ANALYSIS HW 5

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1

Let X be a normed linear space and Y a linear subspace. The set of all continuous linear functionals on X that are zero on Y is called the *annihilator* of Y and denoted by $Y^\perp$. Show that $Y^\perp$ is a closed linear subspace of the dual space, X', of X. Show that $\overline{Y}^\perp = Y^\perp$.

Proof. To see that $Y^\perp$ is a linear subspace, suppose $l_1, l_2 \in Y^\perp$. Then, if $y \in Y$

$$(l_1 + l_2)(y) = l_1(y) + l_2(y) = 0$$

so $l_1 + l_2 \in Y^\perp$. Also, if $c \in \mathbb{R}$, $(cl_1)(x) = cl_1(x) = 0$, so $cl_1 \in Y^\perp$. To see that $Y^\perp$ is closed, suppose $l_k \in Y^\perp$ and $l_k \to L$. Let $\epsilon > 0$. Then, for any $y \in Y$, there exists $N \in \mathbb{N}$ such that, for $n > N$,

$$|l_n(y) - L(y)| < \epsilon/2.$$

Hence,

$$|L(y)| = |0 - L(y)| = |l_n(y) - L(y)| < \epsilon/2.$$

Since $|L(y)|$ is smaller than any positive number, it must be the case that $L(y) = 0$, meaning $L \in Y^\perp$, so $Y^\perp$ is closed.

Now, if $\overline{Y}$ signifies the completion of Y, then it is clear that

$$\overline{Y}^\perp \subseteq Y^\perp,$$

since $Y \subseteq \overline{Y}$. On the other hand, suppose $L \in Y^\perp$. Let $y \in \overline{Y}$. Then there exists a sequence $y_j \in Y$ such that $y_j \to y$. Since L is continuous,

$$y_j \to y \Rightarrow L(y_j) \to L(y).$$

Since $y_j \in Y$, $L(y_j) = 0$ for all $j \in \mathbb{N}$, so their limit $L(y) = 0$. Since our choice of y was arbitrary, we see that $L \in \overline{Y}^\perp$, so $Y^\perp \subseteq \overline{Y}^\perp$.

Since containment goes in both directions, we conclude that $\overline{Y}^\perp = Y^\perp$. □

2

Consider $C[0,1]$ with the uniform norm and let $K \subset C[0,1]$ be the set of functions with the property that $\int_0^{1/2} f(t)dt - \int_{1/2}^1 f(t)dt = 1$. Show that K is a closed convex set not containing the origin but K has no point closest to the origin.

Proof. Suppose there exists a sequence $f_n \in K$ such that the f_n converge to f in $C[0,1]$. Now, since this convergence is uniform, we are justified in switching limits and integrals:

$$\begin{aligned}
&\int_0^{1/2} f(t)dt - \int_{1/2}^1 f(t)dt \\
&= \int_0^{1/2} \lim_{n\to\infty} f_n(t)dt - \int_{1/2}^1 \lim_{n\to\infty} f_n(t)dt \\
&= \lim_{n\to\infty} \int_0^{1/2} f_n(t)dt - \lim_{n\to\infty} \int_{1/2}^1 f_n(t)dt \\
&= \lim_{n\to\infty} \left(\int_0^{1/2} f_n(t)dt - \int_{1/2}^1 f_n(t)dt\right) \\
&= 1.
\end{aligned}$$

Hence, $f \in K$, so K is closed. Also, for $0 \le \alpha \le 1$ and $f, g \in K$,

$$\begin{aligned}
& \int_0^{1/2}(\alpha f(t) + (1-\alpha)g(t))dt - \int_{1/2}^1(\alpha f(t) + (1-\alpha)g(t))dt \\
= & \ \alpha\left(\int_0^{1/2} f(t)dt - \int_{1/2}^1 f(t)dt\right) + \left(\int_0^{1/2} g(t)dt - \int_{1/2}^1 g(t)dt\right) \\
& - \left(\int_0^{1/2} g(t)dt - \int_{1/2}^1 g(t)dt\right) \\
= & \ \alpha + 1 - \alpha \\
= & \ 1.
\end{aligned}$$

Hence, K is convex.

