DIFFERENTIAL GEOMETRY HW 12

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3

Find the Lie algebra $\mathfrak{so}(n)$ of the special orthogonal group $SO(n)$, and the explicit formula for the Lie bracket there.

Proof. Since $SO(n)$ is a subgroup of $GL(n)$, we know that $\mathfrak{so}(n) \subset \mathfrak{gl}(n) \simeq M(n)$, the $n \times n$ matrices. Consider the matrix $I + tA$, where I is the $n \times n$ identity matrix and $A \in \mathfrak{so}(n)$. If $I + tA \in SO(n)$, then

$$I = (I+tA)(I+tA)^T = (I+tA)(I^T+tA^T) = (I+tA)(I+tA^T) = I^2+tA+tA^T+t^2AA^T.$$

Hence, differentiating both sides with respect to t,

$$0 = A + A^T + 2tAA^T.$$

For $t = 0$, $I + tA \in SO(n)$, so, plugging in $t = 0$ to the above equation,

$$0 = A + A^T.$$

Therefore, since our choice of $A \in \mathfrak{so}(n)$ was arbitrary, we see that $\mathfrak{so}(n)$ consists of skew-symmetric matrices, where $[A, B] = AB - BA$ is just regular matrix multiplication (as it is in $\mathfrak{gl}(n)$). In fact, since the dimension of the space of skew-symmetric matrices is $\frac{n(n-1)}{2}$, which is equal to the dimension of $SO(n)$ and, therefore, $\mathfrak{so}(n)$, $\mathfrak{so}(n)$ consists precisely of the skew-symmetric $n \times n$ matrices. □

2

Show that $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$\mathbf{x}(\theta, \phi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\phi, \sin\phi), \quad (\theta, \phi) \in \mathbb{R}^2$$

is an immersion of $\mathbb{R}^2$ into the unit sphere $S^3(1) \subset \mathbb{R}^4$, whose image $\mathbf{x}(\mathbb{R}^2)$ is a torus T^2 with sectional curvature zero in the induced metric.

Proof. First, note that

$$d\mathbf{x} = \begin{pmatrix} -\frac{1}{\sqrt{2}}\sin\theta & 0 \\ \frac{1}{\sqrt{2}}\cos\theta & 0 \\ 0 & -\frac{1}{\sqrt{2}}\sin\phi \\ 0 & \frac{1}{\sqrt{2}}\cos\phi \end{pmatrix},$$

which is certainly of rank 2, so $\mathbf{x}$ is an immersion. Moreover,

$$\|\mathbf{x}(\theta,\phi)\| = \frac{1}{2}(\cos^2\theta + \sin^2\theta + \cos^2\phi + \sin^2\phi) = \frac{1}{2}(2) = 1,$$

so the image of $\mathbf{x}$ is contained in $S^3(1)$. Since $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$ parameterize the unit circle, the image of $\mathbf{x}$ is $S^1 \times S^1 = T^2$. Now, if $X_i = \frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial\theta} = \frac{1}{\sqrt{2}}(-\sin\theta X_1 + \cos\theta X_2)$ and $\frac{\partial}{\partial\phi} = \frac{1}{\sqrt{2}}(-\sin\phi X_3 + \cos\phi X_4)$. Note that $\overline{K}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\right) = 0$ since all sectional curvatures of $\mathbb{R}^4$ are zero. Hence, using the Gauss Theorem,

$$\begin{aligned}
K\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\right) &= K\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\right) - \overline{K}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\right) \\
&= \left\langle B\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right), B\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}\right)\right\rangle - \left|B\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\right)\right|^2 \\
&= \left\langle \overline{\nabla}_{\overline{\partial/\partial\theta}}\overline{\frac{\partial}{\partial\theta}} - \nabla_{\partial/\partial\theta}\frac{\partial}{\partial\theta}, \overline{\nabla}_{\overline{\partial/\partial\phi}}\overline{\frac{\partial}{\partial\phi}} - \nabla_{\partial/\partial\phi}\frac{\partial}{\partial\phi}\right\rangle - \left|\overline{\nabla}_{\overline{\partial/\partial\theta}}\overline{\frac{\partial}{\partial\phi}} - \nabla_{\partial/\partial\theta}\frac{\partial}{\partial\phi}\right|^2 \qquad (1) \\
&= \left\langle \overline{\nabla}_{\overline{\partial/\partial\theta}}\overline{\frac{\partial}{\partial\theta}}, \overline{\nabla}_{\overline{\partial/\partial\phi}}\overline{\frac{\partial}{\partial\phi}}\right\rangle - \left|\overline{\nabla}_{\overline{\partial/\partial\theta}}\overline{\frac{\partial}{\partial\phi}}\right|^2
\end{aligned}$$

