DIFFERENTIAL GEOMETRY HW 3

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3.

Let S be the "saddle surface" $z = y^2 - x^2$, officially known as a *hyperbolic paraboloid*. Parametrize S by the map $X : \mathbb{R}^2 \to S \subset \mathbb{R}^3$, $X(u,v) = (u, v, v^2 - u^2)$, which is the usual way to parametrize the graph of a function from $\mathbb{R}^2 \to \mathbb{R}^3$. Choose $N(u,v)$ to be the roughly upward pointing unit normal vector to S at the point $X(u,v)$.

(1) Show that:

Proof. Since $X(u,v) = (u, v, v^2 - u^2)$, $X_u = (1, 0, -2u)$ and $X_v = (0, 1, 2v)$. Then

$$\begin{aligned} N(u,v) &= \frac{X_u \times X_v}{|X_u \times X_v|} \\ &= \frac{(2u, -2v, 1)}{|(2u, -2v, 1)|} \\ &= \frac{1}{\sqrt{4u^2 + 4v^2 + 1}}(2u, -2v, 1). \end{aligned}$$

Then

$$\begin{aligned} dN_p(X_u) &= N_u \\ &= \frac{1}{4u^2+4v^2+1}\left(2\sqrt{4u^2+4v^2+1} + \frac{8u^2}{\sqrt{4u^2+4v^2+1}}, \frac{8uv}{\sqrt{4u^2+4v^2+1}}, \frac{-4u}{\sqrt{4u^2+4v^2+1}}\right) \\ &= \frac{1}{(4u^2+4v^2+1)^{3/2}}(8v^2 + 2, 8uv, -4u). \end{aligned}$$

Also,

$$\begin{aligned} dN_p(X_v) &= N_v \\ &= \frac{1}{4u^2+4v^2+1}\left(\frac{-8uv}{\sqrt{4u^2+4v^2+1}}, -2\sqrt{4u^2+4v^2+1} + \frac{8v^2}{\sqrt{4u^2+4v^2+1}}, \frac{-4v}{\sqrt{4u^2+4v^2+1}}\right) \\ &= \frac{1}{(4u^2+4v^2+1)^{3/2}}(-8uv, -8u^2 - 2, -4v). \end{aligned}$$

□

(2) Check that N_u and N_v are both orthogonal to N.

Proof. Note that

$$\langle N, N_u \rangle = \frac{1}{(4u^2+4v^2+1)^2}(2u(8v^2+2) - 2v(8uv) - 4u) = 0$$

and

$$\langle N, N_v \rangle = \frac{1}{(4u^2+4v^2+1)^2}(2u(-8uv) - 2v(-8u^2-2) - 4v) = 0,$$

so N_u and N_v are orthogonal to N. □

(3) Show that at $p =$ origin, the map $dN_p : T_pS \to T_pS$ is given by the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

with respect to the basis $X_u = (1, 0, 0)$, $X_v = (0, 1, 0)$ of $T_{\text{origin}}S$.

Proof. At the origin, T_pS lies in the uv-plane, so, with the given basis, we can ignore the z-coordinate of dN_p. Hence,

$$dN_p = \begin{pmatrix} 8v^2+2 & 8uv \\ -8uv & -8u^2-2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},$$

since $u = v = 0$ at the origin. □

6.

Verify the calculations for X_u, X_v, N, N_u, N_v, etc. for the helicoid

$$X(u, v) = (u\cos v, u\sin v, v).$$

Proof. First, note that

$$X_u = (\cos v, \sin v, 0) \text{ and } X_v = (-u\sin v, u\cos v, 1).$$

Hence,

$$\begin{aligned} N(u, v) &= \frac{X_u \times X_v}{|X_u \times X_v|} \\ &= \frac{(\sin v, -\cos v, u\cos^2 v + u\sin^2 v)}{|(\sin v, -\cos v, u\cos^2 v + u\sin^2 v)|} \\ &= \frac{1}{\sqrt{1+u^2}}(\sin v, -\cos v, u). \end{aligned}$$