Finally, consider the sequence f_n as drawn below:

Then, certainly, $f_n \in K$. Now, for each n,

$$||f_n||_{\text{unif}} = \max |f_n(x)| = \frac{n}{n-2}.$$

Hence, $\inf ||f_n||_{\text{unif}} = \lim_{n\to\infty} \frac{n}{n-2} = 1$. However, suppose there exists $f \in K$ such that $||f||_{\text{unif}} \le 1$. Then

$$\int_0^{1/2} f(t)dt - \in_{1/2}^1 f(t)dt \le \int_0^{1/2} 1dt - \int_0^{1/2} -1dt = 1/2 + 1/2 = 1.$$

Since $f \in K$, this inequality must actually be an equality, meaning that

$$f(t) = \begin{cases} 1 & t \le 1/2 \\ -1 & t > 1/2 \end{cases}$$

However, this is not a continuous function, so $f \notin K$. Hence

$$\inf_{f\in K} ||f||_{\text{unif}} = 1$$

but this infimum is never achieved, so there is no point in K closest to the origin. □

3

Let $K \subset L^1(0,1)$ be the set of all f with $\int_0^1 f(t)dt = 1$. Show that K is a closed convex set (of $L^1(0,1)$) that does not contain the origin but there are infinitely many points in K that minimize the distance to the origin.

Proof. To see that K is closed, suppose that f is the limit of a sequence $f_k \in K$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that, if $n > N$,

$$||f - f_n||_1 < \epsilon.$$

Now,

$$\begin{aligned} \left|\int_0^1 f(t)dt - 1\right| &= \left|\int_0^1 f(t)dt - \int_0^1 f_n(t)dt\right| \\ &= \left|\int_0^1 (f(t) - f_n(t))dt\right| \\ &\leq \int_0^1 |f(t) - f_n(t)|dt \\ &= ||f - f_n||_1 \\ &< \epsilon. \end{aligned}$$

Since $\left|\int_0^1 f(t)dt - 1\right|$ is arbitrarily small, we conclude that $\int_0^1 f(t)dt = 1$, which is to say that $f \in K$. Hence, K is closed. To see that K is convex, suppose $f, g \in K$. Then, for $0 \leq \alpha \leq 1$

$$\begin{aligned} \int_0^1 (\alpha f(t) + (1 - \alpha)g(t))dt &= \int_0^1 (\alpha f(t) + g(t) - \alpha g(t))dt \\ &= \alpha \int_0^1 f(t)dt + \int_0^1 g(t)dt - \alpha \int_0^1 g(t)dt \\ &= \alpha + 1 - \alpha \\ &= 1. \end{aligned}$$

Hence, K is convex. Certainly the origin is not in K, since $\int_0^1 0dt = 0 \neq 1$. However, for any $f \in K$,

$$||f - 0||_1 = ||f||_1 = \int_0^1 |f(t)|dt \geq \left|\int_0^1 f(t)dt\right| = 1$$

Furthermore, if $g \in K$ such that $f > 0$,

$$||g - 0||_1 = ||g||_1 = \int_0^1 |g(t)|dt = \int_0^1 g(t)dt = 1$$

There are infinitely many such g, so there are infinitely many points in K that minimize the distance to the origin. □

4

Consider points $x = (x_1, x_2)$ in $\mathbb{R}^2$ with the norm $||x|| = |x_1| + |x_2|$. Let the subspace V be the x_1 axis and define the linear functional ℓ by $\ell((x_1, 0)) = x_1$.

a) Show that ℓ is a bounded linear functional on V.

Proof. Let $(x_1, 0), (y_1, 0) \in V$. Then

$$\ell(c(x_1, 0) + d(y_1, 0)) = \ell((cx_1 + dy_1, 0)) = cx_1 + dy_1 = c\ell((x_1, 0)) + d\ell((y_1, 0))$$

for any $c, d \in \mathbb{R}$, so ℓ is certainly linear on V. Now, to see that ℓ bounded, we need only note that, if $x = (x_1, 0) \in V$

$$|\ell(x)| = |\ell((x_1, 0))| = |x_1| = |x_1| + 0 = ||x||$$

so $|\ell(x)| \leq ||x||$. □

b) Find *all* extensions of ℓ to a continuous linear map on $\mathbb{R}^2$.

Answer: Consider functionals of the form $F_c((x_1, x_2)) = x_1 + cx_2$ for $0 \leq c \leq 1$. Then, for $x \in V$,

$$F_c(x) = F_c((x_1, 0)) = x_1 + c(0) = x_1 = \ell(x).$$

Furthermore, for any $y = (y_1, y_2) \in \mathbb{R}^2$,

$$|F_c(y)| = |F_c(y_1, y_2)| = |y_1 + cy_2| \leq |y_1| + c|y_2| \leq |y_1| + |y_2| = ||y||.$$

Since, $F_c(1, 0) = 1 = ||(1, 0)||$, we see that, in fact, $||F_c|| = ||\ell||$.

♣

c) Construct a related example for $L^1(0, 1)$.

