ALGEBRA HW 12

CLAY SHONKWILER

1

Let V be a vector space and let $G \subset \text{End } V$ be a finite subgroup. Say that $W \subset V$ is *G-invariant* if it is T-invariant for every $T \in G$. Say that $W \subset V$ is *G-irreducible* if W is G-invariant and the only G-invariant subspaces of W are 0 and W.

(a): If V is a finite dimensional vector space over $\mathbb{C}$, show that V can be written as the direct product of G-irreducible subspaces.

Proof. Let $n = \#G$ and let W be a G-invariant subspace. Let W'' be an arbitrary complement of W. Then $V = W \times W''$; that is, if $v \in V$, then v can be uniquely decomposed as

$$v = w + w''$$

for $w \in W$ and $w'' \in W''$. Define $P : V \to W$ by

$$P(v) = P(w + w'') = w.$$

Then, if $v_1, v_2 \in V$ and $a, b \in \mathbb{C}$, then $v_1 = w_1 + w_1''$ and $v_2 = w_2 + w_2''$ for $w_1, w_2 \in W$ and $w_1'', w_2'' \in W''$ and thus

$$\begin{aligned} P(v_1 + v_2) &= P(a(w_1 + w_1'') + b(w_2 + w_2'')) \\ &= P((aw_1 + bw_2) + (aw_1'' + bw_2'')) \\ &= aw_1 + bw_2 \\ &= aP(v_1) + bP(v_2), \end{aligned}$$

so P is linear. Now, define $S : W \to W$ by

$$S(v) = \frac{1}{n} \sum_{T \in G} TPT^{-1}(v).$$

Then, if $v_1, v_2 \in V$ and $a, b \in K$,

$$\begin{aligned} S(av_1 + bv_2) &= \frac{1}{n}\sum_{T\in G} TPT^{-1}(av_1 + bv_2) \\ &= \frac{1}{n}\sum_{T\in G} TP(aT^{-1}(v_1) + bT^{-1}(v_2)) \\ &= \frac{1}{n}\sum_{T\in G} T(aPT^{-1}(v_1) + bPT^{-1}(v_2)) \\ &= \frac{1}{n}\sum_{T\in G} (aTPT^{-1}(v_1) + bTPT^{-1}(v_2)) \\ &= a\frac{1}{n}\sum_{T\in G} TPT^{-1}(v_1) + b\frac{1}{n}\sum_{T\in G} TPT^{-1}(v_2) \\ &= aS(v_1) + bS(v_2), \end{aligned}$$

so S is linear. Now, since $P(v) \in W$ for all $v \in V$, we see that $P(T^{-1}(v)) \in W$ for all v and all $T \in G$ and, since W is G-invariant, $TPT^{-1}(v) \in W$ for all $v \in V$ and $T \in G$. Hence, $S(v) \in W$ for all $v \in V$ and so im $S \subset W$. On the other hand, since P is the identity on W, we see that

$$TPT^{-1}(w) = TT^{-1}(w) = w$$

for all $w \in W$, so

$$S(w) = \frac{1}{n}\sum_{T\in G} TPT^{-1}(w) = \frac{1}{n}\sum_{T\in G} w = w$$

Therefore, $W \subset$ im S. Having shown containment both ways, we conclude that im $S = W$. Furthermore, we just showed that $S|_W$ is the identity; since im $S = W$, this implies that $S^2 = S$. Thus, by our work in PS10#7, we know that

$$V = W \times W'$$

where $W' = \ker S$, so W' is a complement of W. Now, let $w' \in W'$ and $A \in G$. Then

$$\begin{aligned}
S(A(w')) &= \frac{1}{n}\sum_{T\in G} TPT^{-1}(A(w')) \\
&= \frac{1}{n}\sum_{T\in G} (AA^{-1})TP(T^{-1}T')(w') \\
&= \frac{1}{n}\sum_{T\in G} A(A^{-1}T)P(T^{-1}A)(w') \\
&= \frac{1}{n}A\left(\sum_{T\in G}(A^{-1}T)P(A^{-1}T)^{-1}(w')\right) \\
&= A\left(\frac{1}{n}\sum_{T\in G}(A^{-1}T)P(A^{-1}T)^{-1}(w')\right) \\
&= A(S(w')) \\
&= A(0) \\
&= 0,
\end{aligned}$$

so $A(w') \in W'$ and, hence, W' is A-invariant. Since our choice of A was arbitrary, we see that W' is A-invariant for all $A \in G$, so W' is G-invariant.

