which act via LFTs as the identity. Hence we get an induced action of

$$PSL(2, \mathbb{R}) = SL(2, \mathbb{R}) / \{I, -I\}$$

on the upper half-plane \mathbb{H} .

Proof. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ acts as the identity; that is,

$$z = \frac{az + b}{cz + d}$$

for all $z \in \mathbb{C}$. Hence,

$$\frac{-b}{a} = \frac{a \frac{-b}{a} + b}{c \frac{-b}{a} + d} = 0,$$

so $b = 0$. Also,

$$\frac{-d}{c} = \frac{a \frac{-d}{c} + b}{c \frac{-d}{c} + d} = \frac{\frac{-ad}{c} + b}{0},$$

so we must have that $c = 0$. Now,

$$1 = \frac{a(1) + b}{c(1) + d} = \frac{a}{d},$$

so $a = d$. Now, since $ad - bc = 1$, $ad = 1$, so $a = \frac{1}{d}$. Therefore, $a = \frac{1}{a}$, so $d = a = \pm 1$. Therefore, we see that I and $-I$ are the only matrices in $SL(2, \mathbb{R})$ that act as the identity on \mathbb{C} , so the action of $SL(2, \mathbb{R})$ on \mathbb{C} induces an action of $PSL(2, \mathbb{R})$ on \mathbb{C} . \square

14

Define the *cross ratio* of four complex numbers to be

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}.$$

Show that LFTs preserve cross ratios.

Proof. Suppose $f : z \mapsto \frac{az+b}{cz+d}$ is an LFT. Then

$$\begin{aligned} [f(z_1), f(z_2), f(z_3), f(z_4)] &= \frac{(f(z_1) - f(z_2))(f(z_3) - f(z_4))}{(f(z_1) - f(z_3))(f(z_2) - f(z_4))} \\ &= \frac{\left(\frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}\right) \left(\frac{az_3+b}{cz_3+d} - \frac{az_4+b}{cz_4+d}\right)}{\left(\frac{az_1+b}{cz_1+d} - \frac{az_3+b}{cz_3+d}\right) \left(\frac{az_2+b}{cz_2+d} - \frac{az_4+b}{cz_4+d}\right)} \\ &= \frac{(az_1+b)(cz_2+d) - (az_2+b)(cz_1+d)}{(cz_1+d)(cz_2+d)} \cdot \frac{(az_3+b)(cz_4+d) - (az_4+b)(cz_3+d)}{(cz_3+d)(cz_4+d)} \\ &= \frac{(az_1+b)(cz_3+d) - (az_3+b)(cz_1+d)}{(cz_1+d)(cz_3+d)} \cdot \frac{(az_2+b)(cz_4+d) - (az_4+b)(cz_2+d)}{(cz_2+d)(cz_4+d)} \\ &= \frac{((az_1+b)(cz_2+d) - (az_2+b)(cz_1+d))((az_3+b)(cz_4+d) - (az_4+b)(cz_3+d))}{((az_1+b)(cz_3+d) - (az_3+b)(cz_1+d))((az_2+b)(cz_4+d) - (az_4+b)(cz_2+d))} \\ &= \frac{(acz_1z_2 + bcz_2 + adz_1 + bd - acz_1z_2 - bcz_1 - adz_2 - bd)(acz_3z_4 + bcz_4 + adz_3 + bd - acz_3z_4 - bcz_3 - adz_4 - bd)}{(acz_1z_3 + bcz_3 + adz_1 + bd - acz_1z_3 - bcz_1 - adz_3 - bd)(acz_2z_4 + bcz_4 + adz_2 + bd - acz_2z_4 - bcz_2 - adz_4 - bd)} \\ &= \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \\ &= [z_1, z_2, z_3, z_4]. \end{aligned}$$

Since our choice of LFT was arbitrary, we see that LFTs preserve cross ratios. \square

15

Let $w = z + \Delta z$ and $\bar{w} = \bar{z} + \Delta \bar{z}$, where we are using the overscore to denote “complex conjugate”. Let $\Delta z \rightarrow 0$ and show that

$$[z, w, \bar{z}, \bar{w}] \rightarrow \frac{dzd\bar{z}}{(z - \bar{z})^2} = \frac{dx^2 + dy^2}{4y^2}.$$

Conclude that, since LFTs preserve cross ratios, they also preserve

$$\frac{dx^2 + dy^2}{y^2} = ds^2,$$

and hence are isometries.