Construction: Let $V = \{f \in L^1 : f(t) = 0 \text{ for } t \in [1/2.1)\}$ and define ℓ' by

$$\ell'(f) = \int_0^{1/2} f(t)dt.$$

Then ℓ' is certainly linear and bounded on V. To see this, suppose $f, g \in V$, $c, d \in \mathbb{R}$. Then

$$\ell'(cf+dg) = \int_0^{1/2} cf(t)+dg(t)dt = c\int_0^{1/2} f(t)dt+d\int_0^{1/2} g(t)dt = c\ell'(f)+d\ell'(g).$$

For boundedness, if $f \in V$,

$$|\ell'(f)| = \left|\int_0^{1/2} f(t)dt\right| \leq \int_0^{1/2} |f(t)|dt = \int_0^1 |f(t)|dt = ||f||_1.$$

Now, any functional of the form

$$\mathcal{L}(f) = \int_0^{1/2} f(t)dt + c\int_{1/2}^1 f(t)dt$$

where $0 \leq c \leq 1$ will be an extension of ℓ' to all of $L^1(0, 1)$. Since there are infinitely many such, we see that the Hahn-Banach extension is far from unique.

♣

5

Let ℓ^1, ℓ^∞ denote the Banach spaces of all complex sequences $x = \{x_j\}$ with the usual norms $||x||_1 = \sum |x_j| < \infty$ and $||x||_\infty = \sup |x_j| < \infty$ and let c_0 be the subspace of sequences in ℓ^∞ that converge to zero.

a) If $z \in \ell^1$ show that $\ell(x) := \sum z_j x_j$ defines a bounded linear functional on c_0 with $||\ell|| = ||z||_1$.

Proof. Let $x \in \ell^\infty$. Then

$$|\ell(x)| = \left|\sum z_j x_j\right| \leq \sum |z_j x_j| \leq \sum |z_j|||x||_\infty = \left(\sum |z_j|\right) ||x||_\infty = ||z||_1||x||_\infty,$$

so ℓ is bounded. Furthermore, if $x, y \in \ell^\infty$, $c, d \in \mathbb{R}$,

$$\ell(cx+dy) = \sum z_j(cx_j + dy_j) = c\sum z_j x_j + d\sum z_j y_j = c\ell(x) + d\ell(y),$$

so ℓ is linear. To see that $||\ell|| = ||z||_1$, let $\epsilon > 0$. Then, since $\sum |z_j|$ converges, we know there exists $N \in \mathbb{N}$ such that, if $n > N$,

$$\sum_{j=n+1}^{\infty} |z_j| < \epsilon/2.$$

Let $x = (x_j)_{j=1}^\infty$, where $x_j = \overline{z_j}$ for $j \le N$, $x_j = 0$ for $j > N$. Then $x \in c_0$ and

$$|\ell(x) - ||z||_1| = \left|\sum z_j x_j - \sum |z_j|\right| = \left|\sum_{N+1}^{\infty} |z_j|\right| < \epsilon/2.$$

Hence, we see that $||\ell|| = ||z||_1$. □

b) Every bounded linear functional on c_0 is as above. In other words, ℓ^1 is the dual space of c_0.

Proof. Suppose ℓ is a bounded linear functional on c_0. Denote by e_j the vector with a 1 in the j^{th} position and zeros everywhere else. Let $x \in c_0$. Then

$$\begin{aligned} |\ell(x)| &= \left|\ell\left(\textstyle\sum_0^\infty x_j e_j\right)\right| \\ &= \left|\textstyle\sum_0^\infty x_j \ell(e_j)\right| \\ &\le \textstyle\sum_0^\infty |x_j||\ell(e_j)| \\ &\le \textstyle\sum_0^\infty ||x||_\infty |\ell(e_j)| \\ &= ||x||_\infty \textstyle\sum_0^\infty |\ell(e_j)| \\ &= c||x||_\infty \end{aligned}$$

where $\sum |\ell(e_j)| = c < \infty$, so $\{\ell(e_j)\} \subseteq \ell^1$. Hence, we see that the dual space $c_0' \subseteq \ell^1$. Since we showed the reverse containment in part (a), we conclude that $c_0' = \ell^1$. □

c) Show that ℓ^∞ is the dual space of ℓ^1.

Proof. Let $x \in \ell^\infty$. Define $\ell(z) := \sum x_j z_j$. Then, for $y, z \in \ell^1$, $c, d \in \mathbb{C}$,

$$\ell(cy+dz) = \sum x_j(cy_j + dz_j) = c\sum x_j y_j + d\sum x_j z_j = c\ell(y) + d\ell(z),$$

so ℓ is linear on ℓ^1. Furthermore, if $z \in \ell^1$,

$$|\ell(z)| = \left|\sum x_j z_j\right| \le \sum |x_j||z_j| \le \sum ||x||_\infty |z_j| = ||x||_\infty \left(\sum |z_j|\right) = ||x||_\infty ||z||_1,$$

so ℓ is bounded. Furthermore, just as in part (a), $||\ell|| = ||x||_\infty$.