Now, we go by induction. Suppose V is 0-dimensional. Then V is already clearly G-irreducible for all G. If V is 1-dimensional, then V is also necessarily G-irreducible for all finite $G \subset \text{End } V$. Now, suppose that all vector spaces of dimension $\leq k$ are G-invariant for all finite subgroups $G \subset \text{End } V$. Let V be a $(k+1)$-dimensional vector space and let $G \subset \text{End } V$ be a finite group. Then, by what we showed above, if W is a G-invariant subspace of V, then $V = W \times W'$ for some G-invariant subspace W'. Note that G is a subgroup of both End W and End W'. Since $\dim W \leq k$ and $\dim W' \leq k$, so, by the induction hypothesis,

$$W = \prod_{i=1}^{n} W_i$$

and

$$W' = \prod_{i=1}^{m} W_i'$$

for G-irreducible subspaces $W_1, \ldots, W_n, W_1', \ldots, W_m'$. Then

$$V = \left(\prod_{i=1}^{n} W_i\right) \times \left(\prod_{j=1}^{m} W_j'\right) = W_1 \times \ldots \times W_n \times W_1' \times \ldots \times W_m',$$

so V is the product of G-irreducible subspaces.

Having shown the base case and the inductive step, we conclude, by induction, that for any finite-dimensional vector space V and finite group $G \subset \text{End } V$, V can be written as the direct product of G-irreducible subspaces. $\square$

(b): What can you say about the conclusions of parts (a) and of PS11#6 if the field of scalars is not necessarily $\mathbb{C}$?

2

(a): Suppose that $A \in M_n(K)$ is strictly upper triangular. Show that A is nilpotent, and find the index of nilpotence.

Proof. Let

$$A = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,(n-1)} & a_{1,n} \\ 0 & 0 & a_{2,3} & \cdots & a_{2,(n-1)} & a_{2,n} \\ 0 & 0 & 0 & \cdots & a_{3,(n-1)} & a_{3,n} \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{(n-1),n} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Then

$$A^2 = \begin{pmatrix} 0 & 0 & a_{1,2}a_{2,3} & \cdots & \sum_{i=2}^{n-2} a_{1,i}a_{i,(n-1)} & \sum_{i=2}^{n-1} a_{1,i}a_{i,n} \\ 0 & 0 & 0 & \cdots & \sum_{i=3}^{n-2} a_{2,i}a_{i,(n-1)} & \sum_{i=3}^{n-1} a_{2,i}a_{i,n} \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{(n-2),(n-1)}a_{(n-1),n} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

In other words, the i,j entry in A^2 is

$$\sum_{k=i+1}^{j-1} a_{i,k}a_{k,j},$$

so not only is the diagonal of A^2 zero, but the "upper diagonal" (i.e. terms of the form $a_{i,(i+1)}$) is also zero. Now, if we multiply A^2 by A to get A^3, terms of the form $a_{i,(i+2)}$ will be zero. Iterating, we see that in A^m, all terms of the form $a_{i,(i+m-1)}$ (and, obviously, all entries below them) will be zero. Hence, A^n is the zero matrix and, so long as the terms on the "upper diagonal" of A are non-zero, n is the smallest number for which this is the case, so n is the index of nilpotence. $\square$

(b): Show that if S and T are upper triangular, then their *bracket* $[S, T] := ST - TS$ is nilpotent.