Therefore,

$$\begin{aligned} N_u &= \frac{1}{1+u^2}\left(\frac{-u\sin v}{\sqrt{1+u^2}}, \frac{u\cos v}{\sqrt{1+u^2}}, \sqrt{1+u^2} - \frac{u^2}{\sqrt{1+u^2}}\right) \\ &= \frac{1}{(1+u^2)^{3/2}}(-u\sin v, u\cos v, 1) \\ &= \frac{X_v}{(1+u^2)^{3/2}} \end{aligned}$$

and

$$N_v = \frac{1}{\sqrt{1+u^2}}(\cos v, \sin v, 0) = \frac{X_u}{\sqrt{1+u^2}}.$$

Thus,

$$\langle N_u, X_u \rangle = \frac{1}{(1+u^2)^{3/2}}(-u \sin v \cos v + u \sin v \cos v) = 0,$$

$$\langle N_u, X_v \rangle = \frac{1}{(1+u^2)^{3/2}}(u^2 \sin^2 v + u^2 \cos^2 v + 1) = \frac{1+u^2}{(1+u^2)^{3/2}} = \frac{1}{\sqrt{1+u^2}},$$

$$\langle N_v, X_u \rangle = \frac{1}{\sqrt{1+u^2}}(\cos^2 v + \sin^2 v) = \frac{1}{\sqrt{1+u^2}},$$

and

$$\langle N_v, X_v \rangle = \frac{1}{\sqrt{1+u^2}}(-u \sin v \cos v + u \sin v \cos v) = 0.$$

□

7.

On a sphere of any radius, consider a great circle and a smaller circle which are tangent to one another at some point. Let κ_G be the curvature of the great circle and κ_S be the curvature of the small circle. Let θ denote the angle between their principal normals at the common point. Show that

$$\kappa_S = \kappa_G / \cos\theta.$$

Proof. Let r be the radius of a circle. Then $\alpha(\theta) = \left(r\cos\frac{\theta}{r}, r\sin\frac{\theta}{r}\right)$ is an arc-length parametrization of the circle. Hence,

$$\alpha'' = \left(-\frac{1}{r}\cos\frac{\theta}{r}, -\frac{1}{r}\sin\frac{\theta}{r}\right),$$

so

$$\kappa = |\alpha''| = \sqrt{\frac{1}{r^2}} = \frac{1}{r}.$$

Now, suppose R is the radius of the sphere. Then R is also the radius of G, meaning that $\kappa_G = \frac{1}{R}$. Now, suppose r is the radius of the smaller circle S. Then $\kappa_S = \frac{1}{r}$. Then we see the following right triangle:

Hence, $\cos\theta = \frac{r}{R} = \frac{\frac{1}{\kappa_S}}{\frac{1}{\kappa_G}}$, meaning that

$$\kappa_S = \frac{\kappa_G}{\cos\theta}.$$

□

8.

Compute the curvature κ_0 at the origin of this curve, $\alpha_0(t) = (t, 0, f(t, 0))$, and show that

$$\kappa_0 = \frac{|\alpha_0' \times \alpha_0''|}{|\alpha_0'|^3} = |(0, -f_{xx}, 0)| = |f_{xx}|.$$

Proof. First, note that

$$\alpha_0' = (1, 0, f_x) \quad \alpha_0'' = (0, 0, f_{xx}).$$

Note that, since the surface is tangent to the xy-plane at the origin, there is no change in the z-direction at the origin, so $f_x = 0$ at the origin. Hence,

$$|\alpha_0'|^3 = 1$$

and

$$\alpha_0' \times \alpha_0'' = (0, -f_{xx}, 0),$$

so

$$\kappa_0 = \frac{|\alpha_0' \times \alpha_0''|}{|\alpha_0'|^3} = |(0, -f_{xx}, 0)| = |f_{xx}|.$$

□

9.

Compute the curvature κ_α at the origin of the more general curve

$$\alpha(t) = (t, g(t), f(t, g(t))),$$

and show that

$$\kappa_\alpha = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = |(0, -f_{xx}, g'')| = \sqrt{f_{xx}^2 + (g'')^2}.$$

Proof. First, note that

$$\alpha' = (1, g', f_x + f_y g')$$

and

$$\alpha'' = (0, g'', f_{xx} + f_y g'' + f_{xy} g' + f_{yy}(g')^2).$$

Now, since the surface is tangent to the xy-plane, so is the curve $g(t)$, so $g'(0) = 0$ and $f_x = f_y = 0$ at the origin. Hence,

$$|\alpha'|^3 = 1$$

and

$$\begin{aligned}\alpha' \times \alpha'' &= (g'(f_{xx} + f_y g'' + f_{xy} g' + f_{yy}(g')^2) - g''(f_x + f_y g'), -(f_{xx} + f_y g'' + f_{xy} g' + f_{yy}(g')^2, g'') \\ &= (0, -f_{xx}, g'').\end{aligned}$$

Hence,

$$\kappa_\alpha = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = |(0, -f_{xx}, g'')| = \sqrt{f_{xx}^2 + (g'')^2}.$$

□

10.