Proof. By definition,

$$\begin{aligned} [z, w, \bar{z}, \bar{w}] &= \frac{(z-w)(\bar{z}-\bar{w})}{(z-\bar{z})(w-\bar{w})} \\ &= \frac{(z-(z+\Delta z))(\bar{z}-(\bar{z}+\Delta\bar{z}))}{(z-\bar{z})(z+\Delta z-(\bar{z}+\Delta\bar{z}))} \\ &= \frac{\Delta z \Delta\bar{z}}{(z-\bar{z})(z+\Delta z-(\bar{z}+\Delta\bar{z}))} \end{aligned}$$

Therefore, as $\Delta z \rightarrow 0$,

$$[z, w, \bar{z}, \bar{w}] \rightarrow \frac{dzd\bar{z}}{(z-\bar{z})(z-\bar{z})} = \frac{dzd\bar{z}}{(z-\bar{z})^2}.$$

If $z = x + iy$, then the righthand side becomes, as we've seen in Problem 8, $\frac{dx^2+dy^2}{4y^2}$. Therefore,

$$[z, w, \bar{z}, \bar{w}] \rightarrow \frac{dx^2 + dy^2}{4y^2},$$

so, since LFTs preserve the cross ratio, they must also preserve the righthand side, which is $\frac{1}{4}$ the first fundamental form of the upper half-plane model of \mathbb{H} , so we see that LFTs preserve the first fundamental form on \mathbb{H} and, therefore, are isometries of \mathbb{H} . \square

16

Show that $PSL(2, \mathbb{R})$ is the identity component of the group of all isometries of the hyperbolic plane.

Proof. Problems 13 and 15, above, taken together demonstrate that $PSL(2, \mathbb{R})$ is a subgroup of the group of isometries of the hyperbolic plane; furthermore, since the identity element $I \in PSL(2, \mathbb{R})$ acts as the identity isometry on \mathbb{H} , we know that, since $PSL(2, \mathbb{R})$ is connected, it is contained in the identity component of the group of isometries on the hyperbolic plane. Since compositions of translations, rotations, dilations and flips give all isometries of \mathbb{H}^2 and translations, dilations and rotations are orientation-preserving while flips are orientation-reversing, we see that the identity component of the group of isometries of \mathbb{H}^2 is given by compositions of translations, rotations and dilations. Now, as we know from complex analysis, every LFT is the composition of translations, rotations and dilations, so we see that $PSL(2, \mathbb{R})$ is precisely the identity component of the group of isometries of \mathbb{H}^2 . \square

17

- (a): Use the existence and uniqueness theorem to show that any curve on a Riemannian surface which is the fixed point set of an isometry must be a geodesic.

Proof. Let M be a Riemannian surface. Suppose the curve $\alpha(t)$ is the fixed point set of an isometry f . Let p be a point on α ; without loss of generality, we may assume that $\alpha(0) = p$. Then, by the existence and uniqueness theorem for geodesics, there exists a unique geodesic $\gamma : (-\epsilon, \epsilon) \rightarrow M$ (for some $\epsilon > 0$ such that $\gamma(0) = \alpha(0) = p$ and $\gamma'(0) = \alpha'(0)$). Now, since α is fixed by f , $f \circ \alpha = \alpha$, so $(f \circ \alpha)'(t) = \alpha'(t)$ for all t . Furthermore, since $\gamma(0) = \alpha(0)$ and $\gamma'(0) = \alpha'(0)$,

$$(f \circ \gamma)(0) = (f \circ \alpha)(0) = \alpha(0) = \gamma(0)$$

and

$$(f \circ \gamma)'(0) = (f \circ \alpha)'(0) = \alpha'(0) = \gamma'(0).$$

On the other hand, since f is an isometry, $f \circ \gamma$ is a geodesic which, as we've just seen, passes through p and has tangent vector at p given by $\alpha'(0)$. Thus, by the uniqueness of geodesics, $f \circ \gamma = \gamma$ wherever γ is defined. Since γ is fixed by f and α is precisely the fixed point set of f , this implies that $\gamma = \alpha$. Since our choice of α was arbitrary, we see that any curve on a Riemannian surface which is the fixed point set of an isometry must be a geodesic. \square

(b): Conclude that the vertical lines in the upper half-plane model are geodesics, and that the rays through the origin in the Poincaré disk model are geodesics.