Now, if ℓ is a bounded linear functional on ℓ^1 and $x \in \ell^1$, then

$$|\ell(x)| = \left|\ell\left(\sum x_j e_j\right)\right| \le \sum |x_j||\ell(e_j)| \le ||\ell|| \sum |x_j| = ||\ell||||x_j||_1.$$

Since we've shown a correspondance between the linear functionals on ℓ^1 and the elements of ℓ^∞, we conclude that ℓ^∞ is the dual space of ℓ^1. □

d) Show that ℓ^1 is contained in the dual space $(\ell^\infty)'$ of ℓ^∞, but is not all of $(\ell^\infty)'$ since there are elements in $(\ell^\infty)'$ that are zero on all of c_0.

6

a) With the positive linear functional $\ell(f) := \int_{\mathbb{R}} f(t)dt$ on $C_c(\mathbb{R})$ show that the set $\{x \in \mathbb{R} : 0 \leq x < 1\}$ is measurable.

Proof. Let $A = \{x \in \mathbb{R} : 0 \leq x < 1\}$. Let $\epsilon > 0$. Define $\Omega := (-\epsilon/2, 1)$. Then

$$\Omega \Delta A = (\Omega \cup A)\backslash(\Omega \cap A) = (-\epsilon/2, 1)\backslash[0, 1) = (-\epsilon/2, 0).$$

Then

$$M^*(\Omega\Delta A) = \inf_{\Omega\Delta A \subset O, O \text{ open}} \mathrm{Vol}(O) = \mathrm{Vol}((-\epsilon/2, 0)) = \sup_{g \in C_c(\epsilon/2,0)} (I(g))$$

where $0 \leq g(x) \leq 1$. Suppose $f(x)$ is 1 on $(-\epsilon/2, 0)$ and zero everywhere else. Then

$$M^*(\Omega\Delta A) = \sup_{g \in C_c(-\epsilon/2,0)} (\ell(g)) \leq \ell(f) = \int_{-\epsilon/2}^{0} 1dx = -\epsilon/2 < \epsilon.$$

However, this is just the definition of measurable, so we see that A is measurable. □

b) With the positive linear functional $\ell(f) := \int_{\mathbb{R}^2} f(x, y)dxdy$, on $C_c(\mathbb{R}^2)$ show that the set $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 < y < 1\}$ is measurable.

Proof. Let $B = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 < y < 1\}$ and let $\epsilon > 0$. Define

$$\Omega := [0, 1] \times [0, 1].$$

Then

$$\Omega\Delta B = ([0, 1] \times [0, 1]) \cap ([0, 1] \times (0, 1)) = (0 \times (0, 1)) \cup (1 \times (0, 1)).$$

Now,

$$M^*(\Omega\Delta B) = \inf_{\Omega\Delta B \subset O} \mathrm{Vol}(O)$$

where O is open. Now, consider the sequence of open sets $O_n = ((-1/n, 1/n) \times (0, 1)) \cup ((1 - 1/n, 1 + 1/n) \times (0, 1))$. Then

$$\begin{aligned} \mathrm{Vol}(O_n) \quad &= \sup_{g \in Cc(O_n), 0 \leq g(x) \leq 1} (\ell(g)) \leq \textstyle\int_{-1/n}^{1/n} \int_0^1 1dydx + \int_{1-1/n}^{1+1/n} \int_0^1 1dydx \\ &= 2/n + 2/n = 4/n. \end{aligned}$$

Certainly, the sequence $4/n \to 0$, so we see that

$$M^*(\Omega\Delta B) \leq \inf_{n \in \mathbb{N}} \mathrm{Vol}(O_n) = \inf_{n \in \mathbb{N}} \frac{4}{n} = 0.$$

Since $M^*(S)/geq0$ for any set S, we conclude that

$$M^*(\Omega\Delta B) = 0.$$

Therefore, B is measurable, □

c) With the positive linear functional $\ell(f) := f(0) + 2f(1)$ on $C_c(\mathbb{R})$ discuss which sets are measurable. Can you compute their measures?

Answer: Suppose S is a set containing neither 0 nor 1. Then we can construct an open set Ω containing S such that Ω contains neither 0 nor 1. If $f \in C_c(\Omega)$, then $f(0) = f(1) = 0$, meaning $\ell(f) = f(0) + 2f(1) = 0$. Hence

$$\begin{aligned} M^*(S) = \inf_{S\subset O, O \text{ open}} \mathrm{Vol}(O) \quad &\le \mathrm{Vol}(\Omega) \\ &= \sup_{g\in C_c(\Omega), 0\le g(x)\le 1}(\ell(g)) = 0. \end{aligned}$$

Therefore, S has measure 0.