since $\nabla_{\partial/\partial\theta}\frac{\partial}{\partial\theta} = 0$, $\nabla_{\partial/\partial\phi}\frac{\partial}{\partial\phi} = 0$ and $\nabla_{\partial/\partial\theta}\frac{\partial}{\partial\phi} = 0$ (because θ and ϕ give coordinates on $\mathbb{R}^2$). Now, extend $\frac{\partial}{\partial\theta}$ to the vector field V where $V(x_1,x_2,x_3,x_4) = \frac{1}{\sqrt{2}}(-x_2, x_1, -x_4, x_3)$. Then

$$\begin{aligned}
\overline{\nabla}_{\overline{\partial/\partial\theta}}\overline{\frac{\partial}{\partial\theta}} &= \overline{\nabla}_V V \\
&= \frac{1}{2}\nabla_{-x_2X_1+x_1X_2}(-x_2X_1 + x_1X_2) \\
&= \frac{1}{2}\left[-x_2\nabla_{X_1}(-x_2X_1 + x_1X_2) + x_1\nabla_{X_2}(-x_2X_1 + x_1X_2)\right] \\
&= \frac{1}{2}\left[-x_2X_1(x_1)X_2 + x_1X_2(-x_2)X_1\right] \\
&= \frac{1}{2}\left[-x_1X_1 - x_2X_2\right],
\end{aligned}$$

since all the other terms in the expansion of the third line are zero. A similar argument shows that $\overline{\nabla}_{\overline{\partial/\partial\phi}}\overline{\frac{\partial}{\partial\phi}} = \frac{1}{2}\left[-x_3X_3 - x_4X_4\right]$ and that $\overline{\nabla}_{\overline{\partial/\partial\theta}}\overline{\frac{\partial}{\partial\phi}} = 0$. Therefore,

$$K\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\right) = \left\langle \overline{\nabla}_{\overline{\partial/\partial\theta}}\overline{\frac{\partial}{\partial\theta}}, \overline{\nabla}_{\overline{\partial/\partial\phi}}\overline{\frac{\partial}{\partial\phi}}\right\rangle - \left|\overline{\nabla}_{\overline{\partial/\partial\theta}}\overline{\frac{\partial}{\partial\phi}}\right|^2 = \frac{1}{4}\langle -x_1X_1 - x_2X_2, -x_3X_3 - x_4X_4\rangle - 0 = 0.$$

□

6

Let G be a Lie group with a bi-invariant metric. Let H be a Lie group and let $h : H \to G$ be an immersion that is also a homomorphism of groups. Show that h is a totally geodesic immersion.

Proof. Since the metric H inherits from G is bi-invariant on H, we know that the geodesics through the identity in H are one-parameter subgroups of H. However, a one-parameter subgroup of H is also a one-parameter subgroup of G. Since the one-parameter subgroups of G are just the geodesics through the identity, this implies that the geodesics through the identity in H are geodesics in G, which demonstrates that H is a geodesic immersion at the identity. Since left-translation is an isometry of H and isometries take geodesics to geodesics, this in turn means that H is a geodesic immersion at every $p \in H$, so H is a totally geodesic immersion.