Proof. Let

$$S = \begin{pmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n} \\ 0 & s_{2,2} & \cdots & s_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & s_{n,n} \end{pmatrix}, T = \begin{pmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,n} \\ 0 & t_{2,2} & \cdots & t_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & t_{n,n} \end{pmatrix}.$$

Then

$$\begin{aligned} ST &= \begin{pmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n} \\ 0 & s_{2,2} & \cdots & s_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & s_{n,n} \end{pmatrix} \begin{pmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,n} \\ 0 & t_{2,2} & \cdots & t_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & t_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} s_{1,1}t_{1,1} & s_{1,1}t_{1,2} + s_{1,2}t_{2,2} & \cdots & \sum_{i=1}^n s_{1,i}t_{i,n} \\ 0 & s_{2,2}t_{2,2} & \cdots & \sum_{i=2}^n s_{2,i}t_{i,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & s_{n,n}t_{n,n} \end{pmatrix} \end{aligned}$$

On the other hand,

$$\begin{aligned} TS &= \begin{pmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,n} \\ 0 & t_{2,2} & \cdots & t_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & t_{n,n} \end{pmatrix} \begin{pmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n} \\ 0 & s_{2,2} & \cdots & s_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & s_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} t_{1,1}s_{1,1} & t_{1,1}s_{1,2} + t_{1,2}s_{2,2} & \cdots & \sum_{i=1}^n t_{1,i}s_{i,n} \\ 0 & t_{2,2}s_{2,2} & \cdots & \sum_{i=2}^n t_{2,i}s_{i,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & t_{n,n}s_{n,n} \end{pmatrix} \end{aligned}$$

Hence,

$$\begin{aligned} [S,T] &= ST - TS \\ &= \begin{pmatrix} s_{1,1}t_{1,1} & * & \cdots & * \\ 0 & s_{2,2}t_{2,2} & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & s_{n,n}t_{n,n} \end{pmatrix} - \begin{pmatrix} t_{1,1}s_{1,1} & * & \cdots & * \\ 0 & t_{2,2}s_{2,2} & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & t_{n,n}s_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} s_{1,1}t_{1,1} - t_{1,1}s_{1,1} & * & \cdots & * \\ 0 & s_{2,2}t_{2,2} - t_{2,2}s_{2,2} & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & s_{n,n}t_{n,n} - t_{n,n}s_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \end{aligned}$$

is strictly upper triangular and thus, by our work in (a) above, nilpotent. □

(c): Let A_0 be the set of upper triangular matrices in $M_n(K)$. Show that A_0 is a *Lie algebra*, in the sense that it is closed under addition, scalar multiplication, and bracket. Also, inductively define A_i by $A_{i+1} = [A_i, A_i] = \langle [S,T] | S, T \in A_i \rangle$. Show that some $A_r = 0$.

Proof. Let

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n} \end{pmatrix}, B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & a_{1,n} \\ 0 & b_{2,2} & \cdots & b_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b_{n,n} \end{pmatrix}$$

and let $c \in K$. Then

$$\begin{aligned} A + B &= \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n} \end{pmatrix} + \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & a_{1,n} \\ 0 & b_{2,2} & \cdots & b_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} & \cdots & a_{1,n}+b_{1,n} \\ 0 & a_{2,2}+b_{2,2} & \cdots & a_{2,n}+b_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n}+b_{n,n} \end{pmatrix} \end{aligned}$$

is upper triangular and

$$cA = \begin{pmatrix} ca_{1,1} & ca_{1,2} & \cdots & ca_{1,n} \\ 0 & ca_{2,2} & \cdots & ca_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & ca_{n,n} \end{pmatrix}$$

is upper triangular, so we see that A_0 is closed under addition and scalar multiplication. Furthermore, as we saw in (b) above, $[A, B]$ is strictly upper triangular, so A_0 is closed under bracket as well. Hence, A_0 is a Lie algebra.