Proof. Note, first of all, that the vertical line $x = 0$ is the fixed point set of the isometry $(x, y) \mapsto (-x, y)$ in the upper half-plane. Since any vertical line in the upper half-plane can be horizontally translated to the y -axis and horizontal translations have no fixed points, we see that any vertical line in the upper half-plane is the fixed point set of the isometry which is the composition of the following isometries: the horizontal translation that takes that vertical line to the y -axis, followed by the flip $(x, y) \mapsto (-x, y)$, followed by the horizontal translation taking the y -axis back to the given vertical line. Therefore, we see that every vertical line is the fixed point set of an isometry and so, by part (a), is a geodesic.

Turning to the Poincaré disk, we know, from problem 9, that any rotation about the origin is an isometry. Now, any diameter of the disk can be rotated to the diameter along the x -axis. Furthermore, using the inverse of the isometry in problem 8 above (given by $f^{-1}(w) = g(w) = \frac{i(1+w)}{1-w}$ for $w \in D$), the diameter along the x -axis in D is taken to the y -axis in \mathbb{H}^2 . Thus, if h is the isometry $(x, y) \mapsto (-x, y)$ in \mathbb{H}^2 and $f = g^{-1}$, $f \circ h \circ g$ is an isometry of the disk and the diameter along the x -axis is its fixed point set. Thus, we see that any diameter is the fixed set of the isometry obtained by conjugating $f \circ h \circ g$ by the rotation sending the given diameter to the x -axis. Again, by part (a), this implies that all diameters of the disk are geodesics. \square

(a): Show that the remaining geodesics in the upper half-plane model are semi-circles which meet the x -axis orthogonally.

Proof. First, we show that any such semi-circle is a geodesic. Since any such circle can be transformed into the semi-circle centered at the origin of radius 1 by a combination of horizontal translations and scalings, we see that it suffices to show that the semi-circle centered at the origin of radius 1 is the fixed point set of an isometry (since we can simply conjugate by the necessary translation and dilation taking a given semi-circle to the one centered at the origin of radius 1 to get an isometry whose fixed point set is the given circle). Now, $f(z) = \frac{-1}{z}$ is an LFT since $\det \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 1$, so f is an isometry of the upper half-plane. Let $h : x + iy \mapsto -x + iy$.

Now, if $z = x + iy$ such that $|z| = 1$, then $\frac{1}{z} = \bar{z}$ and so

$$(h \circ f)(z) = h(f(z)) = h\left(\frac{-1}{z}\right) = h(-\bar{z}) = h(-x + iy) = x + iy = z.$$

Since h and f are both isometries, we see that $h \circ f$ is an isometry of the upper half-plane whose fixed point set is precisely the semi-circle of radius 1 centered at the origin.

Now, suppose $\gamma : (-\epsilon, \epsilon) \rightarrow \mathbb{H}^2$ is a geodesic, where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. If $\gamma'(0)$ is vertical, then the vertical line $x = \gamma_1(0)$ passes through $\gamma(0)$ and is parallel to $\gamma'(0)$; by the uniqueness of geodesics, this implies that γ is simply a parametrization of this vertical line. If $\gamma'(0)$ is not vertical, then there is a semi-circle centered on the x -axis passing through $\gamma(0)$ and parallel to $\gamma'(0)$; again, since such a semi-circle is a geodesic (as we just proved), the uniqueness of geodesics implies that γ is just a parametrization of this semi-circle. Since our choice of γ was arbitrary, we see that any geodesic in the upper half-plane is either a vertical line or a semi-circle which meets the x -axis orthogonally. \square

(b): Show that the remaining geodesics in the Poincaré disk model are arcs of circles which meet the boundary of the disk orthogonally.

Proof. Let $f : \mathbb{H} \rightarrow D$ be the isometry described in problem 8. Then f is a conformal map and so takes (generalized) circles to circles. Now, since f maps the boundary of \mathbb{H} (that is, the real line) to the boundary of D (that is, the unit circle). Hence, f maps circles which meet the real line orthogonally to circles which meet the unit circle orthogonally, so we see that f maps the geodesics described in part (a) to arcs of circles which meet the boundary of the disk orthogonally. Hence, such arcs are geodesics in D . Since the geodesics of D are just the images under f of the geodesics of \mathbb{H} ,

we see that all geodesics of D are either arcs of circles which meet the boundary of the disk orthogonally or diameters of the disk. \square

DRL 3E3A, UNIVERSITY OF PENNSYLVANIA
E-mail address: `shonkwil@math.upenn.edu`