The metric inherited from G on H is simply $\langle u, v\rangle_H = \langle (dh)_g u, (dh)_g v\rangle_G$, where $u, v \in T_gH$. To see that the metric H inherits is actually bi-invariant on H, suppose $u, v \in T_gH$. Then

$$\begin{aligned}
\langle u, v\rangle_H = \langle (dh_g)u, (dh)_g v\rangle_G &= \langle (dL_{h(g')})_g \circ (dh_g)u, (dL_{h(g')})_g \circ (dh)_g v\rangle_G \\
&= \langle d(L_{h(g')} \circ h)_g u, d(L_{h(g')} \circ h)_g v\rangle_G \\
&= \langle d(h \circ h^{-1}|_{g'} \circ L_{h(g')} \circ h)_g u, d(h \circ h^{-1}|_{g'} \circ L_{h(g')} \circ h)_g\rangle_G \\
&= \langle d(h \circ L_{g'})_g u, d(h \circ L_{g'})_g v\rangle_G \\
&= \langle (dh)_{g'} \circ (dL_{g'})_g u, (dh)_{g'} \circ (dL_{g'})_g v\rangle_G \\
&= \langle (dL_{g'})_g u, (dL_{g'})_g v\rangle_H.
\end{aligned}$$

A similar calculation shows that this inherited metric is right-invariant, so we see that it is, in fact, bi-invariant. □

8

Consider the immersion $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^4$ given in Exercise 2.

(a): Show that the vectors

$$e_1 = (-\sin\theta, \cos\theta, 0, 0), \quad e_2 = (0, 0, -\sin\phi, \cos\phi)$$

form an orthonormal basis of the tangent space, and that the vectors $n_1 = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\phi, \sin\phi)$, $n_2 = \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\phi, \sin\phi)$ form an orthonormal basis of the normal space.

Proof. First,

$$\langle e_1, e_2\rangle = 0,$$

$$\langle e_1, e_1\rangle = \sin^2\theta + \cos^2\theta = 1$$

and

$$\langle e_2, e_2\rangle = \sin^2\phi + \cos^2\phi = 1,$$

so e_1 and e_2 form an orthonormal basis of the tangent space. Moreover,

$$\langle n_1, n_1\rangle = \frac{1}{2}(\cos^2\theta + \sin^2\theta + \cos^2\phi + \sin^2\phi) = 1,$$
$$\langle n_2, n_2\rangle = \frac{1}{2}(\cos^2\theta + \sin^2\theta + \cos^2\phi + \sin^2\phi) = 1$$

and

$$\langle n_1, n_2\rangle = \frac{1}{2}(-\cos^2\theta - \sin^2\theta + \cos^2\phi + \sin^2\phi) = 0.$$

Also,

$$\langle e_1, n_1\rangle = \frac{1}{\sqrt{2}}(-\sin\theta\cos\theta + \sin\theta\cos\theta) = 0,$$
$$\langle e_1, n_2\rangle = \frac{1}{\sqrt{2}}(\sin\theta\cos\theta - \sin\theta\cos\theta) = 0$$

and, similarly, $\langle e_2, n_1\rangle = 0$ and $\langle e_2, n_2\rangle = 0$. Therefore, n_1 and n_2 form an orthonormal basis of the normal space. □

(b): Use the fact that

$$\langle S_{n_k}(e_i), e_j\rangle = -\langle\overline{\nabla}_{e_i} n_k, e_j\rangle = \langle\overline{\nabla}_{e_i} e_j, n_k\rangle,$$

where $\overline{\nabla}$ is the covariant derivative of $\mathbb{R}^4$, and $i, j, k = 1, 2$, to establish that the matrices of S_{n_1} and S_{n_2} with respect to the basis $\{e_1, e_2\}$ are

$$S_{n_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$S_{n_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Proof. Note that, using what we showed in problem 2, $e_1 = \sqrt{2}\frac{\partial}{\partial\theta}$ and $e_2 = \sqrt{2}\frac{\partial}{\partial\phi}$. Since $\langle S_{n_k}(e_i), e_j\rangle = \langle\overline{\nabla}_{e_i} e_j, n_k\rangle$, we have that

$$\begin{aligned}\langle S_{n_1}(e_1), e_1\rangle &= \langle\overline{\nabla}_{e_1} e_1, n_1\rangle \\ &= \left\langle \overline{\nabla}_{\sqrt{2}\partial/\partial\theta}\sqrt{2}\frac{\partial}{\partial\theta}, n_1\right\rangle \\ &= \left\langle -\cos\theta X_1 - \sin\theta X_2, \frac{1}{\sqrt{2}}(\cos\theta X_1 + \sin\theta X_2 + \cos\phi X_3 + \sin\phi X_4)\right\rangle \\ &= \frac{1}{\sqrt{2}}(-\cos^2\theta - \sin^2\theta) \\ &= -\sqrt{1}.\end{aligned}$$

The other 7 calculations are tedious but similar and allow us to conclude that S_{n_1} and S_{n_2} are as above. □

(c): From Exercise 2, $\mathbf{x}$ is an immersion of the torus T^2 into $S^3(1)$ (the Clifford torus). Show that $\mathbf{x}$ is a minimal immersion.