Now, define $A_{i+1} = [A_i, A_i]$. For $i = 1$, $A_1 = [A_0, A_0]$; as we saw in (b) above, $[S, T]$ is strictly upper triangular for all $S, T \in A_0$, so we see that A_1 consists of strictly upper triangular matrices. Now, suppose

$$S = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & 0 & a_{2,3} & \cdots & a_{2,n} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{(n-1),n} \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

and

$$T = \begin{pmatrix} 0 & b_{1,2} & b_{1,3} & \cdots & b_{1,n} \\ 0 & 0 & b_{2,3} & \cdots & b_{2,n} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{(n-1),n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

are elements of A_1. Then

$$ST = \begin{pmatrix} 0 & a_{1,2}b_{1,2} & * & \cdots & * \\ 0 & 0 & a_{2,3}b_{2,3} & \cdots & * \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{(n-1),n}b_{(n-1),n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and

$$TS = \begin{pmatrix} 0 & b_{1,2}a_{1,2} & * & \cdots & * \\ 0 & 0 & b_{2,3}a_{2,3} & \cdots & * \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{(n-1),n}a_{(n-1),n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

so

$$\begin{aligned} [S,T] &= ST - TS \\ &= \begin{pmatrix} 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & \cdots & * \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \end{aligned}$$

So elements of A_2 not only have zero diagonals but zero super-diagonals. Iterating this process, we see that elements of A_3 have zero diagonal, super-diagonal and super-super-diagonal and, thus, that all elements of A_n are zero. □

3

Let V be a real inner product space.

(a): Show that there is a homomorphism $\phi : V \to V^*$ such that $(\phi(v))(w) = \langle v, w \rangle$ for all $v, w \in V$. Show that ϕ is an isomorphism if V is finite dimensional.

Proof. First, we need to show that, for $v \in V$, $\phi(v) \in V^*$. To that end, fix $v \in V$ and let $w_1, w_2 \in V$ and $a, b \in \mathbb{R}$. Then

$$\begin{aligned} \phi(v)(aw_1 + bw_2) = \langle v, aw_1 + bw_2 \rangle &= \langle v, aw_1 \rangle + \langle v, bw_2 \rangle \\ &= \langle aw_1, v \rangle + \langle bw_2, v \rangle \\ &= a\langle w_1, v \rangle + b\langle w_2, v \rangle \\ &= a\langle v, w_1 \rangle + b\langle v, w_2 \rangle \\ &= a\phi(v)(w_1) + b\phi(v)(w_2), \end{aligned}$$

so $\phi(v)$ is, indeed, a linear functional. Since our choice of v was arbitrary, we see that $\phi(v) \in V^*$ for all $v \in V$.

Now, let $u, v, w \in V$ and let $a, b \in \mathbb{R}$. Then

$$\phi(au+bv)(w) = \langle au+bv, w \rangle = \langle au, w \rangle + \langle bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle = a\phi(v)(w) + b\phi(v)(w),$$

so ϕ is linear. Now, suppose $v \in \ker \phi$. Then $\phi(v) = 0$, which is to say that, for any $w \in V$,

$$0 = \phi(v)(w) = \langle v, w \rangle.$$

Specifically,

$$0 = \phi(v)(v) = \langle v, v \rangle,$$

which implies that $v = 0$. Hence, $\ker \phi = 0$ so ϕ is injective. Since ϕ is an injective homomorphism between vector spaces of the same dimension, ϕ is an isomorphism. □

(b): If $W \subset V$ is a subspace, what is $\phi(W^\perp)$?

Proof. Let $W^\perp = \{w \in V | \langle w, v \rangle = 0 \text{ for all } v \in W\}$. Then, for $w \in W^\perp$,

$$\phi(w)(v) = \langle w, v \rangle = 0$$

for all $v \in W$.

