

ALGEBRA HW 3

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1

(a): How many ways can a regular tetrahedron be inscribed in a cube?

Answer: First, we label each of the four vertices of the tetrahedron, v_1, v_2, v_3, v_4 . We inscribe the tetrahedron into the cube such that each vertex of the tetrahedron coincides with a vertex of the cube, which causes each edge of the tetrahedron to coincide with a diagonal on one of the cube's faces, as in the below picture:

Now, let us consider what is done with the vertices v_1 and v_2 . We can place vertex v_1 at any of the eight vertices of the cube; call that vertex c_1 . Now, there are exactly three vertices of the cube which share a face but not an edge with c_1 ; call them c_2, c_3, c_4 . Hence, v_2 , which shares an edge with v_1 (and, thus, must be placed on one of those three vertices of the cube which share a face but not an edge with c_1) can be placed on any of c_2, c_3 , or c_4 . Whichever of these three we choose completely determines where the other two vertices of the tetrahedron, v_3 and v_4 , must be placed, and so we see that the inscription of the tetrahedron is completely determined by the placement of v_1 and v_2 . Since there are eight choices for the placement of v_1 and, contingent on that choice, three for the placement of v_2 , we see that there are $8 \cdot 3 = 24$ ways to inscribe a regular tetrahedron into a cube, assuming we distinguish based on placement of the vertices.

If we don't distinguish in this fashion, then there are really only two ways to inscribe a tetrahedron into a cube. To see this, consider a particular way of inscribing the tetrahedron into the cube, where v_1, v_2, v_3, v_4 are mapped onto c_1, c_2, c_3, c_4 , respectively. Then *any* mapping of v_1 onto one of the elements of $\{c_1, c_2, c_3, c_4\}$ will yield the same tetrahedron, up to a symmetry of the tetrahedron. So there

are $4 \cdot 3 = 12$ ways to yield the same way of inscribing the tetrahedron in this same way, and so there are $24/12 = 2$ different ways of inscribing the tetrahedron into the cube if we don't distinguish up to a symmetry of the tetrahedron.



(b): What homomorphisms between the symmetry groups of a solid regular tetrahedron and of a solid cube, if any, are induced by the inscribing of a tetrahedron in a cube? Describe the kernel and image of any such homomorphism.

Answer: Let us choose one of the ways of inscribing a tetrahedron into a cube. Then we see that each vertex of the tetrahedron corresponds to exactly one of the long diagonals of the cube (and, moreover, that each long diagonal corresponds to a vertex of the tetrahedron). Since a symmetry of the tetrahedron is determined by an admissible permutation of the vertices of the tetrahedron and a symmetry of the cube is determined by a permutation of the long diagonals, we see that however we inscribe the tetrahedron into the cube induces a homomorphism between the symmetries of the tetrahedron and the symmetries of the cube. Since different permutations of tetrahedron vertices correspond to different permutations of cube diagonals, this homomorphism will be an injection. Hence, the kernel will be trivial and the image will be isomorphic to the symmetry group of the tetrahedron.



(c): Repeat parts (a) and (b) with a cube and a regular dodecahedron, rather than with a regular tetrahedron and a cube.

Answer: To inscribe the cube into the dodecahedron, we will want all the vertices of the cube to be assigned to vertices of the dodecahedron and all the edges of the cube to be assigned to a “quasi-diagonal” of the dodecahedron (i.e. to be tangent to one of the faces and connect two vertices on that face that do not share an edge). Let us label the vertices of the cube v_1, \dots, v_8 such that v_2 and v_1 share an edge. To inscribe the cube into the dodecahedron, we first need to assign v_1 to a vertex of the dodecahedron. Since each of the twenty vertices of the dodecahedron are legitimate choices, we see that there are twenty possibilities. Call the vertex we choose d_1 .