On the other hand, if S' is a set containing 0 but not 1, then we can construct an open set Ω' containing S' such that Ω' does not contain 1. If $f \in C_c(O)$ where O is an open set not containing 1, then $f(1) = 0$, so $\ell(f) = f(0) + 2f(1) = f(0)$. Furthermore, if O is an open set containing zero, then there exists a function $h_O \in C_c(O)$ such that $h_O(0) = 1$. Thus,

$$\begin{aligned} M^*(S') \quad &= \inf_{S'\subset O, O \text{ open}} \mathrm{Vol}(O) \\ &\le \inf_{S'\subset O, O \text{ open}} \left(\sup_{g\in C_c(O), 0\le g(x)\le 1}(\ell(g))\right) \\ &\le \inf_{S'\subset O, O \text{ open}}(\ell(h_O)) \\ &= 1. \end{aligned}$$

Thereforre, S' has measure 0.

A parallel argument demonstrates that if S'' contains 1 but not 0, then the measure of S'' is 2.

Finally, if S_1 contains both 0 and 1, then

$$M^*(S_1) = \inf_{S_1\subset O, O \text{ open}} \mathrm{Vol}(O).$$

If Q is any open set containing S_1, there exists $f_Q \in C_c(Q)$ such that $f_Q(0) = f_Q(1) = 1$ and $0 \le f(x) \le 1$. Hence, $\ell(f_Q) = f_Q(0) + 2f_Q(1) = 3$, so

$$\mathrm{Vol}(Q) = \sup_{g\in C_c(Q), 0\le g(x)\le 1}(\ell(g)) \ge 3.$$

If we let g_Q denote the function that is 1 on Q and zero outside, then it is also clear that

$$\mathrm{Vol}(Q) = \sup_{g\in C_c(Q), 0\le g(x)\le 1}(\ell(g)) \le (\ell(g_Q)) = 3.$$

Therefore, for any open Q containing S_1, $\mathrm{Vol}(Q) = 3$. Therefore,

$$M^*(S_1) = \inf_{S_1\subset O, O \text{ open}} \mathrm{Vol}(O) = 3,$$

so S_1 has measure 3.

♣

7

In a Hilbert space H, a map $U : H \to H$ is called *unitary* if

(i). $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all x, y in H.

(ii). U is onto.

Let $Q := \{x \in H : Ux = x\} = \ker(I - U)$ of all points left fixed by U.

a) Show that in finite dimensions property (ii) is a consequence of property (i).

Proof. Let H be finite dimensional and let U be unitary. Suppose $x, y \in H$ such that $Ux = Uy$. Then $0 = Ux - Uy = U(x - y)$, which is to say that

$$0 = \langle U(x - y), U(x - y)\rangle = \langle x - y, x - y\rangle \Rightarrow x - y = 0.$$

Hence, $x = y$, so U is injective. Since $U : H \to H$ is injective, it must be surjective, since H is finite-dimensional. □

b) In the Hilbert space ℓ^2, show that the right shift $S : (x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$ has property (i) but not property (ii).

Proof. Let $x, y \in \ell^2$. Then $x = (x_1, x_2, \ldots,)$, $y = (y_1, y_2, \ldots,)$ for $x_i, y_i \in \mathbb{C}$. Then

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i\overline{y_i} = 0 + \sum_{i=1}^{\infty} x_i\overline{y_i} = \langle Sx, Sy\rangle,$$

so S fulfills property (i). However, There is no $z \in \ell^2$ such that

$$Sz = (1, 0, 0, \ldots,),$$

so S is not unitary. □

c) View functions on the circle S^1 as functions periodic with period 2π. If $H = L^2(S^1)$ and α is real, show that the map $L : f(x) \mapsto f(x + 2\pi\alpha)$ is unitary.

Proof. Suppose $f \in L^2(S^1)$. Then

$$L(f(x - 2\pi\alpha)) = f(x - 2\pi\alpha + 2\pi\alpha) = f(x),$$

so L is surjective.

Now, let $f, g \in L^2(S^1)$. Then

$$\begin{aligned}\langle Lf, Lg\rangle &= \int_{S^1}(Lf)(Lg) \\ &= \int_0^{2\pi} f(x + 2\pi\alpha)g(x + 2\pi\alpha)dx \\ &= \int_{2\pi\alpha}^{2\pi+2\pi\alpha} f(u)g(u)du \\ &= \int_{S^1}(f)(g) \\ &= \langle f, g\rangle\end{aligned}$$

where we let $u = x + 2\pi\alpha$. Therefore, L is unitary. □

d) If α is irrational, find the set Q.