Proof. Note that

$$\left\|\frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\phi, \sin\phi)\right\|^2 = \frac{1}{2}(\cos^2\theta + \sin^2\theta + \cos^2\phi + \sin^2\phi) = 1,$$

so n_1, thought of as a point, lies on the sphere $S^3(1)$, so n_1, thought of as a vector, is orthogonal to $S^3(1)$. Therefore, the subspace of T_pS^3 normal to T_pT^2 (where T_2 is the image of $\mathbf{x}$), is spanned by n_2. Therefore, the condition of being minimal boils down to the condition that trace $S_{n_2} = 0$, which is clearly does, given the computation in part (b) above. □

11

Let $f : \overline{M}^{n+1} \to \mathbb{R}$ be a differentiable function. Define the *Hessian*, Hess f of f at $p \in \overline{M}$ as the linear operator

$$\text{Hess } f : T_p\overline{M} \to T_p\overline{M}, \quad (\text{Hess } f)Y = \overline{\nabla}_Y \text{grad } f, \quad Y \in T_p\overline{M},$$

where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Let a be a regular value of f and let $M^n \subset \overline{M}^{n+1}$ be the hypersurface in $\overline{M}$ defined by $M = \{p \in \overline{M}; f(p) = a\}$. Prove that:

(a): The Laplacian $\overline{\Delta} f$ is given by

$$\overline{\Delta} f = \text{trace Hess } f.$$

Proof. By definition, if $Y \in T_p\overline{M}$,

$$\overline{\Delta} f(Y) = \text{div grad } f(Y) = \text{Trace } \overline{\nabla}_Y \text{grad f} = \text{Trace Hess } f(Y).$$

since our choice of Y was arbitrary, we see that $\overline{Delta} f = \text{Hess } f$. □

(b): If $X, Y \in \mathfrak{X}(\overline{M})$, then

$$\langle(\text{Hess } f)Y, X\rangle = \langle Y, (\text{Hess } fX\rangle.$$

Conclude that Hess f is self-adjoint, hence determines a symmetric bilinear form on $T_p\overline{M}$, $p \in \overline{M}$, given by $(\text{Hess } f)(X, Y) = \langle(\text{Hess } f)X, Y\rangle$, $X, Y \in T_p\overline{M}$.

Proof. Let $p \in \overline{M}$ and let $E_1, \ldots, E_{n+1}$ be a geodesic frame at the point p. Then, for $i, j \in \{1, \ldots, n+1\}$,

$$\begin{aligned}
\langle (\text{Hess } f)E_i, E_j \rangle &= \langle \overline{\nabla}_{E_i} \text{grad } f, E_j \rangle \\
&= \left\langle \overline{\nabla}_{E_i} \sum_{k=1}^{n+1} (E_k(f))E_k, E_j \right\rangle \\
&= \left\langle \sum_{k=1}^{n+1} [E_k(f)\overline{\nabla}_{E_i} E_k + E_i(E_k(f))E_k], E_j \right\rangle \qquad (2)\\
&= \left\langle \sum_{k=1}^{n+1} E_i(E_k(f))E_k, E_j \right\rangle \\
&= E_i(E_j(f)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle E_i, (\text{Hess } f)E_j \rangle &= \langle E_i, \overline{\nabla}_{E_j} \text{grad } f \rangle \\
&= \left\langle E_i, \overline{\nabla}_{E_j} \sum_{k=1}^{n+1} (E_k(f))E_k \right\rangle \\
&= \left\langle E_i, \sum_{k=1}^{n+1} [E_k(f)\overline{\nabla}_{E_j} E_k + E_j(E_k(f))E_k] \right\rangle \qquad (3)\\
&= E_j(E_i(f)).
\end{aligned}$$