Show that $\kappa_\alpha = \kappa_0/|\cos\theta|$.

Proof. Note that the tangent vector to α at the origin is given by $\alpha' = (1, g', f_x + f_y g')$. Therefore, the principal normal is given by

$$\begin{aligned} N_\alpha = \frac{\alpha''}{|\alpha''|} &= \frac{1}{\sqrt{(g'')^2 + (f_{xx} + f_y g'' + f_{xy} g' + f_{yy}(g')^2)^2}}(0, g'', f_{xx} + f_y g'' + f_{xy} g' + f_{yy}(g')^2) \\ &= \frac{1}{\sqrt{(g'')^2 + f_{xx}^2}}(0, g'', f_{xx}) \\ &= \frac{1}{\kappa_\alpha}(0, g'', f_{xx}) \end{aligned}$$

since g', f_x and f_y all vanish at the origin. Hence,

$$|N||N_\alpha|\cos\theta = \langle N, N_\alpha \rangle = \frac{f_{xx}}{\kappa_\alpha}.$$

Since $|N| = |N_\alpha| = 1$, this implies that $\cos\theta = \frac{f_{xx}}{\kappa_\alpha}$. Therefore,

$$\frac{\kappa_0}{|\cos\theta|} = \frac{|f_{xx}|}{\frac{|f_{xx}|}{\kappa_\alpha}} = \kappa_\alpha.$$

□

12.

Consider the curve $\alpha(t)$ on S given by

$$\alpha(t) = \left(t\cos\theta, t\sin\theta, \frac{1}{2}at^2\cos^2\theta + \frac{1}{2}bt^2\sin^2\theta\right).$$

Show that

$$k_\theta = k_1\cos^2\theta + k_2\sin^2\theta,$$

which tells us how the normal curvature varies with direction.

Proof. First, note that

$$\alpha' = (\cos\theta, \sin\theta, at\cos^2\theta + bt\sin^2\theta)$$

and

$$\alpha'' = (0, 0, a\cos^2\theta + b\sin^2\theta).$$

Since S is tangent to the xy-plane at the origin, the z-coordinate of α' is zero; hence,

$$|\alpha'|^3 = (\cos^2\theta + \sin^2\theta)^{3/2} = 1.$$

Also,

$$\alpha' \times \alpha'' = (\sin\theta(a\cos^2\theta + b\sin^2\theta), -\cos\theta(a\cos^2\theta + b\sin^2\theta), 0),$$

so

$$|\alpha' \times \alpha''| = \sqrt{(\sin^2\theta + \cos^2\theta)(a\cos^2\theta + b\sin^2\theta)^2} = a\cos^2\theta + b\sin^2\theta.$$

Therefore,

$$k_\theta = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{a\cos^2\theta + b\sin^2\theta}{1} = k_1\cos^2\theta + k_2\sin^2\theta,$$

since $k_1 = a$ and $k_2 = b$. □

16.

See attached Maple sheets.

19.

See attached Maple sheets.

20.

See attached Maple sheets.

21.

Let V be a small neighborhood of the point p on the regular surface S, and $N(V) \subset S^2$ its image under the Gauss map N. Show that the Gaussian curvature K of S at p is given by the limit

$$\kappa = \lim_{V\to p} \text{area}(N(V))/\text{area}(V).$$

Show that this generalizes an analogous result for the curvature of plane curves.

Proof. By definition, $\text{Area}(V) = \int_V |X_u \times X_v| dA$. Now,

$$\text{Area}(N(V)) = \int_{N(V)} |N_u \times N_v| dA = \int_{N(V)} |K(u,v)||X_u \times X_v| dA.$$

Now, as $V \to p$, $|K(u,v)|$ goes to the constant map $K(p)$, the Gaussian curvature at p, so we can pull it outside the integral. Hence,

$$\lim_{V\to p} \frac{\text{Area}(N(V))}{\text{Area}(V)} = \lim_{V\to p} \frac{\int_{N(V)} |K||X_u \times X_v| dA}{\int_V |X_u \times X_v| dA} = K.$$

□

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