Answer: If α is irrational, then, if $f \in Q$,

$$f(x) = Lf = L^m f = f(x + 2\pi m\alpha)$$

for all $m \in \mathbb{Z}$. Since $Q \subseteq L^2(S^1)$, f must also be 2π-periodic. Hence, either $2\pi m\alpha = 2\pi$ for some integer m or $2\pi\alpha = n2\pi$ for some integer n. Dividing by 2π on both sides of each, we see that

$$m\alpha = 1 \text{ or } \alpha = n.$$

Since α is irrational, neither of these is possible, so we conclude that Q is empty if α is irrational.

♣

e) If α is rational, find the set Q.

Answer: If α is rational, then, again, if $f \in Q$,

$$f(x) = Lf = L^m f = f(x + 2\pi m\alpha)$$

for all $m \in \mathbb{Z}$. Since α is rational, $\alpha = p/q$ for $p \in \mathbb{Z}$, $q \in \mathbb{Z}^*$. Also, if $f \in Q$, f must be 2π-periodic. Hence, the elements of Q will be precisely those functions which are $\frac{2\pi}{nq}$-periodic, where $n \in \mathbb{Z}^*$.

♣

f) If U is unitary, show that $U^* = U^{-1}$ and hence that $Ux = x$ if and only if $U^*x = x$. Consequently $\ker(I - U) = \ker(I - U)^*$.

Proof. Let $x, y \in H$. By definition,

$$\langle x, Uy\rangle = \langle U^*x, y\rangle = \langle UU^*x, Uy\rangle.$$

Hence,

$$0 = \langle x, Uy\rangle - \langle UU^*x, Uy\rangle = \langle x - UU^*x, Uy\rangle.$$

Since our choice of y was arbitrary, we see that it must be the case that $x - UU^*x = 0$, or

$$x = UU^*x \Rightarrow U^{-1}x = U^*x.$$

Since our choice of x was arbitrary, this means that $U^{-1} = U^*$.

Hence, $Ux = x$ if and only if $x = U^{-1}x = U^*x$, and so $\ker(I - U) = \ker(I - U)^*$. □

g) Using $(\text{Image}L)^\perp = \ker L^*$ for any bounded linear operator L in a Hilbert space, conclude that $H = \ker(I - U) \oplus \overline{\text{Image}(I - U)}$.

Proof. From problem 1 above, we know that

$$\overline{\text{Image}(I - U)}^\perp = (\text{Image}(I - U))^\perp = \ker(I - U)^* = \ker(I - U),$$

where we use the given fact to get the second equality and part (g) to get the third. Now, since we are in a Hilbert space,

$$(\ker(I - U))^\perp = (\overline{\text{Image}(I - U)}^\perp)^\perp = \overline{\text{Image}(I - U)}.$$

Applying this result to the conclusion proved in problem 5 of last week's homework, we see that

$$H = \ker(I - U) \oplus (\ker(I - U))^\perp = \ker(I - U) \oplus \overline{\text{Image}(I - U)}.$$

□

8. (Continuation)

von Neumann's *mean ergodic theorem* states that for any $f \in H$, one has

$$\lim_{N \to \infty} \frac{1}{N} \sum_0^{N-1} U^n x = Px \tag{1}$$

where P is the orthogonal projection onto the set $Q := \{x \in H : Ux = x\} = \ker(I - U)$. Equation (1) is clarly true for all x in $Q = \ker(I - U)$.

a) Prove that for all $z \in \text{Image}(I - U)$, that is $z = (I - U)x$ for some $x \in H$, then

$$\left|\left| \frac{1}{N} \sum_0^{N-1} U^n z \right|\right| = \left|\left| \frac{1}{N}(x - U^N x) \right|\right| \to 0. \tag{2}$$

Proof. We see immediately that

$$\begin{aligned} \textstyle\sum_0^{N-1} U^n z &= \textstyle\sum_0^{N-1} U^n (I - U)x \\ &= \textstyle\sum_0^{N-1} U^n x - U^{n+1} x \\ &= (x - Ux) + (Ux - U^2 x) + \ldots + (U^{N-1}x - U^N x) \\ &= x - U^N x. \end{aligned}$$

Hence,

$$\left|\left| \frac{1}{N} \sum_0^{N-1} U^n z \right|\right| = \left|\left| \frac{1}{N}(x - U^N x) \right|\right|.$$

□

b) Conclude that (2) holds for all $z \in \overline{\text{Image}(I - U)}$.

Proof. First, note that, for all $x \in H$,

$$||Ux||^2 = \langle Ux, Ux \rangle = \langle x, x \rangle = ||x||^2,$$

so $||Ux|| = ||x||$.

Now, let $z \in \overline{\text{Image}(I - U)}$. Then there exists a sequence $\{z_j\} \in \text{Image}(I - U)$ such that $z_j \to z$. Let $\epsilon > 0$. Then there exists $M \in \mathbb{N}$ such that, if $k > M$,

$$||z - z_k|| < \epsilon/2.$$

Also, from part (a), we know that for all k there exists $N \in \mathbb{N}$ such that, if $n > N$,

$$\left|\left| \frac{1}{n} \sum_0^{n-1} U^m z_k \right|\right| < \epsilon/2.$$

Let $k > M$, $n > N$. Therefore,

$$\begin{aligned}\left|\left|\frac{1}{n}\sum_0^{n-1} U^m z\right|\right| &= \left|\left|\frac{1}{n}\sum_0^{n-1} U^m (z - z_k) + \frac{1}{n}\sum_0^{n-1} U^m z_k\right|\right| \\ &\leq \left|\left|\frac{1}{n}\sum_0^{n-1} U^m (z - z_k)\right|\right| + \left|\left|\frac{1}{n}\sum_0^{n-1} z_k\right|\right| \\ &\leq \frac{1}{n}\sum_0^{n-1} ||U^m(z - z_k)|| + \left|\left|\frac{1}{n}\sum_0^{n-1} U^m z_k\right|\right| \\ &= \frac{1}{n}\sum_0^{n-1} ||z - z_k|| + \left|\left|\frac{1}{n}\sum_0^{n-1} z_k\right|\right| \\ &= ||z - z_k|| + \left|\left|\frac{1}{n}\sum_0^{n-1} z_k\right|\right| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon.\end{aligned}$$

Hence, for all $z \in \overline{\text{Image}(I - U)}$,

$$\left|\left|\frac{1}{N}\sum_0^{N-1} U^n z\right|\right| \to 0.$$

$\square$

c) Use the last part of the previous problem to show that $Pz = 0$ if and only if $z \in \overline{\text{Image}(I - U)}$. Thus (2) proves (1) for all x that satisfy $Px = 0$ and thus for all x in the Hilbert space.

Proof. Certainly, as we've shown in (b) above, if $z \in \overline{\text{Image}(I - U)}$, then

$$\lim N \to \infty \sum_0^{N-1} U^n z = 0.$$

Furthermore, as we saw in 7(g), if $z \in \overline{\text{Image}(I - U)}$, then $z \in (\ker(I - U)^*)^\perp$, meaning $Pz = 0$. Hence,

$$z \in \overline{\text{Image}(I - U)}.$$

On the other hand, if $Pz = 0$, then $z \in (\ker(I - U)*)^\perp$ and, therefore, by 7(g), $z \in z \in \overline{\text{Image}(I - U)}$. By (b), then,

$$Pz = 0 = \lim_{N\to\infty} \sum_0^{N-1} U^n z.$$

$\square$

9

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $a(x) > 0$ and $f(x)$ be smooth functions periodic with period 2π. Let $H^1(S^1)$ be the completion of smooth 2π periodic functions on the circle S^1 in the norm

$$||\phi||_{H^1}^2 := \int_{S^1} (|\phi'|^2 + \phi^2) dx.$$

Call $u \in H^1(S^1)$ a *weak solution* of $-u'' + a(x)u = f$ on S^1 if

$$\int_{S^1} [u'v' + a(x)uv]dx = \int_{S^1} fvdx \quad \text{for all} \quad v \in H^1(S^1).$$

a) (uniqueness) Show that the equation $-u'' + a(x)u = f$ on S^1 has at most one weak solution.

Proof. Suppose there exist two weak solutions, u_1 and u_2. Then

$$\begin{aligned} 0 = \int_{S^1} fvdx - \int_{S^1} fvdx &= \int_{S^1}[u_1'v' + a(x)u_1v]dx - \int_{S^1}[u_2'v' + a(x)u_2v]dx \\ &= \int_{S^1}[(u_1' - u_2')v' + a(x)(u_1 - u_2)v]dx \end{aligned}$$

for all $v \in H^1(S^1)$. Particularly, when $v = u_1 - u_2$, $v' = u_1' - u_2'$, so

$$\int_{S^1} [(u_1' - u_2')^2 + a(x)(u_1 - u_2)]dx = 0.$$

Since $a(x) > 0$, this means that $u_1 - u_2 = 0$, or $u_1 = u_2$, so any weak solution must be unique. □

b) (existence) Show that the equation $-u'' + a(x)u = f$ on S^1 has a solution.

Proof. Define the norm$|| \cdot ||_H^2$ such that, for $v \in H^1(S^1)$,

$$||v||_H^2 = \int_{S^1} [(v')^2 + a(x)v^2]dx.$$

Define the functional $\ell(v) := \int_{S^1} fv$. Then ℓ is certainly linear and

$$|\ell(v)| = \left| \int_{S^1} fv \right| \leq ||f||_{L^2}||v||_{L^2} \leq c||f||_{L^2}||v||_H^2$$

by the Holder inequality and since this norm is equivalent to the L^2 norm, so ℓ is a bounded linear functional on the Hilbert space. Hence, by the Riesz-Frechet Representation Theorem, there exists $u \in H^1(S^1)$ such that $\ell(v) = \langle v, u \rangle$. Therefore

$$\int_{S^1} fv = \ell(v) = \langle v, u \rangle = \int_{S^1} [v'u' + a(x)vu]dx,$$

so u is a weak solution. □

c) Show that if $b(x)$ is a smooth function on S^1 with $|b(x)|$ sufficiently small, then the equation $-u'' + b(x)u' + a(x)u = f$ on S^1 has exactly one weak solution.

Proof. A weak solution u to this equation must satisfy

$$\int_{S^1} [u'v' - bv'u + a(x)vu]dx = \int_{S^1} fv$$

for all $v \in H^1(S^1)$.

To show uniqueness of weak solutions, suppose u_1 and u_2 are weak solutions. Then, if we let $w = u_1 - u_2$,

$$\begin{aligned} 0 = \int_{S^1} fv - \int_{S^1} fv &= \int_{S^1}[u_1'v' - b(x)v'u_1 + a(x)vu_1]dx \\ &\quad - \int_{S^1}[u_2'v' - b(x)v'u_2 + a(x)vu_2]dx \\ &= \int_{S^1}[w'v' - b(x)v'w + a(x)vw]dx \end{aligned}$$

for all v. Particularly, when $v = w$

$$0 = \int_{S^1}[(w')^2 - b(x)w'w + a(x)w^2]dx.$$

If we integrate the second term by parts, we will get

$$0 = \int_{S^1}[(w')^2 B(x)(w')^2 + a(x)w']dx = \int_{S^1}[(1 + B(x))(w')^2 + a(x)w^2]dx$$

where $B(x) = \int_0^x b(t)dt$. Now, if $|b(x)|$ is suffiently small for all $x \in S^1$, then $|B(x)| > -1$, which in turn implies that $(1 + B(x)) > 0$. Since $a(x) > 0$, all terms in the above integral are nonnegative, so w conclude that $w = 0$, meaning $u_1 = u_2$. Therefore, any weak solution to this equation is unique.

To prove existence, we now need to define another new norm, $||\cdot||^2_{H'}$, where, for $v \in H^1(S^1)$,

$$||v||^2_{H'} = \int_{S^1}[(v')^2 - b(x)v'v + a(x)v^2]dx.$$

Let $\ell(v) = \int_{S^1} fv$. Then, again, ℓ is linear and, using Holder,

$$|\ell(v)| = \left|\int_{S^1} fv\right| \le ||f||_{L^2}||v||_{L^2} \le c||f||_{L^2}||v||^2_{H'}$$

since this norm is equivalent to the L^2 norm. So ℓ s a bounded linear functional on the Hilbert space. Now, in order to construct a weak solution u, we need to introduce a bilinear form $B(\cdot,\cdot)$, where, for $v, w \in H^1(S^1)$,

$$B(v, w) = \int_{S^1}[v'w' - b(x)v'w + a(x)vw]dx.$$

Now, we need to check the following:

(i) B is bounded

(ii) There exists $b > 0$ such that $|B(y, y)| \ge b||y||^2$ for all $y \in H^1(S^1)$.

To see (i), let $v, w \in H^1(S^1)$. Then

$$\begin{aligned} |B(v, w)| &= \left|\int_{S^1}[v'w' - b(x)v'w + a(x)vw]dx\right| \\ &\le \left|\int_{S^1} v'w'dx\right| + \left|\int_{S^1} b(x)v'wdx\right| + \left|\int_{S^1} a(x)vwdx\right| \\ &= \left|\int_{S^1} a(x)vwdx\right| \\ &\le ||av||_{L^2}||w||_{L^2} \\ &\le c||v||^2_{H'}||w||^2_{H'}. \end{aligned}$$

To prove (ii), we merely note that

$$|B(v, v)| = \left|\int_{S^1}[(v')^2 - b(x)v'v + a(x)v^2]dx\right| = |\langle v, v\rangle| = ||v||^2,$$

so $|B(v,v)| \geq ||v||^2$ for all v. Therefore, by the Lax-Milgram Lemma, there exists u such that $\ell(v) = B(v,u)$. This means that

$$\int_{S^1} fv = \ell(v) = B(v,u) = \int_{S^1} [v'u' - b(x)v'u + a(x)vu]dx,$$

meaning u is a weak solution of the given equation. □

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