On the other hand, suppose $f \in V^*$ such that $f(W) = 0$. Then, by our work in (a) above, there exists $w \in V$ such that $f = \phi(w)$. Let $v \in W$. Then

$$0 = \phi(w)(v) = \langle w, v \rangle,$$

so $w \in W^\perp$. Hence, we see that

$$\phi(W^\perp) = \{f \in V^* | f(W) = 0\}.$$

□

(c): If $V = \mathbb{R}^n$ with the usual inner product, describe ϕ explicitly.

Proof. Let $B = \{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$ and let $B' = \{\delta_1, \ldots, \delta_n\}$ be the associated dual basis for $(\mathbb{R}^n)^*$. Then, for $i \in \{1, \ldots, n\}$,

$$\phi(e_i)(e_j) = \langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Hence, $\phi(e_i) = \delta_i$, so ϕ is the isomorphism $\phi_{\mathbb{R}^n,B}$ defined in PS10#5. □

4

Prove the Pythagorean Theorem for inner product spaces: If $a \perp b$ (i.e. $\langle a, b\rangle = 0$) then $||a+b||^2 = ||a||^2 + ||b||^2$. Is the converse true?

Proof. Since inner products are additive in both coordinates, we see that

$$||a+b||^2 = \langle a+b, a+b\rangle = \langle a, a+b\rangle + \langle b, a+b\rangle = \langle a, a\rangle + \langle a, b\rangle + \langle b, a\rangle + \langle b, b\rangle = ||a||^2 + ||b||^2.$$

On the other hand, suppose $||a+b||^2 = ||a||^2 + ||b||^2$ for all some $a, b \in V$ where V is an inner product space. Then the above equality indicates that

$$\langle a, b\rangle = -\langle b, a\rangle.$$

Now, if V is a real inner product space, then the inner product is symmetric, so the above equality holds only if $\langle a, b\rangle = \langle b, a\rangle = 0$. On the other hand, if V is a complex inner product space, then the inner product is conjugate symmetric, so the above indicates that

$$-\langle b, a\rangle = \langle a, b\rangle = \overline{\langle b, a\rangle};$$

hence, since the only complex numbers whose conjugates are their additive inverses are $\pm i$, we see that $\langle a, b\rangle = \pm i$. This can certainly happen; for example, in $\mathbb{C}$,

$$\langle \frac{\sqrt{3}}{2} + \frac{1}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i\rangle = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = i.$$

Now, note that, if $a = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $b = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, then

$$\begin{aligned} ||a+b||^2 &= \left|\left|\frac{\sqrt{3}}{2} + \frac{1}{2}i + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right|\right|^2 \\ &= \left|\left|\frac{\sqrt{3}+1}{2} + \frac{1-\sqrt{3}}{2}i\right|\right|^2 \\ &= \frac{4+2\sqrt{3}}{4} + \frac{4-2\sqrt{3}}{4} \\ &= 2 \\ &= 1+1 \\ &= \left|\left|\frac{\sqrt{3}}{2} + \frac{1}{2}i\right|\right|^2 + \left|\left|\frac{1}{2} - \frac{\sqrt{3}}{2}i\right|\right|^2. \end{aligned}$$

Hence, we conclude that the converse of the Pythagorean Theorem is true in real inner product spaces but not necessarily in complex inner product spaces. □

5

Let V be a finite dimensional complex inner product space. Show that $T \in \text{End } V$ is Hermitian iff $\langle Tv, v\rangle \in \mathbb{R}$ for all $v \in V$.

Proof. Suppose T is Hermitian and let $v \in V$. Then, by definition,

$$\langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\langle Tv, v\rangle};$$

hence, $\langle Tv, v\rangle \in \mathbb{R}$ for all $v \in V$.