Now,

$$0 = \overline{\nabla}_{E_i} E_j - \overline{\nabla}_{E_j} E_i = [E_i, E_j] = E_i E_j - E_j E_i,$$

so we see that $E_i(E_j(f)) = E_j(E_i(f))$. Hence, the expressions in (2) and (3) are equal, so

$$\langle (\text{Hess } f)E_i, E_j \rangle = \langle E_i, (\text{Hess } f)E_j \rangle.$$

Since this holds for all $i, j \in \{1, \ldots, n+1\}$, we conclude that the Hessian is self-adjoint. □

(c): The mean curvature H of $M \subset \overline{M}$ is given by

$$nH = -\text{div}\left(\frac{\text{grad } f}{|\text{grad } f|}\right).$$

Proof. Let $p \in M \subset \overline{M}$. Let $E_1, \ldots, E_n, E_{n+1} = \frac{\text{grad } f}{\|\text{grad } f\|} = \eta$ be an orthonormal frame about p. Note that

$$\langle \overline{\nabla}_\eta \eta, \eta \rangle = \frac{1}{2}\eta\langle \eta, \eta \rangle = \frac{1}{2}\eta(1) = 0.$$

Thus, by definition of divergence and of the mean curvature,

$$\begin{aligned} nH = \text{Trace } S_\eta &= \sum_{i=1}^{n} \langle S_\eta(E_i), E_i \rangle \\ &= -\sum_{i=1}^{n} \langle \overline{\nabla}_{E_i}\eta, E_i \rangle - \langle \overline{\nabla}_\eta \eta, \eta \rangle \\ &= -\sum_{i=1}^{n+1} \langle \overline{\nabla}_{E_i}\eta, E_i \rangle \\ &= -\text{div}_{\overline{M}}\eta \\ &= -\text{div}\left(\frac{\text{grad } f}{\|\text{grad } f\|} \right). \end{aligned}$$

□

(d): Observe that every embedded hypersurface $M^n \subset \overline{M}^{n+1}$ is locally the inverse image of a regular value. Conclude from (c) that the mean curvature H of such a hypersurface is given by

$$H = -\frac{1}{n}\text{div } N,$$

where N is an appropriate local extension of the unit normal vector field on $M^n \subset \overline{M}^{n+1}$.

Proof. Let M be an embedded hypersurface. Then, locally, $M = f^{-1}(r)$ for some regular value r. Hence, for each point $p \in M$, grad $f(p) \neq 0$, so the expression derived in (c) is well-defined on all of M. Hence,

$$H = -\frac{1}{n}\text{div } N,$$

where N is some extension of the unit normal vector field on $M^n \subset \overline{M}^{n+1}$. □

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Let X be a Killing vector field on a Riemannian manifold M. Let $N = \{p \in M; X(p) = 0\}$. Prove that:

(a): If $p \in N$, and $V \subset M$ is a normal neighborhood of p, with $q \in N \cap V$, then the radial geodesic segment γ joining p to q is contained in N. Conclude that $\gamma \cap V \subset N$.

Proof. Let ϕ_t denote the flow of X. If $p \in N$ and $V \subset M$ is a normal neighborhood of p with $q \in N \cap V$, then let γ be the radial geodesic joining p to q. Since $q \in N$, $\phi_t(q) = q$ for all t where the flow is defined. Now, since ϕ_t is an isometry on V, ϕ_t maps geodesics to geodesics; specifically, $\phi_t(\gamma)$ is a geodesic. Since ϕ_t fixes p and q, $\phi_t(\gamma)$ is a geodesic passing through p and q in V; by uniqueness of geodesics, then, $\phi_t(\gamma) = \gamma$. Therefore, for any $q' \in \gamma \cap V$, $\phi_t(q') = q'$, so $X(q') = 0$. Thus, $\gamma \cap V \subset N$. □

(b): If $p \in N$, there exists a neighborhood $V \subset M$ of p such that $V \cap N$ is a submanifold of M (this implies that every connected component of N is a submanifold of M).