Now, our assignment of v_2 to a vertex of the dodecahedron will completely determine how the cube is inscribed into the dodecahedron, so we need only determine how many possible such choices there are. Since d_1 lies on exactly three faces of the dodecahedron, and each of these three faces contains exactly two vertices that do not share an edge with d_1 , so there are a total of 6 possible vertices onto which we can place v_2 . Since there were 20 choices for where to place v_1 and the placement of v_1 and v_2 determine the placement of

all other vertices of the cube, we see that there are $20 \cdot 6 = 120$ ways to inscribe the cube into the dodecahedron, assuming we distinguish on the basis of vertices. If we don't distinguish in this way, then we could map v_1 onto any of the eight vertices corresponding to vertices of the cube in the inscription we described above. Then there are three choices (the three edges adjacent to v_1) for placing v_2 which will yield the same inscription of the cube into the dodecahedron, so there are $8 \cdot 3 = 24$ ways of yielding this same inscription. Hence, there are $120/24 = 5$ different ways of inscribing the cube into the dodecahedron if we don't distinguish up to a symmetry of the cube.

Now, let's turn to the question about homomorphisms. Certainly a rotation of the cube by $\pi/2$ radians about an axis determined by the center points of two opposing faces determines a symmetry of the cube. Let f_1 be one of the faces not fixed by this rotation. Then, with the cube inscribed into the dodecahedron, the end-on view of f_1 looks like one of these two pictures:

If picture (1) obtains, the above-mentioned rotation by $\pi/2$ will result in an end-on view like picture (2) (and *vice versa*). Hence, this rotation is *not* a symmetry of the dodecahedron. Since this fact did not depend on our choice of how to inscribe the cube into the dodecahedron, we see that inscribing the cube into the dodecahedron does not induce a homomorphism between the symmetry groups.



2

(a): Show that the symmetries of a solid regular tetrahedron form the alternating group A_4 .

Proof. Any symmetry of the regular tetrahedron will be completely determined by its action on the four vertices v_1, v_2, v_3, v_4 of the tetrahedron, so we know immediately that the symmetry group will be a subgroup of S_4 . Now, a 3-cycle in S_4 will simply define a rotation of the tetrahedron about an axis determined by the vertex v_i fixed by the 3-cycle and the midpoint of the opposing face. Hence, all of the 3-cycles in S_4 are in the symmetry group of the solid tetrahedron.

On the other hand, we claim that no transposition determines a symmetry of the solid tetrahedron. To see why, let $v_1 = (-1/2, \sqrt{3}/2, 0)$, $v_2 = (-1/2, -\sqrt{3}/2, 0)$ and $v_3 = (1, 0, 0)$. Then the only way to swap v_1 and v_2 while fixing v_3 is by reflection about the xz -axis, which we cannot do without moving into higher dimensions. Hence, (12) is not in the symmetry group of the solid tetrahedron, and, by a similar argument, no other transpositions are, either, so the symmetries of the tetrahedron form a proper subgroup of S_4 . However, we can swap v_1 and v_2 if we also swap v_3 and v_4 : simply rotate in the xy -plane by π radians and then in the xz -plane by $2\pi/3$ radians, as pictures below:

Hence, (12)(34) is in the symmetry group of the solid tetrahedron; a similar argument shows that (13)(24) and (14)(23) are as well. Now, these three permutations, along with the eight 3-cycles and the identity permutation comprise A_4 . Thus, the symmetries of the tetrahedron contain A_4 ; since A_4 is maximal in S_4 and, as we saw, the symmetries of the tetrahedron are proper in S_4 , we see that the symmetries of the tetrahedron are precisely the elements of A_4 . \square

(b): Show that the symmetries of a solid cube form the symmetric group S_4 .

Proof. First, note that a symmetry of the solid cube is completely determined by its action on the long diagonals of the cube; since there are four long diagonals (label them d_1, d_2, d_3, d_4), we see that the symmetry group of the cube is a subgroup of S_4 . Now, let us show that each of the 4-cycles is in the symmetry group of the cube. Consider, for example, the 4-cycle (1324). Then, as illustrated below, a rotation by $\pi/2$ radians corresponds to (1324):

Rotating in the opposite direction yields (1423) and similar rotations (denoted by arrows 2 and 3 in the picture) and their inverses yield the other 4-cycles. Now, if (abc) is a 3-cycle, then

$$(adcb)(acbd) = (abc),$$

so, since all the 4-cycles are in the symmetry group of the cube, all the 3-cycles must be as well. Furthermore, for the three elements of the form $(ab)(cd)$,

$$(acbd)(acbd) = (ab)(cd)$$

so these elements are in the symmetry group of the cube as well. Now, since there are six 4-cycles, eight 3-cycles and three elements of the form $(ab)(cd)$ and one identity permutation, the order of the symmetry group of the cube must be at least

$$6 + 8 + 3 + 1 = 18.$$

Since the only subgroup of S_4 of order ≥ 18 is S_4 itself, we see that the symmetries of the solid cube form the symmetric group S_4 . \square

(c): Show that the symmetries of a solid regular dodecahedron form the alternating group A_5 .

Proof. Consider the 5 different (up to a symmetry of the cube) ways of inscribing a cube into a dodecahedron that we found in 1(c) above. Then any symmetry of the dodecahedron will map one such inscribed cube into another and, furthermore, the symmetries of the dodecahedron will be entirely determined by permutations of these 5 cubes, so we see that the symmetries of the dodecahedron will form a subgroup of the symmetric group on 5 letters, S_5 .

Now, since each of the 5 inscribed cubes has 12 edges, each of which must lie on one of the 12 faces of the dodecahedron, we see that, for any given face of the dodecahedron, it will encompass exactly one edge from each of the 5 inscribed cubes:

Now, consider a rotation of this face by $2\pi/5$ radians in the clockwise direction; this gives a symmetry of the dodecahedron and corresponds to the 5-cycle (14352). In fact, by choosing an appropriate face of the dodecahedron, any 5-cycle in S_5 corresponds to a rotation by a multiple of $2\pi/5$ about this face. Hence, the 24 5-cycles in S_5 are in the symmetry group of the dodecahedron.

Now, let (abc) be a 3-cycle in S_5 . Then we choose a face of the dodecahedron given by picture (1), which has adjacent face (laid flat for easier viewing) given by picture (2):

Now, consider a clockwise rotation in the first face by $4\pi/5$ (corresponding to the 5-cycle $(aedcb)$) followed by an anti-clockwise rotation in the second face of $2\pi/5$ (corresponding to the 5-cycle $(acdeb)$). Then this symmetry corresponds to the product:

$$(acdeb)(aedcb) = (abc).$$

Hence, all of the 20 3-cycles in S_5 are symmetries of the dodecahedron. Now, consider elements of the form $(ab)(cd)$. Then, since all 3-cycles are symmetries of the dodecahedron and

$$(abc)(bcd) = (ab)(cd),$$

all 15 permutations of this form are symmetries of the dodecahedron as well. Hence, the order of the symmetry group of the dodecahedron must be at least

$$24 + 20 + 15 + 1 = 60$$

and, given that all the permutations we've described so far are even, must contain A_5 . Now, since A_5 is maximal in S_5 , we see that the symmetries of the dodecahedron must be either A_5 or S_5 . However, these symmetries cannot be all of S_5 , because there is a homomorphism between S_4 (which we know is the symmetry group of the cube) and S_5 (namely, the inclusion map). However, as we saw in 1(c), there is no homomorphism induced by the inscribing of a cube into a dodecahedron between the symmetries of the cube and the symmetries of the dodecahedron. Since the symmetries of the dodecahedron are determined completely by inscribing a cube into the dodecahedron, this means there is no homomorphism between the symmetries of the cube and the symmetries of the dodecahedron, and so the symmetries of the dodecahedron cannot form the group S_5 . Therefore, we conclude that the symmetries of the solid dodecahedron form the alternating group A_5 . \square

(d): Are the orders of these groups consistent with your answers to Problem 1?

3

(a): Which of the following groups can be written as a direct product of cyclic p -groups (possibly for various p 's)? For those which can be, do so.

$$\mathbb{Z}/24, \mathbb{Z}/6 \times \mathbb{Z}/15, \mathbb{Z}/100 \times \mathbb{Z}/120, D_4, S_3, A_4, Q.$$

Answer: Since 3 and 8 are relatively prime, we know that

$$\mathbb{Z}/24 \simeq \mathbb{Z}/3 \times \mathbb{Z}/8.$$

Both $\mathbb{Z}/3$ and $\mathbb{Z}/8$ are p -groups (for $p = 3$ and $p = 2$, respectively), so $\mathbb{Z}/24$ can be written as the direct product of cyclic p -groups.

Since 3 and 2 are relatively prime, $\mathbb{Z}/6 \simeq \mathbb{Z}/2 \times \mathbb{Z}/3$. Also, since 3 and 5 are relatively prime, $\mathbb{Z}/15 \simeq \mathbb{Z}/3 \times \mathbb{Z}/5$. Hence,

$$\mathbb{Z}/6 \times \mathbb{Z}/15 \simeq \mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}/5.$$

Since $\mathbb{Z}/2$, $\mathbb{Z}/3$ and $\mathbb{Z}/5$ are cyclic p -groups, $\mathbb{Z}/6 \times \mathbb{Z}/15$ can be written as the product of cyclic p -groups.

Since 4 and 25 are relatively prime, $\mathbb{Z}/100 \simeq \mathbb{Z}/4 \times \mathbb{Z}/25$. Also, since 3, 5 and 8 are relatively prime, $\mathbb{Z}/120 \simeq \mathbb{Z}/3 \times \mathbb{Z}/5 \times \mathbb{Z}/8$. Therefore,

$$\mathbb{Z}/100 \times \mathbb{Z}/120 \simeq (\mathbb{Z}/4 \times \mathbb{Z}/25) \times (\mathbb{Z}/3 \times \mathbb{Z}/5 \times \mathbb{Z}/8).$$

Since each of the terms in this product is a cyclic p -group (with some repetition of p 's), $\mathbb{Z}/100 \times \mathbb{Z}/120$ can be written as the product of cyclic p -groups.

Since the direct product of cyclic p -groups (which, being cyclic, must be abelian) must be abelian; since D_4 , S_3 , A_4 and Q are all non-abelian, none of them can be the direct product of cyclic p -groups.



(b): Find all the Sylow subgroups of the above groups, and verify that the Sylow theorems hold.

Answer: The only elements of order 3 in $\mathbb{Z}/24$ are 8 and 16, so we see that there exists a single Sylow-3 subgroup, $\{0, 8, 16\}$. The only elements of orders 2, 4 or 8 are 3, 6, 9, 12, 15, 18, 21, so there exists a single Sylow-2 subgroup, $\{0, 3, 6, 9, 12, 15, 18, 21\}$. We see immediately that the Sylow theorems hold in this case.

In the case of $\mathbb{Z}/6 \times \mathbb{Z}/15$, there is a single element of order 2, $(3, 0)$ and, hence, a single Sylow-2 subgroup $\{(0, 0), (3, 0)\}$; hence, the Sylow theorems clearly hold for $p = 2$. In this same group, there are 4 elements of order 5, $(0, 3), (0, 6), (0, 9), (0, 12)$, so there is a single Sylow-5 subgroup, $\{(0, 0), (0, 3), (0, 6), (0, 9), (0, 12)\}$; again, the Sylow theorems clearly hold for $p = 5$. Now, there are 8 elements of order 3 or 9,

$$(2, 0), (2, 5), (2, 10), (4, 0), (4, 5), (4, 10), (0, 5), (0, 10),$$

so they, along with $(0, 0)$, comprise the only Sylow-3 subgroup. Again, since there is only 1, the Sylow theorems clearly hold.

In the case of $\mathbb{Z}/100 \times \mathbb{Z}/120$, there are only 2 elements of order 3, $(0, 40)$ and $(0, 80)$, so they form the only Sylow-3 subgroup, $\{(0, 0), (0, 40), (0, 80)\}$; clearly, the Sylow theorems hold for $p = 3$. Furthermore, since

$$\begin{aligned} \mathbb{Z}/100 \times \mathbb{Z}/120 &\simeq (\mathbb{Z}/4 \times \mathbb{Z}/25) \times (\mathbb{Z}/3 \times \mathbb{Z}/5 \times \mathbb{Z}/8) \\ &\simeq (\mathbb{Z}/2)^5 \times (\mathbb{Z}/5)^3 \times \mathbb{Z}/3 \end{aligned}$$

and both $(\mathbb{Z}/2)^5$ and $(\mathbb{Z}/5)^3$ are p -groups (for $p = 2$ and $p = 5$), their images in $\mathbb{Z}/100 \times \mathbb{Z}/120$ are the unique Sylow-2 and Sylow-5 subgroups; hence, the Sylow theorems hold for $p = 2$ and $p = 5$.

In the case of D_4 , since $\#D_4 = 8$, D_4 is already a p -group, namely for $p = 2$. Hence, D_4 is already its own Sylow-2 subgroup, so the Sylow theorems trivially hold.

In the case of S_3 , S_3 contains exactly two elements of order 3, (123) and (132) , so it contains a single Sylow-3 subgroup, $\{1, (123), (132)\}$. The Sylow theorems clearly hold in this case. On the other hand, S_3 contains 3 elements of order 2, (12) , (13) and (23) , so it contains three Sylow-2 subgroups, $\{1, (12)\}$, $\{1, (13)\}$ and $\{1, (23)\}$. Since

$$\begin{aligned} (12)(23)(12) &= (13) \\ (23)(12)(23) &= (13) \\ (13)(23)(13) &= (12) \\ (23)(13)(23) &= (12) \\ (13)(12)(13) &= (23) \\ (12)(13)(12) &= (23) \end{aligned}$$

we see that these groups are all conjugate. Furthermore, there are 3 Sylow-2 subgroups, and 3 divides $\#S_3 = 6$ and $3 \equiv 1 \pmod{2}$. Hence, the Sylow theorems all hold.

In the case of A_4 , there are 8 elements of order 3, (123) , (132) , (124) , (142) , (134) , (143) , (234) , (243) , so there are 4 different Sylow-3 subgroups: $\langle(123)\rangle$, $\langle(124)\rangle$ and $\langle(234)\rangle$. Now, since, for $a, b, c, d \in \{1, 2, 3, 4\}$,

$$(abc)(bcd)(acb) = (adc)$$

we see that each of these groups is conjugate to each of the others. Furthermore, 4 divides $12 = \#A_4$ and $4 \equiv 1 \pmod{3}$, so the Sylow theorems hold.

On the other hand, there are 3 elements of order 2 or 4, $(12)(34)$, $(13)(24)$, $(14)(23)$, so there is a single Sylow-4 subgroup, $\{1, (12)(34), (13)(24), (14)(23)\}$, which clearly satisfies the Sylow theorems.

Finally, in the case of Q , $\#Q = 8 = 2^3$, so Q is already a p -group for $p = 2$ and hence the Sylow theorems are trivially true.

4

- (a): Assume that $\#G = pq$, where p and q are prime. Show that one of its Sylow subgroups is normal.

Proof. Suppose, without loss of generality, that $p \geq q$. By the third Sylow theorem, if n denotes the number of Sylow- p subgroups (of which we know there is at least one by the first Sylow theorem), then n divides $\#G = pq$ and $n \equiv 1 \pmod{p}$. Since n divides pq , then $n = 1$, $n = q$, $n = p$ or $n = pq$. However, since $p \equiv 0 \pmod{p}$, $n \neq p$. Similarly, since $pq \equiv 0 \pmod{p}$, $n \neq pq$. Finally, since $q \leq p$ and q prime, $q \equiv q \not\equiv 1 \pmod{p}$, so $n \neq q$. Therefore, we must conclude that $n = 1$. By the second Sylow theorem, then, since there is a unique Sylow- p subgroup, all of its conjugates must simply be itself, which is to say that it must be normal in G . \square

- (b): With G as above, assume $p \geq q$. Show that either G is abelian or else q divides $p - 1$.

Proof. Now, let m denote the number of Sylow- q subgroups of G . We know, by the first Sylow theorem, that $m \geq 1$ and, by the third Sylow theorem, that m divides pq and that $m \equiv 1 \pmod{q}$. Since the only divisors of pq are 1, q , p and pq , we see that these are the only choices for m . However, both q and pq are congruent to 0 (mod q), so the only choices are actually 1 and p . Now, if $m = 1$, then we know, by the second Sylow theorem, that the unique Sylow- q subgroup is normal in G . By part (a) above, we know that the same holds true for the unique Sylow- p subgroup. If we denote these two subgroups by Q and P , respectively, then $P, Q \triangleleft G$ and $P \cap Q = 1$. Furthermore, since p and q are relatively prime,

$$\#(P \times Q) = pq = \#G$$

and so it must be the case that $G \simeq P \times Q$. Since the only groups of prime order are cyclic, P and Q are cyclic and hence abelian and, therefore, G is abelian.

On the other hand, if $m = p$, then, since $m \equiv 1 \pmod{p}$, it must be the case that $p \equiv 1 \pmod{p}$, which is to say that $(p - 1)$ is a multiple of q . \square

- (c): Find all groups of order 51, and all groups of order 55. Which are simple? solvable? nilpotent? abelian? cyclic?

Answer: Let G be a group of order $51 = 17 \cdot 3$. Then, by the first Sylow theorem, it has a Sylow-17 subgroup N . By our work in part (a) above, $N \triangleleft G$. Additionally, we know, again by the first Sylow theorem, that there is a Sylow-3 subgroup $H \subset G$. By our work in part (b) above, the number of Sylow-3 subgroups must be either 1 or 17, and can only be 17 if 3 divides $(17-1)$. Since 3 certainly does not divide 16, we see that H is the unique Sylow-3

subgroup of G . Hence, the argument given in the first case of (b) above demonstrates that $G \simeq N \times H$; since $N \simeq \mathbb{Z}/17$ and $H \simeq \mathbb{Z}/3$, it must be that

$$G \simeq \mathbb{Z}/17 \times \mathbb{Z}/3 \simeq \mathbb{Z}/51.$$

Now, we've demonstrated two normal subgroups, N and H , so it's clear that G is not simple. G is abelian and cyclic. Finally, we can form the composition series

$$G \triangleright N \triangleright 1$$

where

$$G/N \simeq (\mathbb{Z}/17 \times \mathbb{Z}/3)/(\mathbb{Z}/17) \simeq \mathbb{Z}/3,$$

so the composition factors are $\mathbb{Z}/17$ and $\mathbb{Z}/3$. Hence, G is solvable. Furthermore, since the commutator of any abelian group is trivial, G is also nilpotent.

On the other hand, suppose G is a group of order 55. Then, by our work in (a) we know that G contains a unique Sylow-11 subgroup N such that $N \triangleleft G$. Also, G contains a Sylow-5 subgroup H . Now, by our work in (b), we know the number of Sylow-5 subgroups is either 1 or 11 (since $11 \equiv 1 \pmod{5}$). If the number of Sylow-5 subgroups is 1, then $H \triangleleft G$ and so

$$G \simeq N \times H \simeq \mathbb{Z}/11 \times \mathbb{Z}/5 \simeq \mathbb{Z}/55.$$

In this case, G is not simple, but is abelian and cyclic, and therefore has trivial commutator and so is nilpotent and therefore solvable.

On the other hand, if the number of Sylow-5 subgroups is 11, then we have some work to do. Since $(G : N) = 5$ and $N \triangleleft G$, $G/N \simeq H$. Using this isomorphism, we can form the short exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1.$$

Since we are viewing H as both a quotient and a subgroup of H , we can certainly define the map $\sigma : H \rightarrow G$ that maps a generator of H when viewed as a quotient to a generator of H when viewed as a subgroup. σ certainly defines a section on this extension of H by N . Hence, this section induces an action $\alpha : H \rightarrow \text{Aut } N$. Recall that $N \simeq \mathbb{Z}/11$ and $H \simeq \mathbb{Z}/5$ and consider the possible maps

$$\alpha : \mathbb{Z}/5 \rightarrow \text{Aut } \mathbb{Z}/11 \simeq (\mathbb{Z}/11)^* \simeq \mathbb{Z}/10.$$

Since a generator of $\mathbb{Z}/5$ is of order 5, it must map to an element of order 5 or 1 in $\mathbb{Z}/10$. If it maps to an element of order 1, this simply defines the trivial action, and so the induced product will simply be the direct product already examined above. On the other hand, the elements of order 5 in $\mathbb{Z}/10$ are $\{2, 4, 6, 8\}$. Hence,

$$\alpha(1) \in \{2, 4, 6, 8\}$$

Under the isomorphism

$$\begin{aligned} \mathbb{Z}/10 &\rightarrow (\mathbb{Z}/11)^* \\ 1 &\mapsto 2 \end{aligned}$$

so we can consider the elements 2, 4, 6, 8 in $\mathbb{Z}/10$ as corresponding to the elements 4, 5, -2, 3 in $(\mathbb{Z}/11)^*$, respectively. Hence, depending on the choice of α , the semi-direct product

$$\mathbb{Z}/11 \rtimes_{\alpha} \mathbb{Z}/5$$

is given by the presentation

$$\langle n, h \mid n^5 = 1, hnh^{-1} = n^a \rangle,$$

where $a \in \{-2, 3, 4, 5\}$. In any event, all of these possibilities are isomorphic. So we see that there are only two choices for G , $\mathbb{Z}/55$ or the above. The above is not simple, since we showed that N is normal in it, nor is it abelian, since $hn = h^a n$ where $a \neq 1$ and thus it cannot be cyclic, either. G is solvable, though, as we can form the composition series

$$G \triangleright N \triangleright 1$$

where $G/N \simeq \mathbb{Z}/5$ and so the composition factors are the cyclic groups $\mathbb{Z}/11$ and $\mathbb{Z}/5$. Finally, G is not nilpotent, since it is not the direct product of p -groups.



5

Let G' be the commutator subgroup of a finite group G , and let N be a subgroup of G . Show that the following are equivalent:

- (i): N is normal in G and G' is contained in N .
- (ii): N is the kernel of a surjective homomorphism from G to an abelian group.

Proof. (i) \Rightarrow (ii) Let $N \triangleleft G$ and suppose $G' \subset N$. Then G/N is a group and N is certainly the kernel of the surjective homomorphism $\phi : G \rightarrow G/N$. Hence, it suffices to show that G/N is abelian. Let $a, b \in G/N$. Then there exist $\hat{a}, \hat{b} \in G$ such that $\phi(\hat{a}) = a$ and $\phi(\hat{b}) = b$. Hence,

$$aba^{-1}b^{-1} = \phi(\hat{a})\phi(\hat{b})\phi(\hat{a}^{-1})\phi(\hat{b}^{-1}) = \phi(\hat{a}\hat{b}\hat{a}^{-1}\hat{b}^{-1}) = 1,$$

since $\hat{a}\hat{b}\hat{a}^{-1}\hat{b}^{-1} \in G' \subset N$. Therefore, $ab = ba$ and so G/N is abelian, and so N is the kernel of a surjective homomorphism from G to an abelian group.

(ii) \Rightarrow (i) On the other hand, suppose N is the kernel of a surjective homomorphism ϕ from G to an abelian group H . Then, if $g \in G$ and $n \in N$,

$$1 = \phi(1) = \phi(g)\phi(g)^{-1} = \phi(g)\phi(n)\phi(g^{-1}) = \phi(gng^{-1}),$$

so $gng^{-1} \in N$. Since our choices of g and n were arbitrary, we conclude that $N \triangleleft G$. Let $a, b \in G$. Then, since H is abelian,

$$1 = \phi(a)\phi(b)\phi(a)^{-1}\phi(b)^{-1} = \phi(aba^{-1}b^{-1}),$$

so $aba^{-1}b^{-1} \in N$. Since our choice of a and b was arbitrary, we see that this holds for all $a, b \in G$, and so $G' \subset N$. \square

6

Let G be a p -group, let G' be its commutator subgroup, and let Φ be its Frattini subgroup.

(a): Show that if N is a maximal subgroup of G , then N contains G' .

Proof. As we saw in class, every maximal subgroup of a p -group is normal and has index p in the group. Hence, $N \triangleleft G$. Furthermore, since $(G : N) = p$,

$$G/N \simeq \mathbb{Z}/p;$$

in particular, G/N is abelian. Now, since N is the kernel of the surjective homomorphism $G \rightarrow G/N$ and G/N is abelian, the result proved in problem 5 above tells us that $G' \subset N$. \square

(b): Deduce that G' is a subgroup of Φ .

Proof. Since, as we saw in (a) above, every maximal subgroup of G contains G' , G' must lie in the intersection of all maximal subgroups of G . Since Φ is precisely this intersection, it must be the case that $G' \subset \Phi$. \square

(c): Conclude that G/Φ is abelian.

Proof. We saw on last week's homework that Φ is a characteristic subgroup of G and, therefore, Φ is normal in G . In part (b) we saw that $G' \subset \Phi$ so the result in problem 5 above assures us that Φ is the kernel of a surjective homomorphism from G onto an abelian group. Furthermore, in proving the forward direction of 5, we demonstrated that this abelian group is, in fact, the quotient G/Φ , so we conclude that G/Φ is abelian. \square

7

Let G be a finite group and let N be a minimal non-trivial normal subgroup of G . Show that N is isomorphic to a group of the form $S^n = S \times \cdots \times S$ for some (possibly abelian) simple group S and some non-negative integer n .

Proof. First, let us prove the following lemma, which will come in handy.

Lemma 0.1. *If $N, M \triangleleft G$, then $N \cap M \triangleleft G$.*

Proof. $N \cap M$ is certainly a group; we need only show that it is normal in G . To that end, let $h \in N \cap M$ and let $g \in G$. Then, since N and M are both normal in G , $ghg^{-1} \in N$ and $ghg^{-1} \in M$, so $ghg^{-1} \in N \cap M$. Since our choices of h and g were arbitrary, we see that $g(N \cap M)g^{-1} \subset N \cap M$, which is to say that $N \cap M \triangleleft G$. \square

Now, let us turn our attention to the proposition. If N is itself simple, then there is nothing to show. Hence, suppose N is not simple and let S be a minimal non-trivial normal subgroup of N . Since N was a minimal non-trivial subgroup of G , it must be the case that S is not normal in G . Thus, there exists $g \in G$ such that $gSg^{-1} \neq S$. However, since $N \triangleleft G$, $gNg^{-1} = N$, so $gSg^{-1} \subset N$. Now, let $s' \in gSg^{-1}$ and $n' \in N$. Then there exists $s \in S$ such that $s' = gsg^{-1}$. Furthermore, since $N \triangleleft G$, $gNg^{-1} = N$, so there exists $n \in N$ such that $gng^{-1} = n'$, which is to say that

$$gn = n'g.$$

Finally, since $S \triangleleft N$, there exists $\hat{s} \in S$ such that $nsn^{-1} = \hat{s}$. Hence,

$$\begin{aligned} n's'n'^{-1} &= n'(gsg^{-1})n'^{-1} = (n'g)s(n'g)^{-1} = (gn)s(gn)^{-1} = g(nsn^{-1})g^{-1} \\ &= g\hat{s}g^{-1} \in gSg^{-1}, \end{aligned}$$

so $gSg^{-1} \triangleleft N$. Suppose there exists $R \subset gSg^{-1}$ such that $R \triangleleft N$. Then $g^{-1}Rg \triangleleft N$ (by the same reasoning that showed $gSg^{-1} \triangleleft N$) and $g^{-1}Rg \subset S$, which was supposed to be minimal. Hence, we know that gSg^{-1} is also a minimal normal subgroup of N . Furthermore, by the lemma, we know that $S \cap gSg^{-1} \triangleleft N$; since $S \cap gSg^{-1}$ is contained in both S and gSg^{-1} and these are minimal, it must be the case that $S \cap gSg^{-1} = 1$. Hence,

$$S \times gSg^{-1}$$

is isomorphic to the subgroup of N generated by S and gSg^{-1} .

Define

$$S' = \bigcup_{g \in G} gSg^{-1}.$$

Then $S' \subset N$; furthermore, if $a \in S'$ and $g \in G$, then there exist $s \in S$, $g' \in G$ such that $a = g'sg'^{-1}$, so

$$gag^{-1} = g(g'sg'^{-1})g^{-1} = (gg')s(gg')^{-1} \in (gg')S(gg')^{-1} \subset S',$$

so S' is normal in G . Since N is a minimal normal subgroup, it must be the case that $S' = G$. Since for each distinct gSg^{-1} and $g'Sg'^{-1}$,

$$gSg^{-1} \cap g'Sg'^{-1} = 1$$

(for the same reason the intersection of S and gSg^{-1} is trivial) and the groups of the form gSg^{-1} are normal in N and collectively span N , we see that

$$N \simeq S \times g_1Sg_1^{-1} \times \cdots \times g_nSg_n^{-1}$$

where for $g_i \in G$, $i = 1, \dots, n$ (because G is finite). Furthermore, since $gSg^{-1} \simeq S$ for all $g \in G$, we can conclude that

$$G \simeq S \times \cdots \times S = S^n.$$

□

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