On the other hand, suppose $\langle Tv, v\rangle \in \mathbb{R}$ for all $v \in V$. Let $v \in V$ such that $v \neq 0$. Then

$$\begin{aligned} \langle (T - T^*)v, v\rangle &= \langle Tv - T^*v, v\rangle \\ &= \langle Tv, v\rangle - \langle T^*v, v\rangle \\ &= \langle Tv, v\rangle - \langle v, Tv\rangle \\ &= \langle Tv, v\rangle - \overline{\langle Tv, v\rangle} \\ &= \langle Tv, v\rangle - \langle Tv, v\rangle \\ &= 0, \end{aligned}$$

since $\langle Tv, v\rangle \in \mathbb{R}$. Since $v \neq 0$, this means that $(T - T^*)v = 0$; since our choice of v was arbitrary, we see that $(T - T^*)v = 0$ for all $v \in V$, so $T - T^* = 0$ and, hence, $T = T^*$. Therefore, T is Hermitian. □

6

(a): Show that a non-negative matrix $T \in M_r(\mathbb{C})$ has a non-negative n^{th} root, for all $n \in \mathbb{Z}^+$.

Proof. Let T be non-negative. Then, for all $v \in \mathbb{C}^r$,

$$\langle Tv, v\rangle \geq 0.$$

Let $v_1, \dots, v_r$ be an orthonormal basis for $\mathbb{C}^r$. Then, for $i \in \{1, \dots, r\}$,

$$\langle Tv_i, v_i\rangle = a \geq 0$$

and so

$$\langle Tv_i - av_i, v_i\rangle = \langle Tv_i, v_i\rangle - \langle av_i, v_i\rangle = \langle Tv_i, v_i\rangle - a\langle v_i, v_i\rangle = \langle Tv_i, v_i\rangle - a = 0,$$

so $Tv_i - av_i = 0$ and, hence, $Tv_i = av_i$. Thus, we see that T has n linearly independent eigenvectors $v_1, \dots, v_r$ and non-negative eigenvalues $\lambda_1, \dots, \lambda_r$. Hence, T is diagonalizable; that is,

$$T = PDP^{-1}$$

for $P = [v_1 \dots v_r]$ and diagonal $D = \text{diag}(\lambda_1, \dots, \lambda_n)$. Then $\lambda_i \geq 0$ for all i, so we can form the diagonal matrix D_n from the n^{th} roots of the λ_i's. Let $T_n = PD_nP^{-1}$. Then

$$\begin{aligned} T_n^n = (PD_nP^{-1})^n &= (PD_nP^{-1})\cdots(PD_nP^{-1}) \\ &= PD_n(P^{-1}P)D_n(P^{-1}P)\cdots(P^{-1}P)D_nP^{-1} \\ &= PD_n^nP^{-1} \\ &= PDP^{-1} \\ &= T, \end{aligned}$$

so T_n is the n^{th} root of T. Now, $T_nv_i = (\lambda_i^{1/n}v_i$. Hence, for $v \in \mathbb{C}^r$, $v = \sum_{i=1}^r a_iv_i$ and

$$\begin{aligned} \langle T_nv, v\rangle &= \langle T\left(\sum_{i=1}^r a_iv_i\right), \sum_{i=1}^r a_iv_i\rangle \\ &= \langle\sum_{i=1}^r a_iTv_i, \sum_{i=1}^r a_iv_i\rangle \\ &= \sum_{i,j} a_i\overline{a_j}\langle Tv_i, v_j\rangle \\ &= \sum_{i,j} a_i\overline{a_j}\lambda_i^{1/n}\langle v_i, v_j\rangle \\ &= \sum_{i=1}^r a_i\overline{a_i}\lambda^{1/n} \\ &\geq 0, \end{aligned}$$

so T_n is non-negative. □

(b): Show more generally that for such a T we may define, for each real $\alpha \geq 0$, a matrix T^α, such that $T^1 = T$, $(T^\alpha)^\beta = T^{\alpha\beta}$, $T^\alpha T^\beta = T^{\alpha+\beta}$ and T^α varies continuously in α.