Proof. We prove this by induction. If p is an isolated point, then $\{p\}$ is a neighborhood of p and $N \cap \{p\} = \{p\}$ is a 0-submanifold of M. Otherwise, let V be a normal neighborhood of p such that there exists $q_1 \in V \cap N$ not equal to p. Let γ_1 be the radial geodesic from p to q_1. By part (a), $\gamma_1 \cap V \subset N$. If $V \cap N = \gamma_1 \cap V$, then we're done, since $\gamma_1 \cap V$ is a 1-submanifold of M.

If $V \cap N \neq \gamma_1 \cap V$, then let $q_2 \in V \cap N - \{\gamma_1\}$. Let γ_2 be the radial geodesic joining p to q_2. Note that, by the same argument as in (a), $\gamma_2 \cap V \subset N$. Now, let $Q \subset T_pM$ be the subspace generated by $\exp_p^{-1}(q_1)$ and $\exp_p^{-1}(q_2)$; let $N_2 = \exp_p(Q \cap \exp_p^{-1}(V))$. Suppose $v \in N_2$. Then $v = a \exp_p^{-1}(q_1) + b \exp_p^{-1}(q_2)$ for some $a, b \in \mathbb{R}$. Hence,

$$\begin{aligned}
(d\phi_t)_p(v) &= (d\phi_t)_p(a \exp_p^{-1}(q_1) + b \exp_p^{-1}(q_2)) \\
&= a(d\phi_t)_p(\exp_p^{-1}(q_1)) + b(d\phi_t)_p(\exp_p^{-1}(q_2)) \\
&= a \exp_p^{-1}(q_1) + b \exp_p^{-1}(q_2) \\
&= v
\end{aligned}$$

since q_1 and q_2 are fixed by ϕ_t. Therefore, we see that $(d\phi_t)_p$ is the identity on Q. Therefore, if $q \in N_2$,

$$\phi_t(q) = \exp_p((d\phi_t)_p(\exp^{-1}(q))) = \exp_p(\exp_p^{-1}(q)) = q,$$

so we see that $N_2 \subset V \cap N$. If $V \cap N = N_2$, then we're done, since N_2 is a 2-submanifold of M.

Otherwise, if $V \cap N \neq N_2$, we pick $q_3 \in V \cap N - N_2$ and iterate the above procedure. At each stage, either this algorithm terminates with N_i, an i-submanifold of M, or we proceed to the next stage. Since we have only n dimensions to work with, we see that this procedure must terminate, and so $V \cap N$ is a submanifold of M. □

(c): The codimension, as a submanifold of M, of a connected component N_k of N is even. Assume the following fact: if a sphere has a non-vanishing differentiable vector field on it then its dimension must be odd.

Proof. Let $p \in N_k \subset M$. Let $E_p = (T_pN_k)^\perp$ and let $V \subset M$ be a normal neighborhood of p. Let $N_k^\perp = \exp_p(E_p \cap \exp_p^{-1}(V))$. Now, suppose $v \in E_p$. Then $(d\phi_t)_p(v) \in E_p$; otherwise, we would have that $(d\phi_t)_p(v) \in T_pN_k$ and so $\exp_p((d\phi_t)_pv) \in N_k \cap V$, meaning that $\phi_t(\exp_p((d\phi_t)_pv)) = \exp_p((d\phi_t)_pv) = \phi_t(\exp_p(v))$. However, since

$\exp_p(v) \notin N_k$, it can never flow to a point in N_k. Thus, we can conclude that $(d\phi_t)_p : E_p \to E_p$, so X is tangent to $N_k^\perp$.

On the other hand, since p is the unique point in $N_k^\perp$ where X vanishes, we know (from HW 8 Problem 5(b)) that $X \neq 0$ is tangent to the geodesic spheres of $N_k^\perp$ centered at p. Thus, X is a non-vanishing vector field on the geodesic spheres of $N_k^\perp$, which are homeomorphic to plain old spheres. Therefore, by the Hairy Ball Theorem, these geodesic spheres must be odd-dimensional, so $N_k^\perp$ must be even dimensional. Since dim $N_k^\perp =$ codim N_k, we conclude that N_k has even codimension in M. □

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