Proof. Just as before, we can diagonalize

$$T = PDP^{-1}$$

Then D^α is simply the diagonal matrix with non-negative α powers of the entries of D as its entries. Then define

$$T^\alpha = PD^\alpha P^{-1}.$$

Replacing α for $1/n$ in the proof in part (a) that T_n is non-negative shows that T^α is non-negative. Furthermore,

$$T^1 = PD^1P^{-1} = PDP^{-1} = T,$$

and, for $\alpha, \beta \in \mathbb{R}_{\geq 0}$,

$$(T^\alpha)^\beta = (PD^\alpha P^{-1})^\beta = P(D^\alpha)^\beta P^{-1} = PD^{\alpha\beta}P^{-1} = T^{\alpha\beta},$$

(where the second equality is by the definition of the β power) and

$$T^\alpha T^\beta = (PD^\alpha P^{-1})(PD^\beta P^{-1}) = PD^\alpha D^\beta P^{-1} = PD^{\alpha+\beta}P^{-1} = T^{\alpha+\beta}$$

since $a^\alpha a^\beta = a^{\alpha+\beta}$ for all $a \in \mathbb{C}$.

Finally, since $f(x) = z^x$ varies continuously in x for any $z \in \mathbb{C}$, we see that D^α varies continuously in α and, hence,

$$T^\alpha = PD^\alpha P^{-1}$$

varies continuously in α. □

(c): Regard the positive definite matrices in $M_n(\mathbb{C})$ as a subset $P \subset C^{n^2} = \mathbb{R}^{sn^2}$. Show that every element $T \in P$ can be connected to the identity matrix $1 \in P$ by a path in P. So P is a connected set.

Proof. By part (b), we know that

$$I = T^{-1} = T^{1-1} = T^0.$$

Now, define the map $c : [0,1] \to P$ by

$$c(s) = T^(1-s).$$

Then $c(0) = T^1 = T$ and $c(1) = T^0 = I$. Now, since we showed in (b) that T^α was defined for all $\alpha \in \mathbb{R}$ and continuous in α, this is certainly a continuous path. Furthermore, we see that T^α is positive definite for all α, since strict inequalities in (a) and (b) beget strict inequalities, so c is a path in P. Thus, we can connect every element in P to the identity matrix by a continuous path. Hence, if $T, T' \in P$, then we can connect T to the identity with a path c and T' to the identity with a map c'; define $\tilde{c} : [0,1] \to P$ by

$$\tilde{c}(s) = \begin{cases} c(2s) & 0 \leq s \leq 1/2 \\ c'(1-2s) & 1/2 \leq s \leq 1 \end{cases}$$

Then c is a continuous path from T to T'. Since our choice of T and T' was arbitrary, we see that P is connected. □

7

(a): Let $A \in M_n(\mathbb{R})$ be a positive definite matrix. Write $x = (x_1, \ldots, x_n)$. Show that

$$\int_{R^n} e^{-\langle Ax, x\rangle} dx_1 \cdots dx_n = \frac{\pi^{n/2}}{\sqrt{\det A}}.$$

Proof. Since A is positive definite, we know, by our work in problem 6 above, that A has a square root, $\sqrt{A}$. □

(b): Evaluate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+2xy-4y^2-4yz-2z^2}dxdydz$.

Answer: Let

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then, if $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, then

$$Av = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ 4y+4z \\ 2z \end{pmatrix}.$$

Hence,

$$\langle Av, v\rangle = x(x-2y) + y(4y+4z) + z(2z) = x^2 - 2xy + 4y^2 + 4yz + 2z^2 > 0$$

for all $v \neq 0$, so A is positive definite. Therefore, by our work in part (a) above,

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+2xy-4y^2-4yz-2z^2}dxdydz &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\langle Av,v\rangle}dxdydz \\
&= \frac{\pi^{3/2}}{\sqrt{\det A}} \\
&= \frac{\pi^{3/2}}{\sqrt{8}} \\
&= \frac{\pi^{3/2}}{2\sqrt{2}} \\
&= \left(\frac{\pi}{2}\right)^{3/2}.
\end{aligned}$$

♣

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu